

**THE CONDITIONAL AND UNCONDITIONAL MODELS  
OF FACTOR ANALYSIS AND THE NUMERICAL  
SOLUTION OF THEIR ESTIMATING EQUATIONS**

The purpose of this note is to compare and to contrast the estimating equations of the conditional and the unconditional models of factor analysis.

**The Models**

The basic equation underlying both of the models of factor analysis can be written as

$$(1) \quad y_{t.} = \mu_{t.} + \eta_{t.}; \quad t = 1, \dots, T.$$

Here,  $y_{t.}$  is a row vector of observations on  $G$  variables,  $\mu_{t.}$  is vector of unobserved systematic variables and  $\eta_{t.}$  is a vector of disturbances. It is assumed that successive disturbance vectors are uncorrelated and that

$$(2) \quad E(\eta_{t.}) = 0 \quad \text{and} \quad D(\eta_{t.}) = \Omega,$$

where  $\Omega = \text{diag}\{\omega_1, \dots, \omega_M\}$  is a diagonal matrix. It is assumed that the systematic vector  $\mu_{t.}$  is generated by  $K = G - J$  latent factors such that

$$(3) \quad \mu_{t.} = \xi_{t.}B'; \quad t = 1, \dots, T,$$

where  $\xi_{t.}$  is a row vector of  $K$  factors and  $B$  is a matrix of order  $G \times K$  of  $\text{Rank}(B) = K$  comprising the factor loadings.

In the conditional model, the unobserved factors  $\xi_{t.}$  are regarded as fixed quantities. In the unconditional model, they have a statistical distribution such that

$$(4) \quad E(\xi_{t.}) = 0 \quad \text{and} \quad D(\xi_{t.}) = \Phi.$$

It is common practice, when dealing with the unconditional model, to set  $\Phi = I$ . However, we shall assume that both  $\Omega$  and  $\Phi$  are diagonal matrices with  $G$  elements which have positive values.

For ease of notation, the  $T$  realisations of the relationships of (1) and (3) are compiled to give the following matrix equations:

$$(5) \quad Y = M + H, \quad M = \Xi B'.$$

Here  $Y$ ,  $M$  and  $H$  are matrices of order  $T \times G$  which comprise, respectively, the realisations of  $y_{t.}$ ,  $\mu_{t.}$  and  $\eta_{t.}$ , whilst  $\Xi$  is a matrix of order  $T \times K$  comprising the values of the factors  $\xi_{t.}$

## FACTOR ANALYSIS

### The Estimating Equations of the Conditional Model

On the assumption that the disturbance vector is normally distributed, we may write the likelihood function of the conditional model as

$$(6) \quad L(B, \Omega) = (2\pi)^{-GT/2} |\Omega|^{T/2} \exp\left\{\frac{1}{2} \text{Trace}(Y - M)'(Y - M)\Omega^{-1}\right\}.$$

The criterion for estimating  $B$  is to find the value which minimises

$$(7) \quad \text{Trace}\{(Y - \Xi B')'(Y - \Xi B')\Omega^{-1}\} = (Y - \Xi B')^c' (\Omega^{-1} \otimes I)(Y - \Xi B')^c,$$

The first step towards minimising the function is to find an expression for the minimising value of  $M = \Xi B'$  when  $B$  is given. The value is

$$(8) \quad \Xi B' = Y\Omega^{-1}B(B'\Omega^{-1}B)^{-1}B'.$$

Putting this into (7) indicates that the criterion function takes the form of

$$(9) \quad \begin{aligned} & \text{Trace}\{[I - B(B'\Omega^{-1}B)^{-1}B'\Omega^{-1}]Y'Y[I - \Omega^{-1}B(B'\Omega^{-1}B)^{-1}B']\Omega^{-1}\} \\ & = \text{Trace}\{Y'Y\Omega^{-1} - Y'Y\Omega^{-1}B(B'\Omega^{-1}B)^{-1}B'\Omega^{-1}\} \end{aligned}$$

In order to identify a unique estimate of  $B$ , it is necessary to specify some further aspects of this matrix. Therefore, the following normalisation is imposed:

$$(10) \quad B'\Omega^{-1}B = I.$$

In that case, the criterion function for estimating  $B$ , which is derivable from (9), takes the form of

$$(11) \quad L(B) = \text{Trace}\{B'\Omega^{-1}Y'Y'\Omega^{-1}B\} - \text{Trace}\{\Lambda B'\Omega^{-1}B\},$$

where  $\Lambda$  is a diagonal matrix of Lagrangean multipliers. Differentiating the function with respect to  $B^c$ —the long vector formed from  $B$ —gives

$$(12) \quad \begin{aligned} \frac{\partial L}{\partial B^c} &= (B'\Omega^{-1}Y'Y'\Omega^{-1})^r + (\Omega^{-1}Y'Y'\Omega^{-1}B)^r K \\ &+ (\Lambda B'\Omega^{-1})^r + (\Omega^{-1}B\Lambda)^r K, \end{aligned}$$

where the symbol  $r$  denotes the operation of forming a row vector from the rows of a matrix and  $K$  stands for the tensor commutator defined by  $A^r K = A'^r$ .

