MAXIMA AND MINIMA

The Maxima and Minima of Twice-Differentiable Functions

Much of economic theory is concerned with problems of optimisation where it is required to find the maximum or the minimum of a twice-differentiable function. In the following discussion, we shall be concerned primarily with the minimisation of a function of a single variable. Since a problem of maximisation can be solved by minimising the negative of the function in question, there is no real omission in ignoring problems of maximisation. Later, we shall consider the optimisation of functions of several variables, including cases where the variables are subject of constraints, ie where the values which can be assigned to the variables are not wholly independent of each other.

We should begin by giving a precise definition of the minimum of a univariate function.

(1) A point ξ is said to be a *strict minimum* or an *isolated minimum* of the function $f(x)$ if $f(\xi) < f(x)$ for all x in an neighbourhood $(\xi - \epsilon, \xi + \epsilon)$ of ξ or, equivalently, if $f(\xi) < f(\xi + h)$ whenever $|h| < \epsilon$ for some small $\epsilon > 0$. The point is said to be a *weak minimum* of $f(x)$ if $f(\xi) \leq f(x)$ for all x in the neighbourhood.

In effect, the point ξ is a strict minimum if the value of f increases with any small departure from ξ, whereas it is a weak minimum if the fails to decrease. In general, a function may exhibit these properties at a number of points which are described as *local minima.* If there is a unique point at which the function is lowest, then this is called a *global minimum.*

It is not possible to demonstrate that an analytic function has a global minimum without a complete knowledge of its derivatives of all orders. The conditions which are sufficient for the existence of a local minimum are modest by comparison.

(2) *Conditions for a Minimum.* A continuous and twice differentiable function $f(x)$ has a strict minimum at the point ξ if and only if $f'(\xi) = 0$ and $f''(\xi) > 0$.

The condition that $f'(\xi) = 0$ is described as a *condition of stationarity* which, in the present context, becomes the *first-order condition* for the minimum. The condition that $f''(\xi) > 0$ is described as the *second-order condition* for the minimum.

To understand the first-order condition, we observe that, if $f(x)$ attains a minimum at the point ξ, then it cannot be decreasing at that point. This implies that its derivative at ξ must be non-negative, which is the condition that $f'(\xi) \geq 0$. Similarly, the function cannot be increasing at ξ, which implies that its derivate must be non-positive. Thus it is also required that $f'(\xi) \leq 0$. These inequalities can be reconciled only if $f(\xi) = 0$.

To understand the second-order condition, we must recognise that, if f is to attain a minimum at the point ξ, then its slope $f'(x)$ must change its sign from negative to positive as the increasing value of x passes the point ξ. The second-order derivative f'' represents the rate of change of the slope f'; and, since this slope is increasing, we must have $f'' > 0$.

The second-order condition for a minimum is sometimes described as the condition that the function f is concave at the point ξ. The difficulty with such terminology is that the concavity or convexity of a function is determined by the observer's point of view; and to be unambiguous we ought also to specify the viewpoint. In this case, we would say that the function is convex when viewed from below.

The arguments relating to the second-order condition are reversed in the case of a maximum. To summarise both cases, we may state that

(a) If $f'(\xi) = 0$ and $f''(\xi) > 0$, then ξ is an isolated minimum of f,

(b) If $f'(\xi) =$ and $f''(\xi) < 0$, then ξ is an isolated maximum of f.

Example. The costs of a manufacturing firm, as a function of its output q, are given by

$$C = \frac{1}{3}q^3 - 6q^2 + 30q + 50. \tag{3}$$

We shall assume that conditions of perfect competition prevail such that the price $p = 10$ is not affected by the quantity which the firm brings to the market. Then, assuming that everything which is produced is also sold, the sales revenue of the firm is $R = 10q$ and the profits are given by

$$\pi(q) = R - C = 10q - \frac{1}{3}q^3 + 6q^2 - 30q - 50. \tag{4}$$

To find the value of q which maximises profit, we differentiate π with respect of q; and we proceed to set the result to zero in fulfilment of the first-order condition:

$$\frac{d\pi}{dq} = 10 - q^2 + 12q - 30 = 0. \tag{5}$$

This can be rearranged to give

$$\begin{aligned} 0 &= q^2 - 12q + 20 \\ &= (q-2)(q-10). \end{aligned} \tag{6}$$

There are two solutions: $q = 2, 10$. To determine their status, we must evaluate the second derivative at either point:

$$\frac{d^2\pi}{dq^2} = -2q + 12. \tag{7}$$

At $q = 2$, the second derivative is positive, which indicates a minimum. At $q = 10$, it is negative, which indicates a maximum. The level of the profits at that point is $\pi = 227\frac{2}{3}$.

Points of Inflection.

