The Cholesky Decomposition of a Symmetric Matrix of 5 Diagonals

Consider the following Cholesky decomposition of symmetric matrix Γ of five bands:

$$\begin{bmatrix} d_0 & s_0 & q_0 & 0 & 0 & 0 \\ s_0 & d_1 & s_1 & q_1 & 0 & 0 \\ q_0 & s_1 & d_2 & s_2 & q_2 & 0 \\ 0 & q_1 & s_2 & d_3 & s_3 & q_3 \\ 0 & 0 & q_2 & s_3 & d_4 & s_4 \\ 0 & 0 & 0 & q_3 & s_4 & d_5 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * & * \\ 2 & 4 & * & * & * & * \\ 3 & 5 & 7 & * & * & * \\ * & 6 & 8 & 10 & * & * \\ * & * & 9 & 11 & 13 & * \\ * & * & * & 12 & 14 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} g_0 & 0 & 0 & 0 & 0 & 0 \\ g_0b_0 & g_1 & 0 & 0 & 0 & 0 \\ g_0c_0 & g_1b_1 & g_2 & 0 & 0 & 0 \\ 0 & g_1c_1 & g_2b_2 & g_3 & 0 & 0 \\ 0 & 0 & g_2c_2 & g_3b_3 & g_4 & 0 \\ 0 & 0 & 0 & g_3c_3 & g_4b_4 & g_5 \end{bmatrix} \begin{bmatrix} 1 & b_0 & c_0 & 0 & 0 & 0 \\ 0 & 1 & b_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 & b_3 & c_3 \\ 0 & 0 & 0 & 0 & 1 & b_4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, we are factorising the matrix as $\Gamma = (MG)M'$, where G is a diagonal matrix. The sequence of computations that finds the elements of the lower triangular matrix M is indicated by the matrix with numbers and asterisks, which is beside the matrix to be factorised:

The sequence of computations is listed below. To produce the third column, we make the substitutions $g \rightarrow d$, $b \rightarrow s$ and $c \rightarrow q$. These are to signify that, in a computer program, we should be overwriting the corresponding elements:

(1) $$d_0 = g_0 \qquad g_0 = d_0 \qquad d_0 = d_0$$

(2) $$s_0 = g_0 b_0 \mu_{11} \qquad b_0 = s_0/g_0 \qquad s_0 = s_0/d_0$$

(3) $$q_0 = g_0 c_0 \qquad c_0 = q_0/g_0 \qquad q_0 = q_0/d_0$$

(4) $$d_1 = g_0 b_0^2 + g_1 \qquad g_1 = d_1 - g_0 b_0^2 \qquad d_1 = d_1 - d_0 s_0^2$$

(5) $$s_1 = g_0 c_0 b_0 + g_1 b_1 \qquad b_1 = (s_1 - g_0 c_0 b_0)/g_1 \qquad s_1 = (s_1 - d_0 q_0 s_0)/g_1$$

(6) $$q_1 = g_1 c_1 \qquad c_1 = q_1/g_1 \qquad q_1 = q_1/d_1$$

(7) $$d_2 = g_0 c_0^2 + g_1 b_1^2 + g_2 \qquad g_2 = d_2 - g_0 c_0^2 - g_1 b_1^2 \qquad d_2 = d_2 - d_0 q_0^2 - d_1 s_1^2$$

(8) $$s_2 = g_1 c_1 b_1 + g_2 b_2 \qquad b_2 = (s_2 - g_1 c_1 b_1)/g_2 \qquad s_2 = (s_2 - d_1 q_1 s_1)/d_2$$

(9) $$q_2 = g_2 c_2 \qquad c_2 = q_2/g_2 \qquad q_2 = q_2/d_2$$

The generic equation is

$$
\begin{aligned}
&d[j] := d[j] - d[j-1] * Sqr(s[j-1]) - d[j-2] * Sqr(q[j-2]);\\
&s[j] := (s[j[-d[j-1] * q[j-1] * s[j-1])/d[j];\\
&q[j] := q[j])/d[j];
\end{aligned}
$$

In addition, if we begin with $j = 0$, then we require the following initial conditions:

$$
d[-2] := 0.0, \quad d[-1] := 0.0, \quad s[-1] := 0.0, \quad q[-2] := 0.0, \quad q[-1] := 0.0.
$$

The Solution of a Set of Linear Equations

The purpose of the factorisation is to enable us to solve a set of linear equations in the form of $\Gamma x = MDM'x = y$. The technique is to cast the equations as $\Gamma x = LM'x = Lz = y$ where $L = MD$. Then the equations $Lz = y$ may be solved for z by a process of recursive forward substitution that runs through the system from top to bottom. Thereafter, the equations $M'x = z$ can be solved for x by a process that runs from bottom to top.