Transfer Functions

Temporal regression models are more easily intelligible if they can be represented by equations in the form of

$$y(t) = \sum \omega_i x(t-i) + \sum \psi_i \varepsilon(t-i), \tag{1}$$

where there is no lag scheme affecting the output sequence $y(t)$. This equation depicts $y(t)$ as a sum of a systematic component $h(t) = \sum \omega_i x(t-i)$ and a stochastic component $\eta(t) = \sum \psi_i \varepsilon(t-i)$. Both of these components comprise transfer-function relationships whereby the input sequences $x(t)$ and $\varepsilon(t)$ are translated, respectively, into output sequences $h(t)$ and $\eta(t)$.

In the case of the systematic component, the transfer function describes how the observable signal $x(t)$ is commuted into the sequence of systematic values which explain a major part of $y(t)$ and which may be used in forecasting it.

In the case of the stochastic component, the transfer function describes how a white-noise process $\varepsilon(t)$, comprising a sequence of independent random elements, is transformed into a sequence of serially correlated disturbances. In fact, the elements of $h(t)$ represent efficient predictors of the corresponding elements of $y(t)$ only when $\eta(t) = \psi_0 \varepsilon(t)$ is white noise.

A fruitful way of characterising a transfer function is to determine the response, in terms of its output, to a variety of standardised input signals. Examples of such signals, which have already been presented, are the unit-impulse, the unit-step and the sinusoidal and complex exponential sequences defined over a range of frequencies.

The impulse response of the systematic transfer function is given by the sequence $h(t) = \sum_i \omega_i \delta(t-i)$. Since the sequence of coefficients $\omega(i) = \{\omega_0, \omega_1, \ldots\}$ is zero-valued for all $i \leq 0$, it follows that $h(t) = 0$ for all $t < 0$. By setting $t = \{0, 1, 3, \ldots\}$, we generate the a sequence beginning with

$$\begin{aligned} h_0 &= \omega_0, \\ h_1 &= \omega_1, \\ h_2 &= \omega_2. \end{aligned} \tag{2}$$

The impulse-response function is therefore nothing but the sequence of coefficients which define the transfer function.

The response of the transfer function to the unit-step sequence is given by $h(t) = \sum_i \omega_i u(t-i)$. By setting $t = \{0, 1, 3, \ldots\}$, we generate a sequence beginning with

$$\begin{aligned} h_0 &= \omega_0, \\ h_1 &= \omega_0 + \omega_1, \\ h_2 &= \omega_0 + \omega_1 + \omega_2. \end{aligned} \tag{3}$$

Thus the step response is obtained simply by cumulating the impulse response.

In most applications, the output sequence $h(t)$ of the transfer function should be bounded in absolute value whenever the input sequence $x(t)$ is bounded. This is described as the condition of bounded input–bounded output stability or *BIBO* stability.

If the coefficients $\{\omega_0, \omega_1, \ldots, \omega_p\}$ of the transfer function form a finite sequence, then a necessary and sufficient condition for such stability is that $|\omega_i| < \infty$ for all i, which is to say that the impulse-response function must be bounded.

If $\omega(i) = \{\omega_1, \omega_2, \ldots\}$ is a infinite sequence, then it is necessary, in addition, that $|\sum \omega_i| < \infty$, which is the condition that the step-response function is bounded. Together, the two conditions are equivalent to the single condition that $\sum |\omega_i| < \infty$.

To confirm that the latter is a sufficient condition for stability, let us consider any input sequence $x(t)$ which is bounded such that $|x(t)| < M$ for some finite M. Then

$$|h(t)| = \left|\sum \omega_i x(t-i)\right| \leq M \left|\sum \omega_i\right| < \infty, \tag{4}$$

and so the output sequence $h(t)$ is bounded. To show that the condition is necessary, imagine that the $\sum |\omega_i|$ is unbounded. Then a bounded input sequence can be found which gives rise to an unbounded output sequence. One such input sequence is specified by

$$x_{-i} = \begin{cases} \dfrac{\omega_i}{|\omega_i|}, & \text{if } \omega_i \neq 0; \\ 0, & \text{if } \omega_i = 0. \end{cases}$$

This gives

$$h_0 = \sum \omega_i x_{-i} = \sum |\omega_i|, \tag{5}$$

and so $h(t)$ is unbounded.