Signal Extraction in the Case of a Random Walk Observed with Error

Consider an observable random vector

(1) $$y = \xi + \eta$$

where ξ contains the values of an unobserved signal sequence and where η contains the values of a noise corruption. Imagine that ξ and η have a know covariance structure. Then the simple theory of conditional expectations indicates that an optimal estimate of the signal would be provided by the formula

(2) $$E(\xi|y) = E(\xi) + C(\xi, y)D^{-1}(y)\{y - E(y)\},$$

where $D(y)$ stands for the variance–covariance matrix of y and $C(\xi, y)$ stands for the matrix of the covariances of y and ξ. We shall assume that ξ is generated by a random walk such that

(3) $$\xi = S\varepsilon + i\xi_0,$$

where S is a summation matrix whose tth row has t units as its leading elements and $T - t$ zeros in the following positions and where i is the summation vector comprising T units. The vector ε contains a sequence of independently and identically distributed elements from a zero-mean white-noise sequence with variance σ_ε^2, whilst ξ_0 is a presample element from the process generating ξ. Then

(4) $$E(\xi) = iE(\xi_0) \qquad \text{and} \qquad D(\xi) = \sigma_\varepsilon^2 SS' + p_0 ii',$$

where $p_0 = V(\xi_0)$. We assume that the elements of the noise vector η are generated by a zero-mean white-noise sequence with has a variance of σ_η^2. Therefore, the vector y has the same expected value as the vector ξ, which is $E(y) = E(\xi) = iE(\xi_0)$. From these assumptions, it follows that

(5) $$D(y) = D(\xi) + \sigma_\eta^2 I \qquad \text{and} \qquad C(\xi, y) = D(\xi).$$

Now the inverse of the summation matrix S is the differencing matrix $\nabla = S^{-1}$ which has units on the diagonal, negative units on the first subdiagonal and zeros elsewhere. It follows that

(6) $$\begin{aligned} C(\xi, y) &= S(\sigma_\varepsilon^2 I + p_0 e_1 e_1')S' \quad \text{and} \\ D(y) &= S(\sigma_\varepsilon^2 I + p_0 e_1 e_1' + \sigma_\eta^2 \nabla\nabla')S', \end{aligned}$$

where $e_1 e_1' = \nabla ii'\nabla'$ is a matrix with a unit in the leading position and with zeros elsewhere. On substituting these details into equation (2), we find that

(7)
$$E(\xi|y) = iE(\xi_0) + S(\sigma_\varepsilon^2 I + p_0 e_1 e_1')(\sigma_\varepsilon^2 I + p_0 e_1 e_1' + \sigma_\eta^2 \nabla\nabla')^{-1}\{\nabla y - e_1 E(\xi_0)\},$$