RATIONAL TRANSFER FUNCTIONS

BIBO Stability

In most applications, the output sequence $h(t)$ of the transfer function should be bounded in absolute value whenever the input sequence $x(t)$ is bounded. This is described as the condition of bounded input–bounded output (BIBO) stability.

If the coefficients $\{\omega_0, \omega_1, \ldots, \omega_p\}$ of the transfer function form a finite sequence, then a necessary and sufficient condition for BIBO stability is that $|\omega_i| < \infty$ for all i, which is to say that the impulse-response function must be bounded. If $\{\omega_0, \omega_1, \ldots\}$ is an indefinite sequence, then it is necessary, in addition, that $|\sum \omega_i| < \infty$, which is the condition that the step-response function is bounded. Together, the two conditions are equivalent to the single condition that $\sum |\omega_i| < \infty$, which is to say that the impulse response is absolutely summable.

To confirm that the latter is a sufficient condition for stability, let us consider any input sequence $x(t)$ which is bounded such that $|x(t)| < M$ for some finite M. Then

$$(25)\qquad |h(t)| = \left|\sum \omega_i x(t-i)\right| \leq M\left|\sum \omega_i\right| < \infty,$$

and so the output sequence $h(t)$ is bounded. To show that the condition is necessary, imagine that the $\sum |\omega_i|$ is unbounded. Then a bounded input sequence can be found which gives rise to an unbounded output sequence. One such input sequence is specified by

$$(26)\qquad x_{-i} = \begin{cases} \dfrac{\omega_i}{|\omega_i|}, & \text{if } \omega_i \neq 0; \\ 0, & \text{if } \omega_i = 0. \end{cases}$$

This gives

$$(27)\qquad h_0 = \sum \omega_i x_{-i} = \sum |\omega_i|,$$

and so $h(t)$ is unbounded.

A summary of this result may be given which makes no reference to the specific context in which it has arisen:

(28) The convolution product $h(t) = \sum \omega_i x(t-i)$, which comprises a bounded sequence $x(t) = \{x_t\}$, is itself bounded if and only if the sequence $\{\omega_i\}$ is absolutely summable such that $\sum_i |\omega_i| < \infty$.

The Expansion of a Rational Function

In time-series analysis, models are often encountered which contain transfer functions in the form of $y(t) = \{\delta(L)/\gamma(L)\}x(t)$. For this to have a meaningful interpretation, it is normally required that the rational operator $\delta(L)/\gamma(L)$ should obey the BIBO stability condition; which is to say that $y(t)$ should be a bounded sequence whenever $x(t)$ is bounded.

The necessary and sufficient condition for the boundedness of $y(t)$ is that the series expansion $\{\omega_0+\omega_1 z+\cdots\}$ of $\delta(z)/\gamma(z)$ should be convergent whenever $|z| \leq 1$. We can determine whether or not the series will converge by expressing the ratio $\delta(z)/\gamma(z)$ as a sum of partial fractions.

Imagine that $\gamma(z) = \gamma_m \prod(z-\lambda_i) = \gamma_0 \prod(1-z/\lambda_i)$ where the roots may be complex. Then, assuming that there are no repeated roots, and taking $\gamma_0 = 1$, the ratio can be written as

$$\frac{\delta(z)}{\gamma(z)} = \frac{\kappa_1}{1 - z/\lambda_1} + \frac{\kappa_2}{1 - z/\lambda_2} + \cdots + \frac{\kappa_m}{1 - z/\lambda_m}. \tag{22}$$

Since any scalar factor of $\gamma(L)$ may be absorbed in the numerator $\delta(L)$, setting $\gamma_0 = 1$ entails no loss of generality.

If the roots of $\gamma(z) = 0$ are real and distinct, then the conditions for the convergence of the expansion of $\delta(z)/\gamma(z)$ are straightforward. For the rational function converges if and only if the expansion of each of its partial fractions in terms of ascending powers of z converges. For the expansion

$$\frac{\kappa}{1 - z/\lambda} = \kappa\bigl\{1 + z/\lambda + (z/\lambda)^2 + \cdots\bigr\} \tag{23}$$

to converge for all $|z| \leq 1$, it is necessary and sufficient that $|\lambda| > 1$.

In the case where a real root occurs with a multiplicity of n, as in the expression under (20), a binomial expansion is available:

$$\frac{1}{(1 - z/\lambda)^n} = 1 - n\frac{z}{\lambda} + \frac{n(n-1)}{2!}\Bigl(\frac{z}{\lambda}\Bigr)^2 - \frac{n(n-1)(n-2)}{3!}\Bigl(\frac{z}{\lambda}\Bigr)^3 + \cdots. \tag{24}$$

Once more, it is evident that $|\lambda| > 1$ is the necessary and sufficient condition for convergence when $|z| \leq 1$.

