# CONTINUOUS-TIME STOCHASTIC PROCESSES

**Discrete-Time Random Walk** . A standardised random walk defined over the set of integers $\{t = 0, \pm 1, \pm 2, \ldots\}$, is a sequence by $w(t) = \{w_t; t = 0, \pm 1, \pm 2, \ldots\}$ in which

$$w_{t+1} = w_t + \varepsilon_{t+1}, \tag{1}$$

where $\varepsilon(t) = \{\varepsilon_t; t = 0, \pm 1, \pm 2, \ldots\}$ is a sequence of independently distributed of standard normal random elements with $E(\varepsilon_t) = 0$ and $V(\varepsilon_t) = 1$, for all $t$, described as a white-noise process.

By a process of back-substitution, the following expression can be derived:

$$w_t = w_0 + \big\{\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1\big\}. \tag{2}$$

This is the sum of an initial value $w_0$ and of an accumulation of stochastic increments. If $w_0$ has a fixed finite value, then the mean and the variance of $w_t$, conditional of this value, are be given by

$$E(w_t|w_0) = w_0 \quad \text{and} \quad V(w_t|w_0) = t. \tag{3}$$

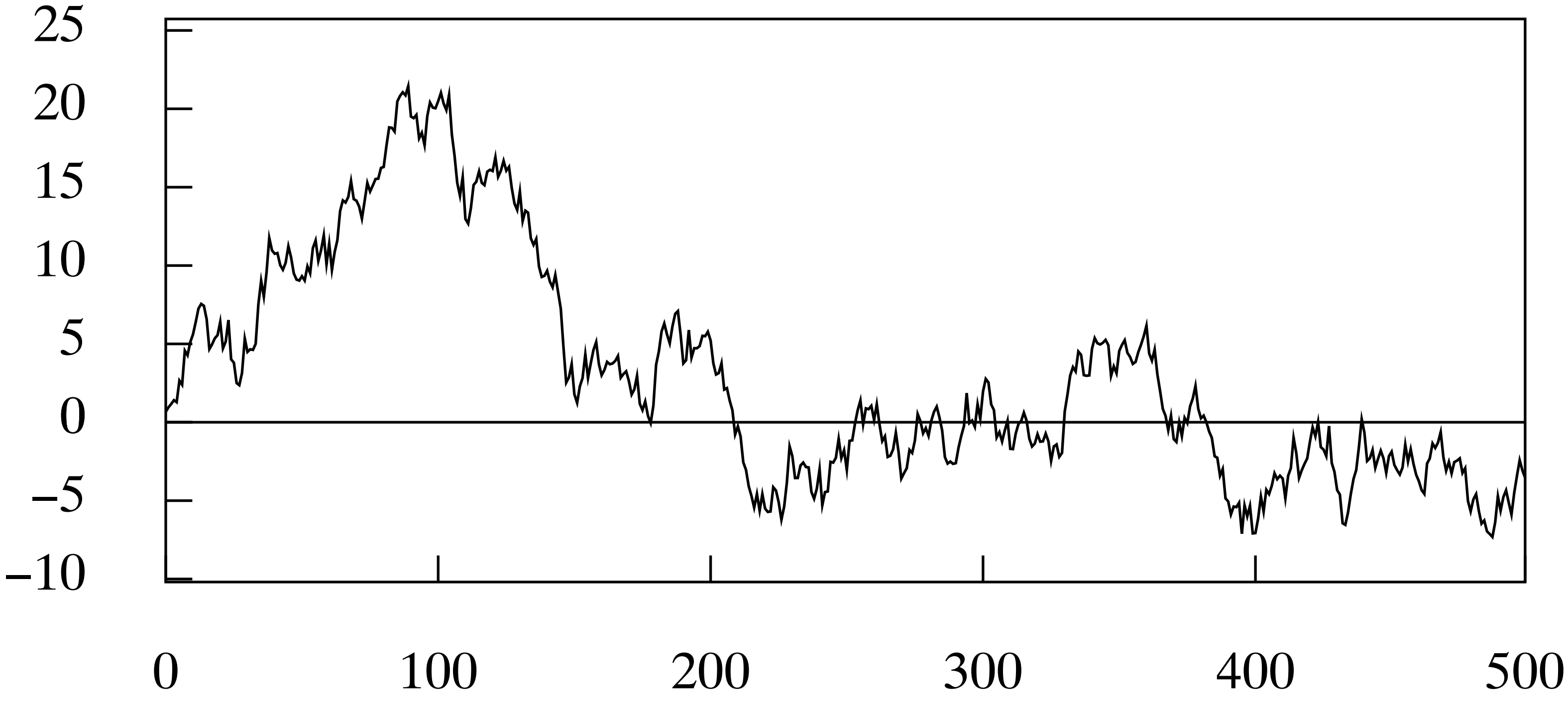


**Figure 1.** The graph of 500 observations on simulated random-walk process generated by the equation $y_t = y_{t-1} + \varepsilon_t$.

