DIFFERENCE AND DIFFERENTIAL EQUATIONS COMPARED

Frequency Response Function: Continuous Systems

Given the unit impulse response $h(t)$ of a continuous-time system, the mapping from the system's input $x(t)$ to its output $y(t)$ can be expressed via the following convolution integral:

$$y(t) = \int_{\infty}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{1}$$

The frequency response of the system is the output associated with a sinusoidal input $2\cos(\omega t) = e^{i\omega t} + e^{-i\omega t}$ or with a complex exponential input $x(t) = e^{i\omega t}$. The convolution integral gives

$$\begin{aligned} y(t) &= \int_{\infty}^{\infty} h(\tau)e^{i\omega(t-\tau)}d\tau \\ &= e^{i\omega t}\int_{\infty}^{\infty} h(\tau)e^{-i\omega\tau}d\tau \\ &= e^{i\omega t}H(i\omega) \end{aligned} \tag{2}$$

Here, $H(i\omega)$ constitutes the frequency response function of the system. The relationship

$$H(i\omega) = \int_{\infty}^{\infty} h(\tau)e^{-i\omega\tau}d\tau \tag{3}$$

and its inverse indicate that the impulse response and the frequency response are a Fourier pair, and we may write $h(\tau) \longleftrightarrow H(i\omega)$.

Imagine that the system is governed by a differential equation that can be written as

$$\phi(D)y(t) = \theta(D)x(t), \tag{4}$$

where $\phi(D)$ and $\theta(D)$ are polynomials in the differential operator $D = d/dt$. Then, according the result (5.65)(i) of T.S.D., there is $\phi(D)e^{\kappa t} = \phi(\kappa)e^{\kappa t}$. Therefore, on setting $x(t) = e^{i\omega t}$ and $y(t) = e^{i\omega t}H(i\omega)$ in (4), we get

$$H(i\omega)\phi(i\omega)e^{i\omega t} = \theta(i\omega)e^{i\omega t}, \tag{5}$$

from which it follows that

$$H(i\omega) = \frac{\theta(i\omega)}{\phi(i\omega)}. \tag{6}$$

Usually, we depict the effect of the frequency response function in terms of its gain effect and its phase effect. This entails expressing the complex function $H(i\omega) = H^{re}(i\omega) + iH^{im}(i\omega)$ in polar form:

$$\begin{aligned} &H(i\omega) = |H(i\omega)|e^{i\theta(\omega)}, \qquad \text{where} \quad \tan\theta(\omega) = \frac{H^{im}(i\omega)}{H^{re}(i\omega)} \\ &\text{and} \quad |H(i\omega)|^2 = H(i\omega)H(-i\omega) = \{H^{re}(i\omega)\}^2 + \{H^{im}(i\omega)\}^2. \end{aligned} \tag{7}$$

Example. Consider a second-order oscillatory system subject to viscous damping and with a sinusoidal forcing function:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + hy = \cos(\omega t). \tag{8}$$

This can be written as $\phi(D)y(t) = x(t)$, where $\phi(D) = mD^2 + cD + h$. we wish to find $|H(i\omega)|^2 = \{\phi(i\omega)\phi(-i\omega)\}^{-1}$. The following table is helpful in performing the multiplications:

	$m(-i\omega)^2$	$c(-i\omega)$	h
$m(i\omega)^2$	$m^2\omega^4$	$-icm\omega^3$	$-hm\omega^2$
$c(i\omega)$	$icm\omega^3$	$c^2\omega^2$	$ihc\omega$
h	$-hm\omega^2$	$-ihc\omega$	h^2

The outcome is that

$$|H(i\omega)|^2 = \frac{1}{(h - m\omega^2)^2 + (c\omega)^2} \quad \text{and} \quad \tan\theta = \frac{c\omega}{h - m\omega^2}. \tag{9}$$

In giving this expression an intepretation, we may observe that the natural frequency of an undamped system, obtained by setting $c = 0$ in (8), is just $\omega_n = \sqrt{h/m}$. This gives $h = m\omega_n^2$. Putting the latter into (9) shows that the amplitude gain is greatest when the driving frequency ω coincides with the natural resonant frequency ω_n of the system.

