## The Cholesky Decomposition of a Symmetric Matrix of 5 Diagonals

Consider the following Cholesky decomposition of symmetric matrix  $\Gamma$  of five bands:

$\lceil d \rceil$	$s_0$	$q_0$	0 0	ך 0		Γ1	*	*	*	*	* -	1
$ s_0 $	$d_1$	$s_1$	$q_1 = 0$	0		2	4	*	*	*	*	
$q_0$	$s_1$	$d_2$	$s_2  q_2$	0		3	5	7	*	*	*	
	$q_1$	$s_2$	$d_3$ $s_3$	$q_3$		*	6	8	10	*	*	
0	0	$q_2$	$s_3  d_4$	$s_4$		*	*	9	11	13	*	
	0	0	$q_3$ $s_4$	$d_5 \rfloor$		L *	*	*	12	14	$15_{-}$	
	$\sub{g_0}$	0	0	0	0	ך 0	Γ1	$b_0$	$c_0$	0	0	ך 0
	$\begin{bmatrix} g_0 \\ g_0 b_0 \end{bmatrix}$	$0 \\ g_1$	$\begin{array}{c} 0 \\ 0 \end{array}$	0 0	0 0	$\begin{bmatrix} 0\\0 \end{bmatrix}$	$\begin{bmatrix} 1\\ 0 \end{bmatrix}$	$b_0$ 1	$c_0 \\ b_1$	$\begin{array}{c} 0 \\ c_1 \end{array}$	$\begin{array}{c} 0 \\ 0 \end{array}$	$\begin{bmatrix} 0\\ 0 \end{bmatrix}$
_	$\begin{bmatrix} g_0 \\ g_0 b_0 \\ g_0 c_0 \end{bmatrix}$	$egin{array}{c} 0 \ g_1 \ g_1 b_1 \end{array}$	$egin{array}{c} 0 \ 0 \ g_2 \end{array}$	0 0 0	0 0 0	$\begin{bmatrix} 0\\0\\0 \end{bmatrix}$	$\begin{bmatrix} 1\\0\\0 \end{bmatrix}$	$egin{array}{c} b_0 \ 1 \ 0 \end{array}$	$egin{array}{c} c_0 \ b_1 \ 1 \end{array}$	$\begin{array}{c} 0 \\ c_1 \\ b_2 \end{array}$	$\begin{array}{c} 0 \\ 0 \\ c_2 \end{array}$	$\begin{bmatrix} 0\\0\\0 \end{bmatrix}$
=	$\begin{bmatrix} g_0 \\ g_0 b_0 \\ g_0 c_0 \\ 0 \end{bmatrix}$	$egin{array}{c} 0 \ g_1 \ g_1 b_1 \ g_1 c_1 \end{array}$	$egin{array}{c} 0 \ 0 \ g_2 \ g_2 b_2 \end{array}$	$egin{array}{c} 0 \ 0 \ 0 \ g_3 \end{array}$	0 0 0 0	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}$	$b_0 \\ 1 \\ 0 \\ 0$	$egin{array}{c} c_0 \ b_1 \ 1 \ 0 \end{array}$	$egin{array}{c} 0 \ c_1 \ b_2 \ 1 \end{array}$	$egin{array}{c} 0 \ 0 \ c_2 \ b_3 \end{array}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ c_3 \end{bmatrix}$
=	$\begin{bmatrix} g_0 \\ g_0 b_0 \\ g_0 c_0 \\ 0 \\ 0 \end{bmatrix}$	$egin{array}{c} 0 \\ g_1 \\ g_1 b_1 \\ g_1 c_1 \\ 0 \end{array}$	$egin{array}{c} 0 \ 0 \ g_2 \ g_2 b_2 \ g_2 c_2 \end{array}$	$egin{array}{c} 0 \ 0 \ 0 \ g_3 \ g_3 b_3 \end{array}$	$egin{array}{c} 0 \ 0 \ 0 \ 0 \ g_4 \end{array}$	$\begin{bmatrix} 0\\0\\0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix} 1\\0\\0\\0\\0 \end{bmatrix}$	$b_0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0$	$c_0 \\ b_1 \\ 1 \\ 0 \\ 0$	$egin{array}{c} 0 \\ c_1 \\ b_2 \\ 1 \\ 0 \end{array}$	$egin{array}{c} 0 \\ 0 \\ c_2 \\ b_3 \\ 1 \end{array}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ c_3 \\ b_4 \end{bmatrix}$

Here, we are factorising the matrix as  $\Gamma = (MG)M'$ , where G is a diagonal matrix. The sequence of computations that finds the elements of the lower triangular matrix M is indicated by the matrix with numbers and asterisks, which is beside the matrix to be factorised:

The sequence of computations is listed below. To produce the third column, we make the substitutions  $g \to d$ ,  $b \to s$  and  $c \to q$ . These are to signify that, in a computer program, we should be overwriting the corresponding elements:

(1) 
$$d_0 = g_0 \qquad g_0 = d_0 \qquad d_0 = d_0$$

(2) 
$$s_0 = g_0 b_0 \mu_{11}$$
  $b_0 = s_0/g_0$   $s_0 = s_0/d_0$ 

(3) 
$$q_0 = g_0 c_0$$
  $c_0 = q_0/g_0$   $q_0 = q_0/d_0$ 

(4) 
$$d_1 = g_0 b_0^2 + g_1$$
  $g_1 = d_1 - g_0 b_0^2$   $d_1 = d_1 - d_0 s_0^2$ 

(5) 
$$s_1 = g_0 c_0 b_0 + g_1 b_1$$
  $b_1 = (s_1 - g_0 c_0 b_0)/g_1$   $s_1 = (s_1 - d_0 q_0 s_0)/g_1$ 

(6) 
$$q_1 = g_1 c_1 \qquad c_1 = q_1/g_1 \qquad q_1 = q_1/d_1$$

(7) 
$$d_2 = g_0 c_0^2 + g_1 b_1^2 + g_2$$
  $g_2 = d_2 - g_0 c_0^2 - g_1 b_1^2$   $d_2 = d_2 - d_0 q_0^2 - d_1 s_1^2$ 

(8) 
$$s_2 = g_1 c_1 b_1 + g_2 b_2$$
  $b_2 = (s_2 - g_1 c_1 b_1)/g_2$   $s_2 = (s_2 - d_1 q_1 s_1)/d_2$ 

(9) 
$$q_2 = g_2 c_2$$
  $c_2 = q_2/g_2$   $q_2 = q_2/d_2$ 

The generic equation is

$$\begin{split} &d[j] := d[j] - d[j-1] * Sqr(s[j-1]) - d[j-2] * Sqr(q[j-2]); \\ &s[j] := (s[j[-d[j-1] * q[j-1] * s[j-1])/d[j]; \\ &q[j] := q[j])/d[j]; \end{split}$$

In addition, if we begin with j = 0, then we require the following initial conditions:

$$d[-2] := 0.0, \quad d[-1] := 0.0, \quad s[-1] := 0.0, \quad q[-2] := 0.0, \quad q[-1] := 0.0.$$

## The Solution of a Set of Linear Equations

The purpose of the factorisation is to enable us to solve a set of linear equations in the form of  $\Gamma x = MDM'x = y$ . The technique is to cast the equations as  $\Gamma x = LM'x = Lz = y$  where L = MD. Then the equations Lz = y may be solved for z by a process of recursive forward substitution that runs through the system from top to bottom. Thereafter, the equations M'x = z can be solved for x by a process that runs from bottom to top.