BIVARIATE SPECTRAL ANALYSIS

Let $x(t)$ and $y(t)$ be two stationary stochastic processes with $E\{x(t)\} = E\{y(t)\} = 0$. The processes have the following spectral representations:

$$(1)\qquad \begin{aligned} x(t) &= \int_0^\pi \left\{\cos(\omega t)dA_x(\omega) + \sin(\omega t)dB_x(\omega)\right\},\\ y(t) &= \int_0^\pi \left\{\cos(\omega t)dA_y(\omega) + \sin(\omega t)dB_y(\omega)\right\}. \end{aligned}$$

The weighting functions $A_j(\omega), B_j(\omega); j = x, y$ are a set of mutually independent stochastic processes defined over the interval $[0, \pi]$. Realisations of these weighting processes correspond to realisations of the temporal processes $x(t)$ and $y(t)$ to which they pertain.

Several conditions must be fulfilled by $A_j(\omega), B_j(\omega)$ to ensure that $x(t)$ and $y(t)$ are stationary and that their autocovariances are invariant over time. A further set of conditions must be fulfilled to ensure that the cross-covariances between the two processes are time-invariant. We shall begin with the assumptions that are internal to the two processes. Then we shall examine the assumptions that relate one process to the other.

The Assumptions Internal to the Processes

The first assumption to be made is that the functions $dA_j(\omega)$ and $dB_j(\omega)$ represent a pair of stochastic processes of zero mean which are indexed on the continuous frequency parameter $\omega \in [0, \pi]$. Thus

$$(2)\qquad E\big\{dA_j(\omega)\big\} = E\big\{dB_j(\omega)\big\} = 0.$$

Next, it is assumed that the two processes $A_j(\omega)$ and $B_j(\omega)$ are mutually uncorrelated and that non-overlapping increments within each process are uncorrelated. Thus

$$(3)\qquad \begin{aligned} E\big\{dA_j(\omega)dB_j(\lambda)\big\} &= 0 \quad \text{for all} \quad \omega, \lambda,\\ E\big\{dA_j(\omega)dA_j(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda,\\ E\big\{B_j(\omega)dB_j(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda. \end{aligned}$$

The final assumption affecting the individual processes $x(t)$ or $y(t)$ is that the variance of the increments is given by

$$(4)\qquad \begin{aligned} V\big\{dA_j(\omega)\big\} = V\big\{dB_j(\omega)\big\} &= 2dF_j(\omega)\\ &= 2f_j(\omega)d\omega. \end{aligned}$$

We can see that, unlike $A_j(\omega)$ and $B_j(\omega)$, $F_j(\omega)$ is a continuous differentiable function. The function $F_j(\omega)$ and its derivative $f_j(\omega)$ are the spectral distribution function and the spectral density function, respectively.

To understand the effect of these various assumptions, we may begin with (4). This is the assumption that the increments of $A_j(\omega)$ and $B_j(\omega)$ have the same variance. Its effect is that the phase angle of the trigonometrical function at frequency ω, which is formed from the weighed combination of the sine and the cosine, is distributed uniformly the interval $[-0,\pi]$. It would be difficult to justify a different assumption.

Next, we may examine the role of the assumptions in connection with the autocovariances of the processes. Consider

$$(5)\qquad \begin{aligned} E(y_ty_s)=\int_0^{\pi}\int_0^{\pi}\big[&\cos(\omega t)\cos(\lambda s)E\{dA_y(\omega)dA_y(\lambda)\}\\ &+\cos(\omega t)\sin(\lambda s)E\{dA_y(\omega)dB_y(\lambda)\}\\ &+\sin(\omega t)\cos(\lambda s)E\{dB_y(\omega)dA_y(\lambda)\}\\ &+\sin(\omega t)\sin(\lambda s)E\{dB_y(\omega)dB_y(\lambda)\}\big].\end{aligned}$$

Given the assumptions of (3) concerning the absence of correlations amongst non-overlapping increments of $A_y(\omega)$ and $B_y(\omega)$ and the absence of correlation between the two processes, this becomes

$$(6)\qquad \begin{aligned} E(y_ty_s)&=2\int_0^{\pi}\big\{\cos(\omega t)\cos(\omega s)f_y(\omega)+\sin(\omega t)\sin(\omega s)f_y(\omega)\big\}d\omega\\ &=2\int_0^{\pi}\cos(\omega[t-s])f_y(\omega)d\omega,\end{aligned}$$

which follows in view of the identity $\cos(A-B)=\cos A\cos B+\sin A\sin B$. Thus, in consequence of the assumptions, the autocovariance $E(y_ty_s)=\gamma^{yy}_{|t-s|}$ is a function of the temporal separation of the elements y_t, y_s which is independent of their absolute dates. This is a necessary condition for the stationarity of the process $y(t)$.

