Processes with Autoregressive Conditionally Heteroskedastic (ARCH) Disturbances

In the classical time-series models of the ARMA variety, it is assumed that the disturbances constitute a white-noise sequence of independently and identically distributed random variables. If the values of the structural parameters—ie of the autoregressive and the moving-average coefficients—are known, then the disturbances are synonymous with the one-step-ahead prediction errors.

In the analysis of economic data, it is sometimes observed that large and small forecast errors occur in clusters; which suggests that, in certain periods, the indices become more or less volatile than usual. It was suggested by Engle (1982) that this phenomenon could be modelled by making the variance of the disturbances the subject of a stochastic process.

The effect should be to achieve a model which fits the data better within the sample as well as giving a more adequate representation of the reliability of the forecasts. The latter feature might be valuable if, for example, the series in question where that of an interest rate or a stock price. This is because the attractiveness of a financial asset is both directly related to its expected rate of return and inversely related to its risk, which is a function of the variance of the rate of return.

Let $\varepsilon(t) = \{\varepsilon_t\}$ be the sequence of disturbances to an ARMA process $y(t)$ described by the equation $\alpha(L)y(t) = \mu(L)\varepsilon(t)$. Then the simplest of ARCH models is one which proposes that

$$\varepsilon_t = \upsilon_t h_t \qquad \text{where} \qquad h_t^2 = \theta_0 + \theta_1 \varepsilon_{t-1}^2, \tag{1}$$

with $\upsilon(t) = \{\upsilon_t\}$ as a white-noise process with unit variance. The condition that $V\{\upsilon(t)\} = 1$ imposes no restriction, since the scaling of the process $\varepsilon(t)$ is accomplished via the value of θ_0. Both the conditional and the unconditional expectation of the ARCH process are zeros. That is

$$\begin{aligned} E(\varepsilon_t|\varepsilon_{t-1}) = E(\varepsilon_t|h_t) = 0, \qquad & \text{and} \\ E(\varepsilon_t) = E(\upsilon_t)E(h_t) = 0. & \end{aligned} \tag{2}$$

The conditional variance of the ARCH process is

$$\begin{aligned} V(\varepsilon_t|\varepsilon_{t-1}) &= h_t^2 V(\upsilon_t^2) \\ &= h_t^2 = \theta_0 + \theta_1 \varepsilon_{t-1}^2, \end{aligned} \tag{3}$$

whereas the unconditional variance of the process is

$$\begin{aligned} V(\varepsilon_t) &= E\big\{V(\varepsilon_t|\varepsilon_{t-1})\big\} \\ &= \theta_0 + \theta_1 V(\varepsilon_{t-1}). \end{aligned} \tag{4}$$

Assuming that it is well-defined and stationary with $V(\varepsilon_t) = V(\varepsilon_{t-1})$, it follows that the unconditional variance is given by

$$V(\varepsilon_t) = \frac{\theta_0}{1-\theta_1}. \tag{5}$$

In the sense that its disturbances are mutually independent with a constant variance, the unconditional model is not distinguished from the classical ARMA model. The distinction comes when some account is taken of the *conditional variance*. Give that $\theta_0, \theta_1 > 0$, which is to ensure tha the variance is positive, the necessary and sufficient condition for the existence of the unconditional variance is simply that the ARCH process under (1) should be stable, for which the equation $\theta_0 + \theta_1 z = 0$ must have its root outside the unit circle. This is also the condition for the stability of an ordinary AR(1) process described by the equation $(\theta_0 + \theta_1 L)y(t) = \eta(t)$ wherein $\eta(t)$ is white noise.

Generalised Processes

An obvious direction in which to generalise the ARCH process of (1) is to increase the number of lagged values of $\varepsilon(t)$ which enter the determination of the variance. Thus an ARCH model of order q, as distinct from the first-order model, may be obtained by specifying that

$$\varepsilon_t = \upsilon_t h_t \qquad \text{where} \qquad h_t^2 = \theta_0 + \theta_1\varepsilon_{t-1}^2 + \cdots + \theta_q\varepsilon_{t-q}^2. \tag{6}$$

A more extensive generalisation is the Generalised Autoregressive Conditionally Heteroskedastic GARCH(p, q) model which has been propounded by Bollerslev (1986). In this case, the conditional variance is specified by

$$h_t^2 = \theta_0 + \theta_1\varepsilon_{t-1}^2 + \cdots + \theta_q\varepsilon_{t-q}^2 + \phi_1 h_{t-1}^2 + \cdots + \phi_q h_{t-p}^2. \tag{7}$$

The formulation is that of an ARMA$(p, q-1)$ model with an allowance for a nonzero mean. The model may be written in the form of

$$h^2(t) = \theta_0 + \theta(L)\varepsilon^2(t-1) + \phi(L)h^2(t-1). \tag{8}$$

It is proved by Bollerslev that the GARCH(p, q) process $\varepsilon(t) = h(t)\upsilon(t)$ is stationary with $E\{\varepsilon(t)\} = 0$ and $V\{\varepsilon(t)\} = \mu_0\{1 - \theta(1) - \phi(1)\}$ if and only if $\theta(1) + \phi(1) < 1$.

The advantage of the GARCH model is that, for relatively low orders of p and q, it enables quite complicated patterns of heteroskedasticity to be modelled which would require a high order of q in the pure ARCH(q) model.

References

Engle, R.F. (1982), Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflations, *Econometrica*, 50, 987–1007,

Bollerslev, T. (1986), Generalised Autoregressive Conditional Heterosked-asticity, *Journal of Econometrics*, 31, 307–327.