BINOMIAL OPTION PRICING MODEL

A One-Step Binomial Model Suppose that a portfolio consists of N units of stock as assets, with a spot price of S_0 per unit, together with a liability of one call option. The initial value of the portfolio at time $t = 0$ will be

$$V_0 = NS_0 - c_{\tau|0}.$$

When the value of the stock rises, so will that of the option.

We envisage two possibilities. Either the price increases to $S_\tau^u = S_0U$ at time $t = \tau$, where $U > 1$, or it decreases to $S_\tau^d = S_0D$, where $D < 1$.

The value of the call option will be $S_\tau - K_{\tau|0}$, if $S_\tau > K_{\tau|0}$, in which case it will be exercised, or it will be worthless, if $S_\tau \leq K_{\tau|0}$. Thus

$$c_{\tau|\tau} = \max(S_\tau - K_{\tau|0}, 0) = (S_\tau - K_{\tau|0})^+.$$

Let c_τ^u and c_τ^d be the values of the option corresponding to S_τ^u and S_τ^d, respectively. Then, the value of the portfolio at time $t = \tau$ will be

$$V_\tau = \begin{cases} NS_0U - c_\tau^u, & \text{if } S_\tau = S_\tau^u = S_0U; \\ NS_0D - c_\tau^d, & \text{if } S_\tau = S_\tau^d = S_0D. \end{cases}$$

The number of units of the asset can be chosen so that the values of the portfolio are the same in these two cases. Therefore, the portfolio is riskless; and there are two equations for V_τ :

$$V_\tau = S_0UN - c_\tau^u \quad \Longleftrightarrow \quad c_\tau^u = S_0UN - V_\tau,$$

$$V_\tau = S_0DN - c_\tau^d \quad \Longleftrightarrow \quad c_\tau^d = S_0DN - V_\tau.$$

Their solutions for N and V_τ are

$$N = \frac{c_\tau^u - c_\tau^d}{S_0(U - D)} \qquad \text{and} \qquad V_\tau = \frac{c_\tau^u D - c_\tau^d U}{U - D}.$$

The initial value the riskless portfolio is $V_0 = V_\tau e^{-r\tau}$, and that of the option is

$$\begin{aligned} c_{\tau|0} &= NS_0 - V_0 \\ &= NS_0 - V_\tau e^{-r\tau}, \end{aligned}$$

Putting the solutions for N and V_τ into this equation gives

$$\begin{aligned} c_{\tau|0} &= \left\{ \frac{c_\tau^u - c_\tau^d}{S_0(U-D)} \right\} S_0 - \left\{ \frac{c_\tau^u D - c_\tau^d U}{U-D} \right\} e^{-r\tau} \\ &= e^{-r\tau} \left\{ \frac{e^{r\tau}(c_\tau^u - c_\tau^d) - (c_\tau^u D - c_\tau^d U)}{U-D} \right\} \\ &= e^{-r\tau} \left\{ \left(\frac{e^{r\tau} - D}{U-D} \right) c_\tau^u + \left(\frac{U - e^{r\tau}}{U-D} \right) c_\tau^d \right\}. \end{aligned}$$

Therefore, the value of the call option at time $t = 0$ is

$$c_{\tau|0} = e^{-r\tau} \left\{ c_\tau^u p + c_\tau^d (1-p) \right\},$$

where

$$p = \frac{e^{r\tau} - D}{U-D} \qquad \text{and} \qquad 1 - p = \frac{U - e^{r\tau}}{U-D}.$$

Interpretations

We may regard p and $1-p$ as the probabilities of c^u_τ and c^d_τ, respectively. This adds another dimension to the model. In that case, we can write

$$\begin{aligned} c_{\tau|0} &= e^{-r\tau}\left\{c^u_\tau p + c^d_\tau(1-p)\right\} \\ &= e^{-r\tau}E(c_{\tau|\tau}), \end{aligned}$$

which is the expectation of the realised value of the option discounted to time $t=0$.

Observe that, if $p = 1-p = 0.5$, then

$$e^{r\tau} - D = U - e^{r\tau} \quad \text{or, equivalently} \quad R - D = U - R.$$

Thus, equal departures from the riskless return R have equal probability. This is also a characteristic of geometric (log-normal) Brownian motion.

Interpretations (continued)

Let x be a point binomial random variable with

$$P(x = 1) = p \quad \text{and} \quad P(x = 0) = 1 - p.$$

Then

$$\begin{aligned} c_{\tau|\tau} &= (S_\tau - K_{\tau|0})^+ \\ &= (S_0 U^x D^{1-x} - K_{\tau|0})^+, \end{aligned}$$

and $c_{\tau|0} = e^{-r\tau} E(c_{\tau|\tau})$ can be written as

$$\begin{aligned} c_{\tau|0} &= e^{-r\tau} \left\{ c_\tau^u p + c_\tau^d (1 - p) \right\} \\ &= e^{-r\tau} \left\{ (S_0 U - K_{\tau|0})^+ p + (S_0 D - K_{\tau|0})^+ (1 - p) \right\} \\ &= e^{-r\tau} E \left\{ (S_0 U^x D^{1-x} - K_{\tau|0})^+ \right\}. \end{aligned}$$

This can serve as a prototype for developing a multi-step binomial model.

