Computing the Values of Trigonometrical Functions

The trigonometrical functions take a relatively large amount of time to compute. In some applications, such as the fast Fourier transform (FFT), it is required to evaluate the sines and cosines of a sequence of harmonically related angles denoted by $\theta \times j; j = 0, 1, \ldots, n$. In such cases, it may be efficient to use a simple recursive scheme based on the trigonometrical identities:

$$\begin{aligned}\cos(j\theta) &= 2\cos(\theta)\cos([j-1]\theta) - \cos([j-2]\theta),\\ \sin(j\theta) &= 2\sin(\theta)\cos([j-1]\theta) + \sin([j-2]\theta).\end{aligned}$$

These follow, respectively, from the trigonometrical identities

$$\begin{aligned}\cos A \cos B &= \frac{1}{2}\big\{\cos(A+B) + \cos(A-B)\big\},\\ \cos A \sin B &= \frac{1}{2}\big\{\sin(A+B) - \sin(A-B)\big\}.\end{aligned}$$

which are to be found under (13.126). We set $A = [j-1]\theta$ and $B = \theta$. The recursive schemes depend only on single evaluations of $\cos(\theta)$ and $\sin(\theta)$ made at the start.

Several alternative schemes are available, of which we show a further two. In the first of these, we generate the cosine and sine values as independent sequences:

$$\begin{aligned}C_{j+1} &= R \times \cos(j\theta) + C_j,\\ \cos([j+1]\theta) &= \cos(j\theta) + C_{j+1}\end{aligned}$$

and

$$\begin{aligned}S_{j+1} &= R \times \sin(j\theta) + S_j,\\ \sin([j+1]\theta) &= \sin(j\theta) + S_{j+1}\end{aligned}$$

where the constant multiplier is

$$R = -4\sin^2(\theta/2)$$

and the initial values are

$$\begin{aligned}C_0 &= 2\sin^2(\theta/2)\\ S_0 &= 2\sin(\theta)\\ \cos(0) &= 1\\ \sin(0) &= 0.\end{aligned}$$

Another method, in which the sine and cosine are computed as a pair with four multiplications instead of two is as follows:

$$\cos([j+1]\theta) = \{C \times \cos(j\theta) - S \times \sin(j\theta)\} + \cos(j\theta)$$

and

$$\sin([j+1]\theta) = \{C \times \sin(j\theta) - S \times \cos(j\theta)\} + \sin(j\theta)$$

where

$$\begin{aligned} C &= 2\sin^2(\theta/2) \\ S &= 2\sin(\theta) \end{aligned}$$

Singleton (1967) reports that both of the latter methods result in a low level of errors, with the second method being marginally more accurate.

Reference

Singleton, R.C., (1967), On Computing the Fast Fourier Transform, Communications of the ACM, 10, 647–654.