Frequency Response Function: Discrete Systems

Given the impulse response $\psi(j) = \{\phi_j\}$ of a discrete-time system, the mapping from the system's input $x(t)$ to its output can be expressed via the following discrete-time convolution:

$$y(t) = \sum_j \phi_j x(t - j). \tag{10}$$

The frequency response of the system may be defined as the mapping $\phi(\omega)$ from a complex exponential input $x(t) = e^{i\omega t}$ to the corresponding output $y(t) = \phi(\omega)e^{i\omega t}$. We have

$$y(t) = \sum_j \phi_j e^{i\omega t - j} = e^{i\omega t} \sum_j \phi_j e^{-i\omega j} = e^{i\omega t}\phi(\omega) \tag{11}$$

from which we get

$$\phi(\omega) = \sum_j \alpha_j e^{-i\omega j}. \tag{12}$$

Consider a difference equation system expressed in terms of the lag operator L by

$$\alpha(L)y(t) = \beta(L)x(t). \tag{13}$$

Substituting $x(t) = e^{i\omega t}$ and $y(t) = \phi(\omega)e^{i\omega t}$ into the equation gives us

$$\begin{aligned} \phi(\omega)\sum \alpha_j e^{i\omega t - j} &= \phi(\omega)e^{i\omega t}\sum \alpha_j e^{-i\omega j} \\ &= \sum \beta_j e^{i\omega t - j} \\ &= e^{i\omega t}\sum \beta_j e^{-i\omega j} \end{aligned} \tag{14}$$

This indicates that the frequency response function can be expressed as

$$\phi(\omega) = \frac{\sum \beta_j e^{-i\omega j}}{\sum \alpha_j e^{-i\omega j}} = \frac{\alpha(e^{-i\omega})}{\beta(e^{-i\omega})}. \tag{15}$$

Here $\alpha(e^{-i\omega})$ and $\beta(e^{-i\omega})$ may be formed from the polynomials $\alpha(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_p z^p$ and $\beta(z)$ by setting $z = e^{-i\omega}$.

As in the case of the continuous-time system, we usually require the frequency response function to be expressed in terms of the factors affecting the gain and the phase of the sinusoidal input. The factorisation is

$$\begin{aligned} &\phi(\omega) = |\phi(\omega)|e^{i\theta(\omega)}, \qquad \text{where} \quad \tan\theta(\omega) = \frac{\phi^{im}(\omega)}{\phi^{re}(\omega)} \\ &\text{and} \quad |\phi(\omega)|^2 = \phi(\omega)\phi(-\omega) = \{\phi^{re}(\omega)\}^2 + \{\phi^{im}(\omega)\}^2. \end{aligned} \tag{16}$$

Example. Consider a second-order difference equation that can we written as

$$(\alpha_0 + \alpha_1 L + \alpha_2 L^2)y(t) = \cos(\omega t) \tag{17}$$

or, more summarily, as $\alpha(L)y(t) = x(t)$. The following table is helpful in performing the multiplications $\alpha(z)\alpha(z^{-1})$, whereafter we may set $z = e^{-i\omega}$ to obtain the squared modulus of the frequency response function:

	α_0	$\alpha_1 z^{-1}$	$\alpha_2 z^{-2}$
α_0	α_0^2	$\alpha_0\alpha_1 z^{-1}$	$\alpha_0\alpha_2 z^{-2}$
$\alpha_1 z$	$\alpha_0\alpha_1 z$	α_1^2	$\alpha_1\alpha_2 z^{-1}$
$\alpha_2 z^2$	$\alpha_0\alpha_2 z^2$	$\alpha_1\alpha_2 z$	α_2^2

On gathering the terms and setting $z + z^{-1} = 2\cos(\omega)$, $z^2 + z^{-2} = 2\cos(2\omega)$, the outcome is

$$|\phi(\omega)|^2 = \frac{1}{(\alpha_0^2 + \alpha_1^2 + \alpha_2^2) + 2\alpha_0\alpha_1\cos(\omega) + 2\alpha_0\alpha_1(2\omega)}. \tag{18}$$

