

REPARAMETRISATION OF DYNAMIC MODELS

Reparametrisation of a Distributed Lag Model

Consider a distributed-lag model of the form

$$(1) \quad y(t) = \beta_0 x(t) + \beta_1 x(t-1) + \dots + \beta_k x(t-k) + \varepsilon(t).$$

This can be written in summary notation as

$$(2) \quad y(t) = \beta(L)x(t) + \varepsilon(t),$$

where

$$(3) \quad \beta(L) = \beta_0 + \beta_1 L + \dots + \beta_k L^k$$

is a polynomial in the lag operator L . We wish to show how this can be reparametrised as an error-correction model.

Define the vectors

$$(4) \quad \beta' = [\beta_0, \beta_1, \dots, \beta_k] \quad \text{and} \quad x = [x_t, x_{t-1}, \dots, x_{t-k}]',$$

and let Λ be an arbitrary nonsingular matrix of order $(k+1) \times (k+1)$. Then

$$(5) \quad \beta' x = \{\beta' \Lambda\} \{\Lambda^{-1} x\} = \delta' z,$$

where $\delta' = \beta' \Lambda$ is a vector of alternative parameters and $z = \Lambda^{-1} x$ is a vector of transformed variables.

A simple reparametrisation of this sort can be used in forming an error correction model:

$$(6) \quad \beta \Lambda = [\beta_0, \beta_1, \dots, \beta_{k-1}, \beta_k] \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} \\ = [\kappa, -\delta_1, \dots, -\delta_{k-1}, -\delta_k]$$

$$(7) \quad \Lambda^{-1} x = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-k+1} \\ x_{t-k} \end{bmatrix} = \begin{bmatrix} x_t \\ -\nabla x_t \\ \vdots \\ -\nabla x_{t-k+2} \\ -\nabla x_{t-k+1} \end{bmatrix}$$

These identities enable us to rewrite equation (2) as

$$(8) \quad y(t) = \beta(L)x(t) + \varepsilon \\ = \beta(1)x(t) + \nabla x(t)\delta(L) + \varepsilon.$$

REPARAMETRISATION OF DYNAMIC MODELS

where

$$\begin{aligned}
 (9) \quad & \beta(1) = \beta_0 + \beta_1 + \cdots + \beta_k = \kappa \quad \text{and} \\
 & \delta(L) = \delta_1 L + \cdots + \delta_{k-1} L^{k-1} \quad \text{with} \\
 & \delta_j = -(\beta_j + \beta_{j+1} + \cdots + \beta_k).
 \end{aligned}$$

Taking $y(t)$ from both sides of equation (8) gives the error-correction formulation of equation (2):

$$(10) \quad \nabla y(t-1) = \left\{ \kappa x(t) - y(t-1) \right\} + \nabla \delta(L)x(t) + \varepsilon(t).$$

A Variety of Reparametrisations

There is a variety of reparametrisations which can serve the same essential purpose of expressing the transfer function $\beta(L)x(t)$ in terms of a level of $x(t)$ and the differences of $x(t)$. Consider first the identity

$$(11) \quad \beta(1) = L\beta(1) + \nabla\beta(1).$$

On substituting this into the equation

$$(12) \quad \beta(L) = \beta(1) + \nabla\delta(L),$$

which is from (8), we get

$$\begin{aligned}
 (13) \quad & \beta(L) = L\beta(1) + \nabla\beta(1) + \nabla\delta(L) \\
 & = L\beta(1) + \nabla\delta_1(L).
 \end{aligned}$$

In particular, it will be observed in reference to (9), that the leading element of $\delta_1(L)$ is

$$(14) \quad \beta(1) = \beta_0.$$

To demonstrate the full range of possibilities, let us consider the identity

$$(15) \quad 1 = L^n + \nabla(1 + L + \cdots + L^{n-1}),$$

where n does not exceed the maximum lag in $\beta(L)$. Then

$$(16) \quad \beta(1) = L^n \beta(1) + \nabla\{1 + L + \cdots + L^{n-1}\}\beta(1),$$

and, on defining

$$(17) \quad \delta_n(L) = \delta(L) + \{1 + L + \cdots + L^{n-1}\}\beta(1),$$

we can write

$$(18) \quad \beta(L) = L^n \beta(1) + \nabla \delta_n(L).$$

The leading element of $\delta_n(L)$ continues to be provided by (14).

Reparametrisation of an Autoregressive Distributed Lag Model

Now consider an equation in the form of

$$(19) \quad y(t) = \phi_1 y(t-1) + \dots + \phi_p y(t-p) + \beta_0 x(t) + \dots + \beta_k x(t-k) + \varepsilon(t).$$

which can be written in summary notation as

$$(20) \quad \alpha(L)y(t) = \beta(L)x(t) + \varepsilon(t).$$

Using the identity of (13) for $\beta(L)$ and a similar identity for $\alpha(L)$, we may write

$$(21) \quad \beta(L) = \beta(1)L + \delta_1(L)\nabla,$$

$$(22) \quad \begin{aligned} \alpha(L) &= \alpha(1)L + \theta_1(L)\nabla \\ &= L\alpha(1) + \{\nabla - \rho(L)\nabla\}. \end{aligned}$$

Here, equation (22) depends on the condition that $\alpha_0 = 1$, which accounts for the fact that ∇ is found on the RHS in association with a unit coefficient. Substituting (21) and (22) in (20) gives

$$(23) \quad \left\{ \alpha(1)L + \nabla - \rho(L)\nabla \right\} y(t) = \left\{ \beta(1)L + \nabla \delta_1(L) \right\} x(t) + \varepsilon(t).$$

This can be rearranged to give

$$(24) \quad \begin{aligned} \nabla y(t) &= \left\{ \beta(1)Lx(t) - \alpha(1)Ly(t) \right\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t) \\ &= \lambda \left\{ \gamma x(t-1) - y(t-1) \right\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t), \end{aligned}$$

where $\lambda = \alpha(1)$ is the so-called adjustment parameter and where $\gamma = \kappa/\lambda = \beta(1)/\alpha(1)$ is the steady-state gain of the rational transfer function $\beta(L)/\alpha(L)$. The term $\gamma x(t-1) - y(t-1)$ is described as the equilibrium error; and the value of the error will tend to zero if a steady state is maintained by $x(t)$ and if there are no disturbances. Equation (24) is the classical form of the error-correction equation.