To reduce the random walk to a stationary process, we take its first differences:

$$w_{t+1} - w_t = \varepsilon_{t+1}. \tag{4}$$

We can take longer steps through time without fundamentally altering the nature of the process. Let $h$ be any integral number of time periods. Then

$$w_{t+h} - w_t = \sum_{j=1}^{h} \varepsilon_{t+j} = \zeta_{t+h}\sqrt{h}, \tag{5}$$

where $\zeta_{t+h}$ is an element of a sequence $\{\zeta_{t+jh}; j = 0, \pm 1, \pm 2, \ldots\}$ of independently and identically distributed standard normal variates.

The factor $\sqrt{h}$ is present because we are now taking steps through time of length $h$ instead of steps of unit length.

A first-order random walk over a surface is know as *Brownian motion.* One can imagine small particles, such a pollen grains, floating on the surface of a viscous liquid. The viscosity is expected to bring the particles to a halt quickly. However, if they are very light, then they will dart hither and thither on the surface of the liquid under the impact of its molecules, which are in constant motion.

**A Wiener process** is the consequence of allowing the intervals of a discrete-time random walk to tend to zero. The dates at which the process is defined become a continuum—and the process becomes continuous almost everywhere, but nowhere differentiable.

When sampled at regular intervals, a Wiener process has the same mathematical description as the discrete-time process. However, whereas the random walk is defined only on the set of integers, the Wiener process is defined for all points on a real line that represents continuous time.

To generalise of equation (5), replace the integer $h$ by an infinitesimally small increment $dt$. Then, the equation becomes

$$dw(t) = w(t + dt) - w(t) = \zeta(t + dt)\sqrt{dt} \tag{6}$$

This equation describes a standard Wiener process. The process fulfils the following conditions:

(a) $w(0)$ is finite,

(b) $E\{w(t)\} = 0$, for all $t$,

(c) $w(t)$ is normally distributed,

(d) $dw(s), dw(t)$ for all $t \neq s$ are independent stationary increments,

(e) $V\{w(t + h) - w(t)\} = h$ for $h > 0$.

**Arithmetic Brownian Motion** The standard Wiener process is inappropriate to much of financial modelling.

A generalisation is the so-called random walk with drift. In discrete time, this can be represented by the equation

$$x(t+1) = x(t) + \mu + \sigma\varepsilon(t+1). \tag{7}$$

The continuous-time analogue of this process is described by

$$dx(t) = \mu dt + \sigma dw(t). \tag{8}$$

A generalisation of the latter is the Ito process, where the drift parameter $\mu$ and the variance or volatility parameter $\sigma^2$ become time-dependent functions of the level of the process:

$$dx(t) = \mu(x,t)dt + \sigma(x,t)dw(t). \tag{9}$$

**Geometric Brownian Motion** The domain of a normally distributed random variable extends from $-\infty$ to $+\infty$. In finance, nominal interest rates must be nonnegative. Also, asset values cannot become negative.

The difficulty can be overcome by taking logarithms of the variables. The logarithmic transformation maps from $[0, \infty)$ to $(-\infty, \infty)$.

The logarithmic version of the random walk is described by

$$\ln x(t+1) = \ln x(t) + \sigma\varepsilon(t+1). \tag{10}$$

The corresponding continuous-time version can be written as

$$d\{\ln x(t)\} = \sigma dw(t). \tag{11}$$

Given that

$$\frac{d}{dt}\ln x = \frac{1}{x}\frac{dx}{dt} \qquad \text{or} \qquad d\{\ln x(t)\} = \frac{dx}{x},$$

it follows that equation (11) can also be written as

$$dx = \sigma x dw(t). \tag{12}$$

Observe that the process has an absorbing barrier at zero. That is to say, if $x = 0$ at any time, then it will remain at that value thereafter.

A more general equation incorporates a drift term:

$$dx = x\mu dt + \sigma x dw(t). \tag{13}$$

On the strength of the preceding reasoning, it might be imagined that this is synonymous with the process described by the equation $d\{\ln y(t)\} = \mu dt + \sigma dw(t)$. It is interesting to discover that this is not the case.

To find the logarithm of the process described by (13), we must use Ito's Lemma. This indicates that, for any process described by equation (9), and for any continuous differentiable function $f(x)$, there is

$$df(x,t) = \left\{ \mu(x,t)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \sigma^2(x,t)\frac{1}{2}\frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma(x,t)\frac{\partial f}{\partial x}dw. \tag{14}$$