MA Processes with Common Autocovariances

The purpose is to show that several moving-average process can share the same autocovariance function. Consider the moving-average process

$$y(t) = \mu(L)\varepsilon(t), \tag{1}$$

where $\mu(z)$ has the factorisation

$$\mu(z) = \prod_{i=1}^{q} \left(1 - \frac{z}{\lambda_i}\right). \tag{2}$$

We assume that $|\lambda_i| > 1$ for all i, which is to say that all of the roots of $\mu(z) = 0$ lie outside the unit circle. Therefore the MA process obeys the conditions of invertibility. The autocovariance generating function is defined by

$$\gamma(z) = \sigma_\varepsilon^2 \mu(z)\mu(z^{-1}), \tag{3}$$

wherein $\sigma_\varepsilon^2 = V\{\varepsilon(t)\}$. The same autocovariances will be generated if we invert some of the roots of $\mu(z)$ and revise the value of σ_ε^2.

To illustrate this, let us consider the case of a real root λ. To invert this root, we may multiply $\mu(z)$ by $1 - \lambda z$ and divide by $1 - z/\lambda$. The result is

$$\tilde{\mu}(z) = \mu(z)\frac{(1 - \lambda z)}{(1 - z\lambda^{-1})}. \tag{4}$$

Hence we get

$$\begin{aligned}\tilde{\mu}(z)\tilde{\mu}(z^{-1}) &= \mu(z)\mu(z^{-1})\frac{(1 - \lambda z)(1 - \lambda z^{-1})}{(1 - z\lambda^{-1})(1 - \{z\lambda\}^{-1})} \\ &= \lambda^2 \mu(z)\mu(z^{-1});\end{aligned} \tag{5}$$

and it follows that

$$\gamma(z) = \tilde{\sigma}_\varepsilon^2 \tilde{\mu}(z)\tilde{\mu}(z^{-1}) \quad \text{where} \quad \tilde{\sigma}_\varepsilon^2 = \sigma_\varepsilon^2/\lambda^2. \tag{6}$$

Clearly, we can can invert an arbitrary selection of the roots of $\mu(z)$ in this way and still retain the same autocovariance generating function. By taking account of all such inversions, we can define the complete class of the processes which share the common autocovariance function. Amongst such a class, there can be no more than one process which satisfies the condition of stationarity which requires every root of $\mu(z) = 0$ to lie outside the unit circle.

As an example, consider the MA(2) process which is specified by

$$
\begin{aligned}
y(t) &= (\mu_0 + \mu_1 L + \mu_2 L^2)\varepsilon(t) \\
&= \left(1 + \frac{5}{6}L + \frac{1}{6}L^2\right)\varepsilon(t),
\end{aligned} \tag{7}
$$

where $\sigma_\varepsilon^2 = 36$. With $\mu_0 = 1$, $\mu_1 = 5/6$ and $\mu_2 = 1/6$, the autocovariances of this process are given by

$$
\begin{aligned}
\gamma_0 &= \sigma_\varepsilon^2(\mu_0^2 + \mu_1^2 + \mu_2^2) = 62, \\
\gamma_1 &= \sigma_\varepsilon^2(\mu_0\mu_1 + \mu_1\mu_2) = 35, \\
\gamma_2 &= \sigma_\varepsilon^2\mu_0\mu_2 = 6.
\end{aligned} \tag{8}
$$

Now consider the factorisation

$$
\begin{aligned}
\mu(z) &= \left(1 + \frac{5}{6}z + \frac{1}{6}z^2\right) \\
&= \left(1 + \frac{1}{3}z\right)\left(1 + \frac{1}{2}z\right).
\end{aligned} \tag{9}
$$

This shows that $\lambda_1 = -3$ and $\lambda_2 = -2$, and both of these roots are outside the unit circle, which satisfies the condition of invertibility. To obtain an non-invertible model which generates the same autocovariances, we can invert the roots so as to give

$$
\begin{aligned}
\mu_*(z) &= (1 + 3z)(1 + 2z) \\
&= \left(1 + 5z + 6z^2\right)
\end{aligned} \tag{10}
$$

and

$$
V\{\varepsilon_*(t)\} = \frac{\sigma_\varepsilon^2}{|\lambda_1\lambda_2|^2} = 1. \tag{11}
$$

There are two other non-invertible models which we can devise which also generated these autocovariances.

A particular feature of the invertible model in comparison with the others is that the corresponding transfer function entails the minimum time delays in the mapping from $\varepsilon(t)$ to $y(t)$. This can be confirmed by comparing the normalised step-response functions which are implied by the various models.