The expansion under (23) applies to complex roots as well as to real roots. To investigate the conditions of convergence in the case of complex roots, it is appropriate to combine the products of the expansion of a pair of conjugate factors. Therefore consider following expansion:

$$\begin{aligned} \frac{c}{1 - z/\lambda} + \frac{c^*}{1 - z/\lambda^*} &= c\bigl\{1 + z/\lambda + (z/\lambda)^2 + \cdots\bigr\} \\ &\qquad + c^*\bigl\{1 + z/\lambda^* + (z/\lambda^*)^2 + \cdots\bigr\} \\ &= \sum_{t=0}^{\infty} z^t(c\lambda^{-t} + c^*\lambda^{*-t}). \end{aligned} \tag{25}$$

The various complex quantities can be represented in terms of exponentials:

$$\begin{aligned} \lambda &= \kappa^{-1}e^{-i\omega}, & \lambda^* &= \kappa^{-1}e^{i\omega}, \\ c &= \rho e^{-i\theta}, & c^* &= \rho e^{i\theta}. \end{aligned} \tag{26}$$

Then the generic term in the expansion becomes

$$\begin{aligned} z^t(c\lambda^{-t} + c^*\lambda^{*-t}) &= z^t\Big\{\rho e^{-i\theta}\kappa^t e^{i\omega t} + \rho e^{i\theta}\kappa^t e^{-i\omega t}\Big\} \\ &= z^t\rho\kappa^t\Big\{e^{i(\omega t-\theta)} + e^{-i(\omega t-\theta)}\Big\} \\ &= z^t 2\rho\kappa^t\cos(\omega t - \theta). \end{aligned} \tag{27}$$

The expansion converges for all $|z| \leq 1$ if and only if $|\kappa| < 1$. But $|\kappa| = |\lambda^{-1}| = |\lambda|^{-1}$; so it is confirmed that the necessary and sufficient condition for convergence is that $|\lambda| > 1$.

The case of repeated complex roots can also be analysed to reach a similar conclusion. Thus a general assertion regarding the expansions of rational function can be made:

(28) The expansion $\omega(z) = \{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}$ of the rational function $\delta(z)/\gamma(z)$ converges for all $|z| \leq 1$ if and only if every root λ of $\gamma(z) = 0$ lies outside the unit circle such that $|\lambda| > 1$.

So far, the condition has been imposed that $|z| \leq 1$. The expansion of a rational function may converge under conditions which are either more or less stringent in the restrictions which they impose of $|z|$. If fact, for any series $\omega(z) = \{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}$, there exists a real number $r \geq 0$, called the radius of convergence, such that, if $|z| < r$, then the series converges absolutely with $\sum |\omega_i| < \infty$, whereas, if $|z| > r$, then the series diverges.

In the case of the rational function $\delta(z)/\gamma(z)$, the condition for the convergence of the expansion is that $|z| < r = \min\{|\lambda_1|, \ldots, |\lambda_m|\}$, where the λ_i are the roots of $\gamma(z) = 0$.

The roots of the numerator polynomial $\delta(z)$ of a rational function are commonly described as the zeros of the function whilst the roots of the denominator function polynomial $\gamma(z)$ are described as the poles.

In electrical engineering, the z-transform of a sequence defined on the positive integers is usually expressed in terms of negative powers of z. This leads to an inversion of the results given above. In particular, the condition for the convergence of the expansion of the function $\delta(z^{-1})/\gamma(z^{-1})$ is that $|z| > r = \max\{|\mu_1|, \ldots, |\mu_m|\}$, where $\mu_i = 1/\lambda_i$ is a root of $\gamma(z^{-1}) = 0$.

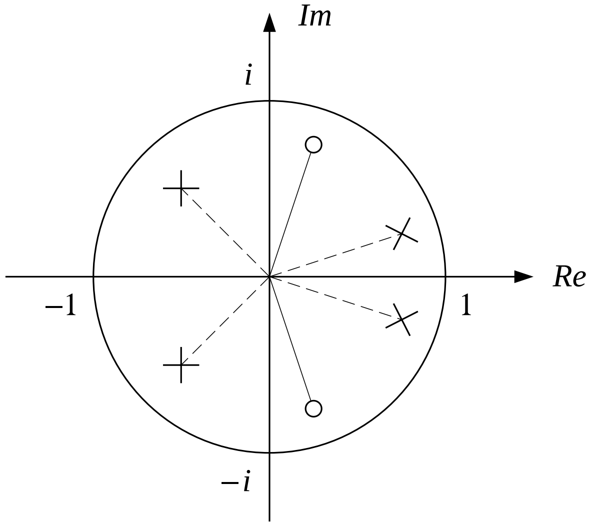

Figure 1. The pole–zero diagram of the stable transfer function

$$\frac{\delta(z^{-1})}{\gamma(z^{-1})} = \frac{\{1-(0.25 \pm i0.75)z^{-1}\}}{\{1-(0.75 \pm i0.25)z^{-1}\}\{1+(0.5 \pm i0.5)z^{-1}\}}.$$

The poles are marked with crosses and the zeros with circles.

Example. It is often helpful to display a transfer function graphically by means of a pole–zero plot in the complex plane; and, for this purpose, there is an advantage in the form $\delta(z^{-1})/\gamma(z^{-1})$ which is in terms of negative powers of z. Thus, if the function satisfies the BIBO stability condition, then the poles of $\delta(z^{-1})/\gamma(z^{-1})$ will be found within the unit circle. The numerator may also be subject to conditions which will place the zeros within the unit circle. On the other hand, the poles of $\delta(z)/\gamma(z)$ will fall outside the unit circle; and they may be located at a considerable distance from the origin, which would make a diagram inconvenient.

Because the pole–zero diagram can be of great assistance in analysing a transfer function, we shall adopt the negative-power z-transform whenever it is convenient to do so.