On recognising that $e^{-i\omega} = \cos(\omega) - i\sin(\omega)$, one can also see that

$$\begin{aligned} \phi^{re}(\omega) &= \alpha_0 + \alpha_1 \cos(\omega) + \alpha_2 \cos(2\omega) \quad \text{and} \\ \phi^{im}(\omega) &= \alpha_0 + \alpha_1 \sin(\omega) + \alpha_2 \sin(2\omega). \end{aligned} \tag{19}$$

2nd-Order Differential and Difference Equations

In the following two sections, we shall display the analytic solutions of a second-order differential equation and a second-order differential equation. We shall concentrate on the cases where the solutions of the auxiliary equations are in terms of a pair of conjugate complex roots.

In this case, the equations generate sinusoidal trajectories. Moreover, provided that the angular velocity of the sinusoids is less than the Nyquist value of π radians per period, the differential and the difference equations can be used interchangeably.

The correspondence between the alternative formulations can be established via a straightforward comparison of the complex exponential solution of the differential equation with the complex geometric solution of the difference equation. The one-to-one mapping between the coefficients of the alternative equations is less straightforward. It cannot be expressed via an algebra that entails only the binary operations of addition and multiplication. It also entails logarithmic and exponential transformations.

The Differential Equation

The second-order homogeneous differential equation can be represented by

$$(\phi_0 D^2 + \phi_1 D + \phi_2)y(t) = 0. \tag{20}$$

The auxiliary equation is

$$\begin{aligned} \phi_0 s^2 + \phi_1 s + \phi_2 &= \phi_0(s - \kappa_1)(s - \kappa_2) \\ &= \phi_0\{s^2 - (\kappa_1 + \kappa_2)s + \kappa_1\kappa_2\}. \end{aligned} \tag{21}$$

The roots are

$$\kappa_1, \kappa_2 = \frac{-\phi_1 \pm \sqrt{\phi_1^2 - 4\phi_0\phi_2}}{2\phi_0} \tag{22}$$

In the case of complex roots, we have

$$\kappa = \gamma + i\omega \qquad \text{and} \qquad \kappa^* = \gamma - i\omega, \tag{23}$$

where

$$\gamma = -\frac{\phi_1}{2\phi_0} \qquad \text{and} \qquad \omega = \frac{\sqrt{4\phi_0\phi_2 - \phi_1^2}}{2\phi_0}, \tag{24}$$

fromm which it follows that

$$\gamma^2 + \omega^2 = \frac{\phi_2}{\phi_0}. \tag{25}$$

The roots contribute to the following expression for the solution of the differential equation:

$$\begin{aligned} q(t) &= ce^{(\gamma+i\omega)t} + c^* e^{(\gamma-i\omega)t} \\ &= e^{\gamma t}\{ce^{i\omega t} + c^* e^{-i\omega t}\} \end{aligned} \tag{26}$$

Here there are the conjugate numbers

$$\begin{aligned} c &= \sigma(\cos\theta + i\sin\theta) = \sigma e^{i\theta}, \\ c^* &= \sigma(\cos\theta - i\sin\theta) = \sigma e^{-i\theta}. \end{aligned} \tag{27}$$

Thus, we have

$$\begin{aligned} q(t) &= \sigma e^{\gamma t}\left\{e^{i(\omega t-\theta)} + e^{-i(\omega t-\theta)}\right\} \\ &= 2\sigma e^{\gamma t}\cos(\omega t - \theta). \end{aligned} \tag{28}$$

The Difference Equation

The second-order homogeneous difference equation can be represented by

$$(\alpha_0 + \alpha_1 L + \alpha_2 L^2)y(t) = 0. \tag{29}$$

The auxiliary equation is

$$\begin{aligned} \alpha_0 z^2 + \alpha_1 z + \alpha_2 &= \alpha_0(z - \mu_1)(z - \mu_2) \\ &= \alpha_0\{z^2 - (\mu_1 + \mu_2)z + \mu_1\mu_2\}. \end{aligned} \tag{30}$$