We can express (6) more elegantly in terms of complex exponentials. Let the domain of the $f_y(\omega)$ be extended over the negative frequencies such that $f_y(-\omega)=f_y(\omega)$. Then, using $\cos(\omega\tau)=\{\exp(i\omega\tau)+\exp(-i\omega\tau)\}/2$, where $\tau=|s-t|$, and denoting the autocovariance by $\gamma^{yy}_\tau=E(y_ty_s)$, we can render (6) as

$$(7)\qquad \gamma^{yy}_\tau=\int_{-\pi}^{\pi}f_y(\omega)e^{i\omega\tau}d\omega.$$

The inverse mapping from the autocovariances to the spectrum is given by

$$f_y(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\gamma_\tau^{yy}e^{-i\omega\tau} = \frac{1}{2\pi}\Big\{\gamma_0^{yy} + 2\sum_{\tau=1}^{\infty}\gamma_\tau^{yy}\cos(\omega\tau)\Big\}. \tag{8}$$

This becomes a cosine Fourier transform in consequence of the symmetry of the autocovariance function whereby $\gamma_\tau^{yy} = \gamma_{-\tau}^{yy}$.

The essential interpretation of the spectral density function is indicated by the equation

$$\gamma_0^{yy} = \int_{-\pi}^{\pi} f_y(\omega)d\omega, \tag{9}$$

which comes from setting $\tau = 0$ in equation (7). This equation shows the measure in which the variance or 'power' of $y(t)$, which is γ_0^{yy}, is attributed by the spectral density function to the various cyclical components of which the process is composed.

The Assumptions Connecting the Processes

In order to determine the relatedness of the two processes $x(t), y(t)$, we need to make some assumptions regarding the covariances across the processes $A_x(\omega), A_y(\omega)$ and $B_x(\omega), B_y(\omega)$. First, there is the assumption that there are no correlations across the frequencies:

$$\begin{aligned} &E\{dA_x(\omega)dA_y(\lambda)\} = 0 \quad \text{if} \quad \omega \neq \lambda,\\ &E\{dA_x(\omega)dB_y(\lambda)\} = 0 \quad \text{if} \quad \omega \neq \lambda,\\ &E\{dB_x(\omega)dB_y(\lambda)\} = 0 \quad \text{if} \quad \omega \neq \lambda. \end{aligned} \tag{10}$$

Next, there are two covariance relationships:

$$\begin{aligned} E\{dA_x(\omega)dA_y(\omega)\} = E\{dB_x(\omega)dB_y(\omega)\} &= dC(\omega)\\ &= c(\omega)d\omega \end{aligned} \tag{11}$$

and

$$\begin{aligned} E\{dA_x(\omega)dB_y(\omega)\} = -E\{dB_x(\omega)dA_y(\omega)\} &= dQ(\omega)\\ &= q(\omega)d\omega. \end{aligned} \tag{12}$$

The first of these functions is co-spectrum $c(\omega)$, which defines the covariances of the amplitudes of cosine components of $x(t)$ and $y(t)$ that are in phase at the frequency ω. The second is the quadrature spectrum $q(\omega)$, which defines the covariance of the amplitudes of the sine and the cosine components of $x(t)$ and $y(t)$ that are in quadrature at the frequency ω, which is to say that they are out of phase to the extent of $\pi/2$ radians.

We may examine the role of these assumptions in connection with the spectral representation of the covariances of $x(t)$ and $y(t)$. From the spectral representations of (1), we get the following quadratic product:

$$\begin{aligned} E(x_t y_s) = \int_0^\pi \int_0^\pi \big[&\cos(\omega t)\cos(\lambda s) E\{dA_x(\omega) dA_y(\lambda)\} \\ &+ \cos(\omega t)\sin(\lambda s) E\{dA_x(\omega) dB_y(\lambda)\} \\ &+ \sin(\omega t)\cos(\lambda s) E\{dB_x(\omega) dA_y(\lambda)\} \\ &+ \sin(\omega t)\sin(\lambda s) E\{dB_x(\omega) dB_y(\lambda)\}\big]. \end{aligned} \tag{13}$$

However, according to (10), the random increments for the frequency ω in one process are uncorrelated with the random increments for the frequency λ in the other process. Therefore, the double integral collapses to an single integral in respect of one frequency, and, using the results from (11) and (12), we get

$$\begin{aligned} E(x_t y_s) = \int_0^\pi \big[&\cos(\omega t)\cos(\omega s) dC(\omega) \\ &- \cos(\omega t)\sin(\omega s) dQ(\omega) \\ &+ \sin(\omega t)\cos(\omega s) dQ(\omega) \\ &+ \sin(\omega t)\sin(\omega s) dC(\omega)\big]. \end{aligned} \tag{14}$$

