

ITO'S LEMMA

Preliminaries Ito's lemma enables us to deduce the properties of a wide variety of continuous-time processes that are driven by a standard Wiener process $w(t)$.

We may begin an account of the lemma by summarising the properties of a Wiener process under six points. First, we may note that

$$\begin{aligned} \text{(i)} \quad & E\{dw(t)\} = 0, \\ \text{(ii)} \quad & E\{dw(t)dt\} = E\{dw(t)\}dt = 0, \\ \text{(iii)} \quad & E\{[dw(t)]^2\} = dt. \end{aligned} \tag{1}$$

The first of these is true by construction, since the increments of a Wiener process over any finite interval have expectations of zero. The second result follows from the fact that the expectation of the product of a random variable $dw(t)$ and a constant dt is the constant times the expectation of the variable. The third result is also part of the construction, since it is the variance of the mean-zero increment $dw(t)$ of the Wiener process.

Next, are some derived properties:

$$\begin{aligned} \text{(iv)} \quad & V\{[dw(t)]^2\} = E\{[dw(t)]^4\} - (E\{[dw(t)]^2\})^2 = 3(dt)^2 - (dt)^2 = 0, \\ \text{(v)} \quad & E\{[dw(t)dt]^2\} = E\{[dw(t)]^2\}(dt)^2 = 0, \\ \text{(vi)} \quad & V\{dw(t)dt\} = E\{[dw(t)dt]^2\} - (E\{dw(t)dt\})^2 = 0. \end{aligned} \tag{2}$$

Here, (iv) follows from the fact that the fourth central moment of a standard normal distribution has the value of 3, together with the fact that we may set $(dt)^2 = 0$, since this is the square of an infinitesimal quantity. The condition (v) follows likewise, and condition (vi) follows from (ii) and (v).

These results have three consequences that are used extensively in the calculus of stochastic processes:

$$\{dw(t)\}^2 = dt, \tag{3}$$

$$dw(t)dt = 0, \tag{4}$$

$$(dt)^2 = 0. \tag{5}$$

As regards (3), we know from (iii) that the expected value of the square increment $\{dw(t)\}^2$ is just dt , and we know from (iv) that its variance is zero. Therefore, the increment can be treated in the manner of a constant, and the expectation operator can be stripped away. The result of (4) follows likewise in

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view of (ii) and (vi). Finally, the result of (5) repeats what we know regarding the square of an infinitesimal quantity.

The Lemma Now consider a differentiable function of a stochastic variable x that is driven by a Wiener process described by the equation

$$dx(t) = \mu(x, t)dt + \sigma(x, t)dw(t). \quad (6)$$

From Taylor's theorem, we have

$$\begin{aligned} f(x + dx) &= f(x) + dx f'(x) + \frac{(dx)^2}{2} f''(x) + \dots \\ &\dots + \frac{(dx)^{(n-1)}}{(n-1)!} f^{(n-1)}(x) + \frac{(dx)^n}{n!} f^n(x + \lambda dx). \end{aligned} \quad (7)$$

If x were a non stochastic variable, and if we were to let $dx \rightarrow 0$, then we should find that

$$df(x) = f(x + dx) - f(x) = f'(x)dx, \quad (8)$$

since the term in $(dx)^2$ vanishes. In the case of a stochastic process, we do need to take account of the term. Assuming that the process x is represented by equation (6), there is

$$\begin{aligned} (dx)^2 &= \mu^2(dt)^2 + \sigma^2(dw)^2 + 2\mu\sigma dt dw \\ &= \sigma^2(dw)^2 = \sigma^2 dt, \end{aligned} \quad (9)$$

where (3) accounts for the final equality. Therefore, in place of (8), we should have

$$df(x) = f'(x)dx + \frac{1}{2} f''(x)(dx)^2. \quad (10)$$

The result is readily extended to the case of a function $f(x, t)$ that contains two variables x and t , of which x is a stochastic process. Then, there is

$$df(x, t) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial t} dt + \frac{1}{2} \left\{ \frac{\partial^2 f}{\partial x^2} dx^2 + \frac{2\partial^2 f}{\partial x \partial t} dx dt + \frac{\partial^2 f}{\partial t^2} dt^2 \right\}. \quad (11)$$

Here we can set $(dt)^2 = 0$ and $dx dt = 0$. Therefore, the equation becomes

$$df(x, t) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dx^2. \quad (12)$$

Now consider the process described by equation (6), which can be used to represent the increment dx . We can continue to use the expression for $(dx)^2 = \sigma^2 dt$ that is provided by equation (9). Substituting these details into (12) gives

$$df(x, t) = \left\{ \mu \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \frac{\sigma \partial f}{\partial x} dw. \quad (13)$$

This result is known as Ito's lemma.

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