The roots are

$$\mu_1, \mu_2 = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}}{2\alpha_0}. \tag{31}$$

In the case of complex roots, we have

$$\begin{aligned} \mu &= \beta + i\delta = \rho(\cos\omega + i\sin\omega) = \rho e^{i\omega}, \\ \mu^* &= \beta - i\delta = \rho(\cos\omega - i\sin\omega) = \rho e^{-i\omega}. \end{aligned} \tag{32}$$

where

$$\beta = -\frac{\alpha_1}{2\alpha_0}, \quad \delta = \frac{4\alpha_0\alpha_2 - \alpha_1^2}{2\alpha_0}, \quad \rho = \beta^2 + \delta^2 \quad \text{and} \quad \tan\omega = \frac{\delta}{\beta}. \tag{33}$$

The roots contribute to the following expression for the solution of the difference equation:

$$q(t) = c(\rho e^{i\omega})^t + c^*(\rho e^{-i\omega})^t, \tag{34}$$

Here there are the conjugate numbers

$$\begin{aligned} c &= \sigma(\cos\theta + i\sin\theta) = \sigma e^{i\theta}, \\ c^* &= \sigma(\cos\theta - i\sin\theta) = \sigma e^{-i\theta}. \end{aligned} \tag{35}$$

Thus, we have

$$\begin{aligned} q(t) &= \sigma\rho^t\left\{e^{i(\omega t-\theta)} + e^{-i(\omega t-\theta)}\right\} \\ &= 2\sigma\rho^t\cos(\omega t - \theta). \end{aligned} \tag{36}$$

The Comparison

The relationship between the differential and difference equations can be made in terms of equations (28) and (36). First there is the damping term. In (28), it is ρ^t whereas, in (36) it is $e^{\gamma t}$. Equating the two gives $\gamma = \ln\rho$. Then there is the angular velocity ω, which is common to the two equations. From (23), we have $\omega = -i(\kappa - \kappa^*)/2$, whereas (33) gives $\tan\omega = \delta/\beta$, where $\delta = -i(\mu - \mu^*)/2$.

The parmeters σ and θ, which represent, respectively, the amplitude and the phase of the damped sinusoial solution of the homogeneous equations, are products of the initial conditions and they can be ignored.

We may also observe that, since they are homogeneous, both equation (20) and (28) are amenable to arbitrary normalisations. It is makes sense to set $\phi_0 = 1$ in (20) and to set $\alpha_0 = 1$ in (29).

Since the system is liable to be calibrated by estimating the parameters of the difference equation from discrete time data, we are more likely to wish to express the parameters of the differential equation in terms of those of the difference equation than vice versa. The essential equations are (24) and (25). The route is from ρ to $\gamma = \ln\rho$. Then, we can use $\gamma = -\phi_1/2\phi_0$ to find ϕ_1 subject to the normalisation of ϕ_0. Finally, knowing ω, we can use $\phi_2/\phi_0 = \gamma^2 + \omega^2$ to find ϕ_2 subject to the same normalisation of ϕ_0.

The mapping from the parameters of the differential equation to those of the difference equation is more straightforward. Given the normalisation $\alpha_0 = 1$, we have

$$\begin{aligned} 0 &= z^2 - (\mu_1 + \mu_2)z + \mu_1\mu_2 \\ &= z^2 + \alpha_1 z + \alpha_2 \\ &= z^2 + 2\rho\cos(\omega) + \rho^2. \end{aligned}$$

Therefore

$$\alpha_2 = \rho^2 = e^{2\gamma} = e^{\kappa_1+\kappa_2}$$

Also

$$\alpha_1 = 2\rho\cos(\omega) = 2e^{\gamma}\left(\frac{e^{i\omega}+e^{-i\omega}}{2}\right).$$

But

$$e^{\kappa_1}+e^{\kappa_2} = e^{\gamma}(e^{i\omega}+e^{-i\omega}) = 2e^{\gamma}\cos(\omega),$$

so it follows that

$$\alpha_1 = e^{\kappa_1}+e^{\kappa_2}$$