Finally, by virtue of two trigonometrical identities, which are the cosine identities quoted above, and the analogous sine identity $\sin(A-B) = \sin A\cos B - \cos A \sin B$, we find that

$$E(x_t y_s) = \int_0^\pi \big\{ \cos(\omega[t-s]) dC(\omega) + \sin(\omega[t-s]) dQ(\omega)\big\}. \tag{15}$$

Let us set $dC(\omega) = c(\omega)d\omega$ and $dQ(\omega) = q(\omega)d\omega$ in accordance with the assumption that the spectral functions are differentiable, which will be true in the absence of perfectly regular periodic components in the processes $x(t)$ and $y(t)$. Then, on setting $t - s = \tau$ and using the notation $\gamma_\tau^{xy} = E(x_t, y_{t-\tau})$, we can rewrite equation (15) as

$$\gamma_\tau^{xy} = \int_0^\pi \big\{ \cos(\omega\tau) c(\omega) d\omega + \sin(\omega\tau) q(\omega)\big\} d\omega. \tag{16}$$

In order to express (16) in terms of complex exponentials, we define the so-called cross-spectrum

$$g^{xy}(\omega) = c(\omega) - iq(\omega). \tag{17}$$

Also, we may extend the domain of the integration for $[0, \pi]$ to $(-\pi, \pi]$ by regarding $c(\omega)$ as an even function such that $c(\omega) = c(-\omega)$ and by regarding $q(\omega)$ as an odd function such that $q(\omega) = -q(-\omega)$. Then, since $\cos(\omega t) = \{\exp(i\omega t) + \exp(-i\omega t)\}/2$ and $\sin(\omega t) = -i\{\exp(i\omega t) - \exp(-i\omega t)\}/2$, we have

$$\gamma_\tau^{xy} = \int_{-\pi}^{\pi} g^{xy}(\omega)e^{i\omega\tau}d\omega. \tag{18}$$

The inverse of this relationship indicates that the cross spectrum is the Fourier transform of the covariances of $x(t)$ and $y(t)$ in the same way that the spectral density function of $y(t)$, which is to be found under (8), is the Fourier transform of the sequence of its autocovariances:

$$g^{xy}(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \gamma_\tau^{xy} e^{-i\omega\tau}. \tag{19}$$

Notice, however, that since, in general, $\gamma_\tau^{xy} \neq \gamma_{-\tau}^{xy}$, this does not entail a cosine Fourier transform as does the corresponding definition of the spectrum as the transform of the autocovariance function.

The even function $c(\omega)$ is the cosine portion of $g^{xy}(\omega)$ and the odd function $-q(\omega)$ is its sine portion. These quantities are defined separately by

$$c(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \frac{\gamma_\tau^{xy} + \gamma_{-\tau}^{xy}}{2} e^{-i\omega\tau}, \tag{20}$$

$$-q(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \frac{\gamma_\tau^{xy} - \gamma_{-\tau}^{xy}}{2} e^{-i\omega\tau}, \tag{21}$$

where $(\gamma_\tau^{xy} + \gamma_{-\tau}^{xy})/2$ is an even or symmetric function and $(\gamma_\tau^{xy} - \gamma_{-\tau}^{xy})/2$ is an odd or anti-symmetric function. Observe that, in (20), we may replace the complex exponential by $\cos(\omega t)$ whereas, in (21), we may replace it by $\sin(\omega t)$.

An insight into the nature of the cross spectrum is indicated by the equation

$$\begin{aligned}\gamma_0^{xy} &= \int_{-\pi}^{\pi} g_{xy}(\omega)d\omega \\ &= \int_{-\pi}^{\pi} c(\omega)d\omega,\end{aligned} \tag{22}$$

which comes from setting $\tau = 0$ in equation (18). This equation shows the ordinary contemporaneous covariance between the processes $x(t)$ and $y(t)$ relates only to the cyclical components that are in phase. The concept of spectral coherence, which we shall expound in the next section, also accommodates relationships between components which are out of phase.

Measures of Relatedness between Two Series

We can measure the relatedness of two stationary stochastic processes in terms of their spectral coherence. The coherence function, which is defined over the interval $[0, \pi]$, gives the correlation of the cyclical components of the series at each frequency. The coherence of $x(t)$ and $y(t)$ at the frequency ω is defined by

$$\rho(\omega) = \frac{|g^{xy}(\omega)|}{\{f^{xx}(\omega)f^{yy}(\omega)\}^{1/2}} = \left\{\frac{c^2(\omega) + q^2(\omega)}{f^{xx}(\omega)f^{yy}(\omega)}\right\}^{1/2}. \tag{23}$$

Since it takes account of both the co-spectrum and the quadrature spectrum, the measure is unaffected by the relative phase alignment of the two components. One should recall, in this connection, that the ordinary correlation coefficient of two sinusoids of the same frequency that are in quadrature would be zero.

