Deseasonalising Wiener–Kolmogorov Filters

According to the assumptions of the structural time-series model, a trended econometric data sequence can be described by the equation

$$y(t) = \frac{\zeta(t-1)}{\nabla^2(L)} + \frac{\eta(t)}{\nabla(L)} + \frac{\omega(t)}{S(L)} + \varepsilon(t), \tag{1}$$

where $\zeta(t)$, $\eta(t)$, $\omega(t)$ and $\varepsilon(t)$ are mutually independent white-noise processes. In the numerators, there are the lag-operator polynomials $\nabla(L) = 1 - L$ and $S(L) = 1 + L + \cdots + L^{s-1} = (1 - L^s)/(1 - L)$, where s is the number of months or quarters in the year, depending on the rate of sampling.

By combining the leading terms on the RHS of (1), the equation may be written as

$$\begin{aligned} y(t) &= \frac{\xi(t)}{\nabla^2(L)} + \frac{\omega(t)}{S(L)} + \varepsilon(t) \\ &= \kappa(t) + \rho(t) + \varepsilon(t), \end{aligned} \tag{2}$$

where

$$\xi(t) = \nabla(L)\eta(t) + \zeta(t-1) = (1 - \mu L)\nu(t) \tag{3}$$

follows a first-order moving-average process.

The first term on the RHS of (2) represents the trend and the second term represents the seasonal fluctuations. It can be seen that the trend follows an integrated moving-average IMA(2, 1) model, which is a second-order random walk driven by a first-order moving-average forcing function.

The autocovariance generating functions of the components of equation (2) are

$$\gamma_\kappa(z) = \sigma_\nu^2 \frac{\mu(z)\mu(z^{-1})}{\nabla^2(z)\nabla^2(z^{-1})}, \qquad \gamma_\rho(z) = \frac{\sigma_\omega^2}{S(z)S(z^{-1})} \quad \text{and} \quad \gamma_\varepsilon(z) = \sigma_\varepsilon^2, \tag{4}$$

where $\sigma_\nu^2 = V\{\nu(t)\}$, $\sigma_\omega^2 = V\{\omega(t)\}$ and $\sigma_\varepsilon^2 = V\{\varepsilon(t)\}$ are the variances of the various white-noise forcing functions and where $\mu(z) = 1 - \mu z$. On the assumption that the forcing functions are statistically independent, the autocovariance generating function of the observable process $y(t)$ is

$$\gamma_\kappa(z) = \gamma_\kappa(z) + \gamma_\rho(z) + \sigma_\varepsilon^2. \tag{5}$$

The frequency response function of the Wiener–Kolmogorov filter for extracting the seasonal component is

$$\begin{aligned} \beta_\rho(z) = \frac{\gamma_\rho(z)}{\gamma_y(z)} &= \frac{\sigma_\omega^2/S(z)S(z^{-1})}{\{\sigma_\nu^2\mu(z)\mu(z^{-1})/\nabla^2(z)\nabla^2(z^{-1})\} + \{\sigma_\omega^2/S(z)S(z^{-1})\} + \sigma_\varepsilon^2} \\ &= \frac{\sigma_\omega^2\nabla^2(z)\nabla^2(z^{-1})}{\sigma_\nu^2\mu(z)\mu(z^{-1})S(z)S(z^{-1}) + \sigma_\omega^2\nabla^2(z)\nabla^2(z^{-1}) + \sigma_\varepsilon^2 S(z)S(z^{-1})\nabla^2(z)\nabla^2(z^{-1})}. \end{aligned} \tag{6}$$

The filter for deseasonalising the data is $\beta_s(z) = 1 - \beta_\rho(z)$. Thus,

(7)
$$\begin{aligned}\beta_s(z) &= \frac{\sigma_\nu^2\mu(z)\mu(z^{-1})S(z)S(z^{-1}) + \sigma_\varepsilon^2 S(z)S(z^{-1})\nabla^2(z)\nabla^2(z^{-1})}{\sigma_\nu^2\mu(z)\mu(z^{-1})S(z)S(z^{-1}) + \sigma_\omega^2\nabla^2(z)\nabla^2(z^{-1}) + \sigma_\varepsilon^2 S(z)S(z^{-1})\nabla^2(z)\nabla^2(z^{-1})}\\ &= \frac{\{\sigma_\nu^2\mu(z)\mu(z^{-1}) + \sigma_\varepsilon^2\nabla^2(z)\nabla^2(z^{-1})\}S(z)S(z^{-1})}{\sigma_\omega^2\nabla^2(z)\nabla^2(z^{-1}) + \{\sigma_\nu^2\mu(z)\mu(z^{-1}) + \sigma_\varepsilon^2\nabla^2(z)\nabla^2(z^{-1})\}S(z)S(z^{-1})}.\end{aligned}$$

This is a complicated expression, and we may endeavour to simplify it. We may write

(8)
$$\sigma_\eta^2\theta(z)\theta(z^{-1}) = \sigma_\nu^2\mu(z)\mu(z^{-1}) + \sigma_\varepsilon^2\nabla^2(z)\nabla^2(z^{-1}),$$

which stands for the autocovariance generating function of a derived second-order moving average process $\zeta = \theta(L)\eta(t)$. Then

(9)
$$\begin{aligned}\beta_s(z) &= \frac{\sigma_\eta^2\theta(z)\theta(z^{-1})S(z)S(z^{-1})}{\sigma_\omega^2\nabla^2(z)\nabla^2(z^{-1}) + \sigma_\eta^2\theta(z)\theta(z^{-1})S(z)S(z^{-1})}\\ &= \frac{S(z)S(z^{-1})}{\{\sigma_\omega^2\nabla^2(z)\nabla^2(z^{-1})/\sigma_\eta^2\theta(z)\theta(z^{-1})\} + S(z)S(z^{-1})}.\end{aligned}$$

The essential factor of this filter is the operator $S(z)$ of the numerator that comprises zeros at the seasonal frequencies, which have the effect of nullifying the seasonal component.

The filter can be compared with some simpler special cases. Consider setting $\mu = 0$ and $\sigma_\varepsilon^2 = 0$. Then, we should have

(10)
$$\beta_s(z) = \frac{\sigma_\nu^2 S(z)S(z^{-1})}{\sigma_\nu^2 S(z)S(z^{-1}) + \sigma_\omega^2\nabla^2(z)\nabla^2(z^{-1})}.$$

This is the filter that is appropriate for the purpose of deseasonalising the process

(11)
$$y(t) = \kappa(t) + \rho(t) = \frac{\nu(t)}{\nabla^2(L)} + \frac{\omega(t)}{S(L)}$$

and, equally, for the purpose of extracting $\nu(t)$ from the process

(12)
$$z(t) = \nabla^2(L)y(t) = \nu(t) + \frac{\nabla^2(L)}{S(L)}\omega(t).$$

At the simplest level, we may consider the filter that is appropriate to extracting $\nu(t)$ from the process

(13)
$$q(t) = \nu(t) + \frac{\omega(t)}{S(L)}$$

This the so-called comb filter

(14)
$$\beta_n(z) = \frac{\sigma_\nu^2 S(z)S(z^{-1})}{\sigma_\omega^2 + \sigma_\nu^2 S(z)S(z^{-1})} = \frac{S(z)S(z^{-1})}{\lambda + S(z)S(z^{-1})}.$$