

THE EQUATIONS OF THE KALMAN FILTER

The state-space model, which underlies the Kalman filter, consists of two equations

$$y_t = H_t \xi_t + \eta_t, \quad \text{Observation Equation} \quad (1)$$

$$\xi_t = \Phi_t \xi_{t-1} + \nu_t, \quad \text{Transition Equation} \quad (2)$$

where y_t is the observation on the system and ξ_t is the state vector. The observation error η_t and the state disturbance ν_t are mutually uncorrelated random vectors of zero mean with dispersion matrices

$$D(\eta_t) = \Omega_t \quad \text{and} \quad D(\nu_t) = \Psi_t. \quad (3)$$

It is assumed that the matrices H_t , Φ_t , Ω_t and Ψ_t are known for all $t = 1, \dots, n$ and that an initial estimate x_0 is available for the state vector ξ_0 at time $t = 0$ together with a dispersion matrix $D(\xi_0) = P_0$. The empirical information available at time t is the set of observations $\mathcal{I}_t = \{y_1, \dots, y_t\}$.

The Kalman-filter equations determine the state-vector estimates $x_{t|t-1} = E(\xi_t | \mathcal{I}_{t-1})$ and $x_t = E(\xi_t | \mathcal{I}_t)$ and their associated dispersion matrices $P_{t|t-1}$ and P_t . From $x_{t|t-1}$, the prediction $\hat{y}_{t|t-1} = H_t x_{t|t-1}$ is formed which has a dispersion matrix F_t . A summary of these equations is as follows:

$$x_{t|t-1} = \Phi_t x_{t-1}, \quad \text{State Prediction} \quad (4)$$

$$P_{t|t-1} = \Phi_t P_{t-1} \Phi_t' + \Psi_t, \quad \text{Prediction Dispersion} \quad (5)$$

$$e_t = y_t - H_t x_{t|t-1}, \quad \text{Prediction Error} \quad (6)$$

$$F_t = H_t P_{t|t-1} H_t' + \Omega_t, \quad \text{Error Dispersion} \quad (7)$$

$$K_t = P_{t|t-1} H_t' F_t^{-1}, \quad \text{Kalman Gain} \quad (8)$$

$$x_t = x_{t|t-1} + K_t e_t, \quad \text{State Estimate} \quad (9)$$

$$P_t = (I - K_t H_t) P_{t|t-1}. \quad \text{Estimate Dispersion} \quad (10)$$

Alternative expressions are available for P_t and K_t are available on the assumption that Ω_t is nonsingular:

$$P_t = (P_{t|t-1}^{-1} + H_t' \Omega_t^{-1} H_t)^{-1}, \quad (11)$$

$$K_t = P_t H_t' \Omega_t^{-1}. \quad (12)$$

By applying the well-known matrix inversion lemma to the expression on the RHS of (11), we obtain the original expression for P_t given under (10). To verify the identity $P_{t|t-1} H_t' F_t^{-1} = P_t H_t' \Omega_t^{-1}$ which equates (8) and (12), we write it as $P_t^{-1} P_{t|t-1} H_t' = H_t' \Omega_t^{-1} F_t$. The latter is readily confirmed using the expression for P_t from (11) and the expression for F_t from (7).

Derivation of the Kalman Filter. The equations of the Kalman filter may be derived using the ordinary algebra of conditional expectations which

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indicates that, if x, y are jointly distributed variables which bear the linear relationship $E(y|x) = \alpha + B\{x - E(x)\}$, then

$$E(y|x) = E(y) + C(y, x)D^{-1}(x)\{x - E(x)\}, \quad (13)$$

$$D(y|x) = D(y) - C(y, x)D^{-1}(x)C(x, y), \quad (14)$$

$$E\{E(y|x)\} = E(y), \quad (15)$$

$$D\{E(y|x)\} = C(y, x)D^{-1}(x)C(x, y), \quad (16)$$

$$D(y) = D(y|x) + D\{E(y|x)\}, \quad (17)$$

$$C\{y - E(y|x), x\} = 0. \quad (18)$$

Of the equations listed under (4)—(10), those under (6) and (8) are merely definitions.

To demonstrate equation (4), we use (15) to show that

$$\begin{aligned} E(\xi_t|\mathcal{I}_{t-1}) &= E\{E(\xi_t|\xi_{t-1})|\mathcal{I}_{t-1}\} \\ &= E\{\Phi_t\xi_{t-1}|\mathcal{I}_{t-1}\} \\ &= \Phi_t x_{t-1}. \end{aligned} \quad (19)$$

We use (17) to demonstrate equation (5):

$$\begin{aligned} D(\xi_t|\mathcal{I}_{t-1}) &= D(\xi_t|\xi_{t-1}) + D\{E(\xi_t|\xi_{t-1})|\mathcal{I}_{t-1}\} \\ &= \Psi_t + D\{\Phi_t\xi_{t-1}|\mathcal{I}_{t-1}\} \\ &= \Psi_t + \Phi_t P_{t-1} \Phi_t'. \end{aligned} \quad (20)$$

To obtain equation (7), we substitute (1) into (6) to give $e_t = H_t(\xi_t - x_{t|t-1}) + \eta_t$. Then, in view of the statistical independence of the terms on the RHS, we have

$$\begin{aligned} D(e_t) &= D\{H_t(\xi_t - x_{t|t-1})\} + D(\eta_t) \\ &= H_t P_{t|t-1} H_t' + \Omega_t = D(y_t|\mathcal{I}_{t-1}). \end{aligned} \quad (21)$$

To demonstrate the updating equation (9), we begin by noting that

$$\begin{aligned} C(\xi_t, y_t|\mathcal{I}_{t-1}) &= E\{(\xi_t - x_{t|t-1})y_t'\} \\ &= E\{(\xi_t - x_{t|t-1})(H_t\xi_t + \eta_t)'\} \\ &= P_{t|t-1} H_t'. \end{aligned} \quad (22)$$

It follows from (13) that

$$\begin{aligned} E(\xi_t|\mathcal{I}_t) &= E(\xi_t|\mathcal{I}_{t-1}) + C(\xi_t, y_t|\mathcal{I}_{t-1})D^{-1}(y_t|\mathcal{I}_{t-1})\{y_t - E(y_t|\mathcal{I}_{t-1})\} \\ &= x_{t|t-1} + P_{t|t-1} H_t' F_t^{-1} e_t. \end{aligned} \quad (23)$$

The dispersion matrix under (10) for the updated estimate is obtained via equation (14):

$$\begin{aligned} D(\xi_t|\mathcal{I}_t) &= D(\xi_t|\mathcal{I}_{t-1}) - C(\xi_t, y_t|\mathcal{I}_{t-1})D^{-1}(y_t|\mathcal{I}_{t-1})C(y_t, \xi_t|\mathcal{I}_{t-1}) \\ &= P_{t|t-1} - P_{t|t-1} H_t' F_t^{-1} H_t P_{t|t-1}. \end{aligned} \quad (24)$$