The coherence at any frequency ω is, in effect, the ordinary measure of correlation which would be obtained by bringing the components of the two series into phase alignment. Consider the components

$$\begin{aligned} x(\omega, t) &= \cos(\omega t)dA_x(\omega) + \sin(\omega t)dB_x(\omega), \\ y(\omega, t) &= \cos(\omega t - \theta)dA_y(\omega) + \sin(\omega t - \theta)dB_y(\omega), \end{aligned} \tag{24}$$

which are extracted from the spectral representation of the two series given under (1). The second series has been translated through a phase angle of θ which, so we assume, can be adjusted to maximise the covariance of the two series. By the algebra that has given rise to equation (14), we find that

$$E\{x(\omega, t), y(\omega, t)\} = c(\omega)\cos\theta + q(\omega)\sin\theta. \tag{25}$$

The condition for a maximum, which is found by differentiating the function with respect to θ and setting the result to zero, is $-c(\omega)\sin\theta + q(\omega)\cos\theta = 0$. For the condition to be satisfied, we must have $\sin(\theta) = \alpha q(\omega)$ and $\cos(\theta) = \alpha c(\omega)$ for some factor α. Since $\sin^2(\theta) + \cos^2(\theta) = 1$, it follows that $\alpha = \{c^2(\omega) + q^2(\omega)\}^{-1/2}$, and we have

$$\sin\theta = \frac{q(\omega)}{\{c^2(\omega) + q^2(\omega)\}^{1/2}} \quad \text{and} \quad \cos\theta = \frac{c(\omega)}{\{c^2(\omega) + q^2(\omega)\}^{1/2}}, \tag{26}$$

On substituting these into (25), it is found that

$$c(\omega)\cos\theta + q(\omega)\sin\theta = \{c^2(\omega) + q^2(\omega)\}^{1/2}. \tag{27}$$

This maximised covariance measure is the numerator of the coherence function $\rho(\omega)$ of (23).

The function

$$\theta(\omega)=\tan^{-1}\left\{\frac{q(\omega)}{c(\omega)}\right\}. \tag{28}$$

constitutes the phase spectrum of $x(t)$ and $y(t)$. It indicates, for each frequency, the extent to which the components of $y(t)$ lead or lag behind those of $x(t)$.

We have demonstrated that the measure of spectral coherence is unaffected by a changes in the phase of the sinusoidal components of which the processes are composed. It is also unaffected by systematic changes in their amplitudes. Thus, if the two processes $x(t)$ and $y(t)$ are transformed by linear filters and if the transfer functions of the filters have none of their poles or zeros on the unit circle, then their coherence is not affected.

Let $\alpha(L)$ and $\beta(L)$ be two filters and let

$$\xi(t)=\alpha(L)x(t)\quad\text{and}\quad\zeta(t)=\beta(L)y(t). \tag{29}$$

Also let $\alpha(\omega)=\sum_j\alpha_j\exp(-i\omega)$ and $\beta(\omega)=\sum_j\beta_j\exp(-i\omega)$ be the corresponding frequency response functions. Then the spectra of $\xi(t)$ and $\zeta(t)$ together with their co-spectrum are given by

$$\begin{aligned}f^{\xi\xi}(\omega)&=|\alpha(\omega)|^2f^{xx}(\omega),\quad f^{\zeta\zeta}(\omega)=|\beta(\omega)|^2f^{yy}(\omega)\\ \text{and}\quad g^{\xi\zeta}(\omega)&=\alpha^*(\omega)\beta(\omega)g^{xy}(\omega).\end{aligned} \tag{30}$$

The modulus of the cross spectrum of $\xi(t)$ and $\zeta(t)$ is

$$|g^{\xi\zeta}(\omega)|=|\alpha(\omega)||\beta(\omega)||g^{xy}(\omega)|. \tag{28}$$

Therefore, the coherence of the transformed processes is

$$\rho^{\xi\zeta}(\omega)=\frac{|g^{\xi\zeta}(\omega)|}{\{f^{\xi\xi}(\omega)f^{\zeta\zeta}(\omega)\}^{1/2}}=\frac{|g^{xy}(\omega)|}{\{f^{xx}(\omega)f^{yy}(\omega)\}^{1/2}}=\rho^{xy}(\omega), \tag{31}$$

which is equal to the original coherence of $x(t)$ and $y(t)$.