A Two-Step Binomial Model In a practical model, there may be many revaluations of the stock between the time $t = 0$, when the option is written, and the time $t = \tau$, when it expires. Then, there will be a wide range of possible eventual prices.

Consider a two-step model. In the first step, the spot price of the stock moves up to $S^u = S_0U$ or down to $S^d = S_0D$. In the next step, there are three possible prices $S^{uu} = S_0U^2$, $S^{ud} = S_0UD$ and $S^{dd} = S_0D^2$:

		$S^{uu} = S_0U^2$
	$S^u = S_0U$	
S_0		$S^{ud} = S_0UD$
	$S^d = S_0D$	
		$S^{dd} = S_0D^2$

To price the call option, we start at the time of expiry and to work backwards to derive prices for the option at the intermediate nodes. From these, we can derive the price $c_{\tau|0}$ of the option at the base of the tree.

The Two-Step Binomial Model (continued)

Let the values of the option corresponding to the outcomes S^{uu}, S^{ud} and S^{dd} be denoted by c^{uu}, c^{ud} and c^{dd}, respectively, and let those corresponding to the intermediate outcomes S^{u}, S^{d} be c^{u}, c^{d}. Then,

$$c^{u} = e^{-r\tau/2}\left\{c^{uu}p + c^{ud}(1-p)\right\}, \quad c^{d} = e^{-r\tau/2}\left\{c^{ud}p + c^{dd}(1-p)\right\}.$$

From these, we derive

$$\begin{aligned}
c_{\tau|0} &= e^{-r\tau/2}\left\{c^{u}p + c^{d}(1-p)\right\} \\
&= e^{-r\tau}\left\{\left[c^{uu}p + c^{ud}(1-p)\right]p + \left[c^{ud}p + c^{dd}(1-p)\right](1-p)\right\} \\
&= e^{-r\tau}\left\{c^{uu}p^{2} + 2c^{ud}p(1-p) + c^{dd}(1-p)^{2}\right\}.
\end{aligned}$$

The Multi-Step Binomial Model The generalisation to n sub periods is as follows:

$$\begin{aligned} c_{\tau|0} &= e^{-r\tau}\left\{\sum_{j=0}^{n}\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}c^{uj,d(n-j)}\right\} \\ &= e^{-r\tau}E(c_{\tau|\tau}). \end{aligned}$$

Here, $c_{\tau|\tau} = c^{uj,d(n-j)}$ is the value of the option after the price has reached $S_\tau = S_0U^jD^{n-j}$ by moving up j times and down $n-j$ times:

$$\begin{aligned} c^{uj,d(n-j)} &= \max(S_\tau - K_{\tau|0}, 0) \\ &= (S_0U^jD^{n-j} - K_{\tau|0})^+. \end{aligned}$$

Taking expectations and discounting back to $t = 0$ gives

$$c_{\tau|0} = e^{-r\tau}E\left\{(S_0U^jD^{n-j} - K_{\tau|0})^+\right\}.$$

Convergence of the Binomial to the Black–Scholes Model As the number n of the subintervals of the period $[0, \tau]$ increases indefinitely, the binomial formula for $c_{\tau|0}$ converges to

$$c_{\tau|0} = e^{-r\tau} E \left\{ S_0 e^w - K_{\tau|0})^+ \right\},$$

where

$$w = (r - \sigma^2/2)\tau + \sigma\sqrt{\tau} z, \quad z \sim N(0, 1)$$

is normal random variable with mean $(r - \sigma^2/2)\tau$ and variance $\sigma^2\tau$.

By evaluating the expectation, we get the Black–Scholes formula for the price of a European call option. This is

$$c_{\tau|0} = S_0 \Phi(d_1) - K_{\tau|0} e^{-r\tau} \Phi(d_2),$$

where $\Phi(d) = P(z \leq d)$ denotes a value from the cumulative Normal distribution function, and where

$$d_1 = \frac{\ln(S_0/K_{\tau|0}) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

Properties of the Call-Option Premium

The call-option premium $c_{\tau|0}$ is a function of five variables S_0, $K_{\tau|0}$ τ, r and σ.

$c_{\tau|0}$ increases as S_0 increases.

$c_{\tau|0}$ decreases as $K_{\tau|0}$ increases.

$c_{\tau|0}$ decreases as τ decreases. The call option is a wager that the price of the stock will increase beyond the exercise price. If τ, which the time available for this to occur, is small, then the option will be less valuable.

$c_{\tau|0}$ increases as σ increases. The option holder will benefit from the increases in the stock price at the time of expiry that might occur with increased volatility. They will not suffer additional losses from the larger decreases below the exercise price that might occur.

$c_{\tau|0}$ increases as r increases. This follows directly from the formula, since the term $-K_{\tau|0}e^{-r\tau}\Phi(d_2)$ is diminished as r increases.

The Substantive Assumptions underlying the Black–Scholes Model

1. There are no transaction costs and the assets are infinitely divisible.

2. Trading is continuous, which, together with the absence of transaction costs, permits a continuous rebalancing of portfolios—as is required in order to create the riskless portfolio.

3. Stock prices are generated by a Geometric Brownian motion stochastic process, characterised by a constant volatility parameter σ and a constant drift parameter μ.

4. There is a risk-free rate of interest r, which is constant over the period in question, at which the investor can borrow without constraint.

5. The stock pays no dividends. This assumption can be relieved easily, if the dates and the sizes of the dividends are known in advance.