On setting the derivative to zero, we obtain, via some minor manipulations, the equation

$$(13) \quad Y'Y\Omega^{-1}B = B\Lambda,$$

which is the estimating equation for  $B$ .

In order to find an estimate of  $\Omega$ , we must consider the criterion function

$$(14) \quad L(\Omega) = -T \log |\Omega| - \text{Trace}\{(Y - M)'(Y - M)\Omega^{-1}\},$$

which is derivable from (6) by taking logarithms. Let  $\omega = [\omega_{11}, \omega_{22}, \dots, \omega_{MM}]$  be the vector of the diagonal elements of  $\Omega = \Omega(\omega)$ . Then, by using a chain rule to differentiate the function in respect of  $\omega$ , we get

$$(15) \quad \frac{\partial L}{\partial \omega} = -T\Omega^r \frac{\partial(\Omega^{-1})^c}{\partial \omega} + \{(Y - M)'(Y - M)\}^r \frac{\partial(\Omega^{-1})^c}{\partial \omega}.$$

By setting this derivative to zero, we can obtain an equation which can be written as

$$(16) \quad T\Omega(\omega) = \text{diag}\{(Y - M)'(Y - M)\}.$$

By using the condition  $B'\Omega^{-1}B = I$ , we can obtain from (8) the expression

$$(17) \quad M = \Xi B' = Y\Omega^{-1}BB'.$$

Substituting the latter into (16) gives

$$(17) \quad \begin{aligned} T\Omega(\omega) &= \text{diag}\{Y'Y - Y'Y\Omega^{-1}BB' - BB'\Omega^{-1}Y'Y \\ &\quad + BB'\Omega^{-1}Y'Y\Omega^{-1}BB'\} \\ &= \text{diag}\{Y'Y - B\Lambda B'\}, \end{aligned}$$

where the second equality follows from using the condition under (13) which also implies that  $\Lambda = B'\Omega^{-1}Y'Y\Omega^{-1}B$ .

Thus, by gathering equations (13) and (17), it can be seen that the estimating equations of the conditional model of factor analysis are given by

$$(19) \quad \begin{aligned} Y'Y\Omega^{-1}B &= B\Lambda, \quad \Lambda = B'\Omega^{-1}Y'Y\Omega^{-1}B \\ T\Omega &= \text{diag}\{Y'Y - B\Lambda B'\}. \end{aligned}$$

### **The Estimating Equations of the Unconditional Model**

The basic equation of the model can be written as

$$(20) \quad y_{t.} = \xi_{t.}B' + \eta_{t.}; \quad t = 1, \dots, T.$$

Under the assumptions of the unconditional model, the dispersion matrix of  $y_{t.}$  becomes

$$(21) \quad \begin{aligned} D(y_{t.}) &= B(\xi_{t.})B' + D(\eta_{t.}) \\ &= B\Phi B' + \Omega. \end{aligned}$$

## FACTOR ANALYSIS

Under the assumption that  $\xi_t$  and  $\eta_t$  are normally distributed, the likelihood function of the unconditional model can be written as

$$(22) \quad L(B, \Phi, \Omega) = (2\pi)^{MT/2} |B\Phi B'|^{T/2} \exp\left\{\frac{1}{2} \text{Trace}[Y'Y(B\Phi B' + \Omega)^{-1}]\right\}.$$

The parameters can be estimated by finding the values which minimise the function

$$(23) \quad L = -T \log |B\Phi B'| - \text{Trace}[Y'Y(B\Phi B' + \Omega)^{-1}].$$

Differentiating with respect to  $B^c$  gives

$$(24) \quad \frac{\partial L}{\partial B^c} = -2T\{\Phi B'(N\Phi B')^{-1}\}^r + 2\{\Phi B'(B\Phi B')^{-1}Y'Y\Phi B'(N\Phi B')^{-1}\}^r.$$