It is interesting consider cases where the condition $f'' = 0$ is fulfilled at a point ξ. These are cases where the function f exhibits zero curvature, and the corresponding value ξ of the function's argument is described as a point of inflection. An example is provided by the function $f(x) = x^3 + x$ which has a point of inflection at $x = 0$.

Maxima and Minima and the Mean-Value Theorem

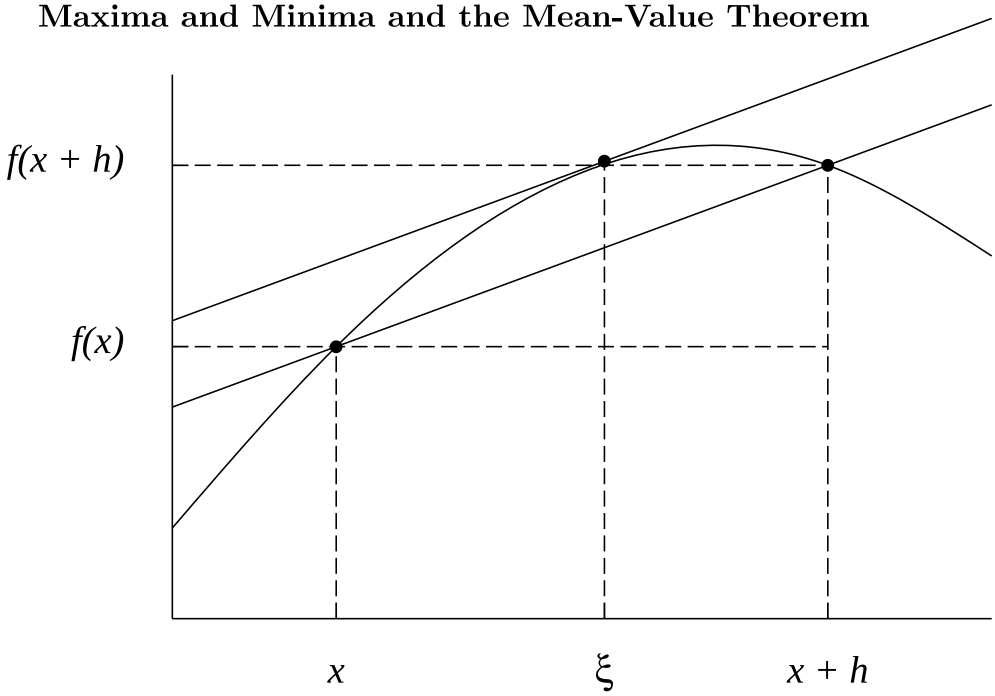

Figure 1. The Mean-Value Theorem

The mean-value theorem asserts that, if f is continuous in the interval $[x, x+h]$, then there exists a point ξ in that interval such that $f'(\xi) = \{f(x+h) - f(x)\}/h$. Hence

$$f(x+h) = f(x) + hf'(\xi). \tag{8}$$

An alternative way of denoting the mean value is to write it as

$$(9) \qquad \xi = x + \lambda h \quad \text{where} \quad \lambda \in [0, 1].$$

The theorem, which is perhaps intuitively obvious, concerns the linear approximation of the function $f(x)$. For some purposes, we may wish to approximate the function by a polynomial of a higher degree than unity. For every such approximation there is a corresponding mean-value theorem. Thus, in the case of a quadratic approximation, we have the following:

(10) *The Quadratic Mean-Value Theorem.* If $f(x)$ is continuous and twice differentiable in the interval $[x, x+h]$ then there is a point $x+\lambda h \in [x, x+h]$ such that

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x + \lambda h).$$

This theorem can be used in a more rigorous proof of the conditions for minimising a function:

(11) *Conditions for a Minimum.* A continuous and twice-differentiable function $f(x)$ has a strict minimum at the point ξ if and only if $f'(\xi) = 0$ and $f''(\xi) > 0$.

Proof. We shall consider only the second-order condition since the necessity of the first-order condition $f'(\xi) = 0$ has already been established to our satisfaction.

The quadratic mean-value theorem indicates that

$$(12) \qquad f(\xi + h) = f(\xi) + hf'(\xi) + \frac{h^2}{2} f''(\xi + \lambda h)$$

for some value $\lambda \in [0, 1]$. Now, if $f'(\xi) = 0$, then the inequality $f(\xi + h) > f(\xi)$ holds for all $|h| < \epsilon$ if and only if

$$(13) \qquad \frac{1}{2} h^2 f''(\xi + h) > 0.$$

Since $h^2 > 0$, this amounts to the condition that $f''(\xi + h) > 0$. Letting $h \to 0$ establishes that $f''(\xi) > 0$ is the necessary second-order condition.