THE MULTIVARIATE NORMAL DISTRIBUTION

We say that the $n \times 1$ random vector x is normally distributed with a mean of $E(x) = \mu$ and a dispersion matrix of $D(x) = \Sigma$ if the probability density function is

$$N(x; \mu, \Sigma) = (2\pi)^{-n/2}|\Sigma|^{-1/2} \exp\big\{-\tfrac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\big\}. \tag{17.22}$$

It is understood that x is non-degenerate with $\text{Rank}(\Sigma) = n$ and $|\Sigma| \neq 0$. To denote that x has this distribution, we can write $x \sim N(\mu, \Sigma)$.

We shall demonstrate two notable features of the normal distribution. The first feature is that the conditional and marginal distributions associated with a normally distributed vector are also normal. The second is that any linear function of a normally distributed vector is itself normally distributed.

We shall base our arguments on two fundamental facts. The first is that

(17.23) If $x \sim N(\mu, \Sigma)$ and if $y = A(x - b)$ where A is nonsingular, then $y \sim N\{A(\mu - b), A\Sigma A'\}$.

This may be illustrated by considering the case where $b = 0$. Then, according to the result in (17.8), y has the distribution

$$\begin{aligned}
&N(A^{-1}y; \mu, \Sigma)\|\partial x/\partial y\| \\
&\quad = (2\pi)^{-n/2}|\Sigma|^{-1/2} \exp\big\{-\tfrac{1}{2}(A^{-1}y-\mu)'\Sigma^{-1}(A^{-1}y-\mu)\big\}\|A^{-1}\| \\
&\quad = (2\pi)^{-n/2}|A\Sigma A'|^{-1/2} \exp\big\{-\tfrac{1}{2}(y-A\mu)'(A\Sigma A')^{-1}(y-A\mu)\big\};
\end{aligned}$$

so, clearly, $y \sim N(A\mu, A\Sigma A')$.

The second of the fundamental facts is that

(17.25) If $x \sim N(\mu, \Sigma)$ can be written in partitioned form as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}\right),$$

then $x_1 \sim N(\mu_1, \Sigma_{11})$ and $x_2 \sim N(\mu_2, \Sigma_{22})$ are independently distributed normal variates.

To see this, we need only consider the quadratic form

$$(x-\mu)'\Sigma^{-1}(x-\mu) = (x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1) + (x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)$$

which arises in this particular case. On substituting in into the expression for $N(x, \mu, \Sigma)$ in (17.22) and using $|\Sigma| = |\Sigma_{11}||\Sigma_{22}|$, we get

$$\begin{aligned}
N(x;\mu,\Sigma) &= (2\pi)^{-m/2}|\Sigma_{11}|^{-1/2}\exp\left\{-\tfrac{1}{2}(x_1-\mu_1)'\Sigma_{11}^{-1}(x_1-\mu_1)\right\}\\
&\quad\times(2\pi)^{(m-n)/2}|\Sigma_{22}|^{-1/2}\exp\left\{-\tfrac{1}{2}(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)\right\}\\
&= N(x_1;\mu_1,\Sigma_{11})N(x_2;\mu_2,\Sigma_{22}).
\end{aligned}$$

The latter can only be the product of the marginal distributions of x_1 and x_2, which proves that these vectors are independently distributed.

The essential feature of the result is that

(17.26) If x_1 and x_2 are normally distributed with $C(x_1, x_2) = 0$, then they are mutually independent.

A zero covariances does not generally imply statistical independence.

Even when x_1, x_2 are not independently distributed, their marginal distributions are still formed in the same way from the appropriate components of μ and Σ. This is entailed in the first of our two main results which is that

(17.27) If $x \sim N(\mu, \Sigma)$ is partitioned as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right),$$

then the marginal distribution of x_1 is $N(\mu_1, \Sigma_{11})$ and the conditional distribution of x_2 given x_1 is

$$N(x_2|x_1;\mu_2+\Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1),\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}).$$

Proof. Consider a non-singular transformation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

such that $C(y_1, y_2) = C(Fx_1 + x_2, x_1) = FD(x_1) + C(x_2, x_1) = 0$. Writing this condition as $F\Sigma_{11} + \Sigma_{21} = 0$ gives $F = -\Sigma_{21}\Sigma_{11}^{-1}$. It follows that

$$E\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1 \end{bmatrix};$$

and, since $D(y_1) = \Sigma_{11}$, $C(y_1, y_2) = 0$ and

$$\begin{aligned} D(y_2) &= D(Fx_1 + x_2) \\ &= FD(x_1)F' + D(x_2) + FC(x_1, x_2) + C(x_2, x_1)F' \\ &= \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{11}\Sigma_{11}^{-1}\Sigma_{12} + \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \\ &= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}, \end{aligned}$$

it also follows that

$$D\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix}.$$

Therefore, according to (17.25), we can write the joint density function of y_1, y_2 as

$$N(y_1; \mu_1, \Sigma_{11})N(y_2; \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1, \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}).$$

Integrating with respect to y_2 gives the marginal distribution of $x_1 = y_1$ as $N(x_1; \mu_1, \Sigma_{11})$.

Now consider the inverse transformation $x = x(y)$. The Jacobian of this transformation is unity. Therefore, to obtain an expression for $N(x; \mu, \Sigma)$, we need only write $y_2 = x_2 - \Sigma_{21}\Sigma_{11}^{-1}x_1$ and $y_1 = x_1$ in the expression for the joint distribution of y_1,, y_2. This gives

$$\begin{aligned} N(x; \mu, \Sigma) = N&(x_1; \mu_1, \Sigma_{11}) \\ &\times N(x_2 - \Sigma_{21}\Sigma_{11}^{-1}x_1; \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1, \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}), \end{aligned}$$

which is the product of the marginal distribution of x_1 and the conditional distribution $N(x_2|x_1; \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$ of x_2 given x_1.

The linear function $E(x_2|x_1) = \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$, which defines the expected value of x_2 for given values of x_1, is described as the regression of x_2 on x_1. The matrix $\Sigma_{21}\Sigma_{11}^{-1}$ is the matrix regression coefficients.