Setting this to zero and rearranging gives the first-order condition

$$(25) \quad T^{-1}Y'Y(B\Phi B' + \Omega)^{-1}B = B.$$

Now consider the identity

$$(26) \quad (B\Phi B' + \Omega)^{-1} = \Omega^{-1} - \Omega^{-1}B(B'\Omega^{-1}B + \Phi^{-1})^{-1}B'\Omega^{-1}.$$

By using the condition  $B'\Omega^{-1}B = I$  and the fact that

$$(27) \quad \{\Phi(I - \Phi^{-1})\}\{I - (I - \Phi^{-1})^{-1}\} = I,$$

it can be seen that

$$(28) \quad \begin{aligned} (B\Phi B' + \Omega)^{-1}B &= \Omega^{-1}B\{I - (B'\Omega^{-1}B + \Phi^{-1})B'\Omega^{-1}B\} \\ &= \Omega^{-1}B\{I - (I + \Phi^{-1})^{-1}\} \\ &= \Omega^{-1}B(I - \Phi)^{-1}. \end{aligned}$$

On substituting this result in (25), we get

$$(29) \quad T^{-1}Y'Y\Omega^{-1}B(I - \Phi)^{-1} = B,$$

or simply

$$(30) \quad T^{-1}Y'Y\Omega^{-1}B = B(I - \Phi),$$

which represents the estimating equation for  $B$ .

Now consider differentiating the criterion of (23) in respect of  $\Omega^c$ . This gives

$$(31) \quad \begin{aligned} \frac{\partial L}{\partial \Omega^c} &= -T(B\Phi B' + \Omega)^{-1r} - Y'Y \frac{\partial (B\Phi B' + \Omega)^{-1c}}{\partial \Omega^c} \\ &= -T(B\Phi B' + \Omega)^{-1r} - (Y'Y)^r \{(B\Phi B' + \Omega)^{-1c} \otimes (B\Phi B' + \Omega)^{-1c}\} \\ &= -T(B\Phi B' + \Omega)^{-1r} - \{(B\Phi B' + \Omega)^{-1}Y'Y(B\Phi B' + \Omega)^{-1}\}^r. \end{aligned}$$

Using this derivative in the condition

$$(32) \quad \frac{\partial L}{\omega} = \frac{\partial L}{\Omega^c} \frac{\partial \Omega^c}{\omega} = 0$$

results in the following estimating equation:

$$(33) \quad T\Omega = \text{diag}\{Y'Y - TB\Phi B'\}.$$

Now consider differentiating the criterion function in respect of  $\Phi$ . This gives

$$(34) \quad \begin{aligned} \frac{\partial L}{\partial \Phi^c} &= -T(B\Phi B' + \Omega)^{-1r}(B \otimes B) \\ &\quad + (Y'Y)^r \{(B\Phi B' + \Omega)^{-1} \otimes (B\Phi B' + \Omega)^{-1}\}(B \otimes B) \\ &= -T\{B'(B\Phi B' + \Omega)^{-1}B\}^r \\ &\quad + T\{B'(B\Phi B' + \Omega)^{-1}Y'Y(B\Phi B' + \Omega)^{-1}B\}^r. \end{aligned}$$

The estimating equation for  $\Phi$  is derived from the condition that

$$(35) \quad \text{diag}\{T B'(B\Phi B' + \Omega)^{-1}B - B'(B\Phi B' + \Omega)^{-1}Y'Y(B\Phi B' + \Omega)^{-1}B\} = 0.$$

Using the identity of (28) and the condition that  $B'\Omega^{-1}B = I$ , this can be written as

$$(36) \quad \text{diag}\{T(I + \Phi)^{-1} + \Omega\}^{-1}B - (I + \Phi)^{-1}B'\Omega\}^{-1}Y'Y\Omega\}^{-1}B(I + \Phi)^{-1}\} = 0.$$

Since  $I + \Phi$  is diagonal if  $\Phi$  is diagonal, this is simply

$$(37) \quad T(I + \Phi) = B'\Omega^{-1}Y'Y\Omega^{-1}B = \Lambda.$$

By gathering the equations (30), (33) and (37), it can be seen that the estimating equations of the unconditional model of factor analysis are given by

$$(19) \quad \begin{aligned} Y'Y\Omega^{-1}B &= TB(I + \Phi) = B'\Lambda, & \Lambda &= B'\Omega^{-1}Y'Y\Omega^{-1}B, \\ T\Omega &= \text{diag}\{Y'Y - TB\Phi B'\}. \end{aligned}$$

Comparison with the equations under (19) shows that the estimating equations for the conditional model and the unconditional model differ only in respect of the estimator of the dispersion matrix  $\Omega$  of the disturbances.