THE BINOMIAL THEOREM

Pascal's Triangle and the Binomial Expansion. To prove the binomial theorem, it is necessary to invoke some fundamental principles of combinatorial calculus. We shall develop the necessary results after derving the general expression for a binomial expansion by a process of induction.

Consider the following binomial expansions:

$$\begin{aligned}(p+q)^0 &= 1,\\ (p+q)^1 &= p+q,\\ (p+q)^2 &= p^2+2pq+q^2,\\ (p+q)^3 &= p^3+3p^2q+3pq^2+q^3,\\ (p+q)^4 &= p^4+4p^3q+6p^2q^2+4pq^3+q^4,\\ (p+q)^5 &= p^5+5p^4q+10p^3q^2+10p^2q^3+5pq^4+q^5.\end{aligned}$$

The generic expansion is in the form of

$$\begin{aligned}(p+q)^n =& p^n + np^{n-1}q + \frac{n(n-1)}{2}p^{n-2}q^2 + \frac{n(n-1)(n-2)}{3!}p^{n-3}q^3 + \\ & \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}p^{n-r}q^r + \cdots \\ & + \frac{n(n-1)(n-2)}{3!}p^3q^{n-3} + \frac{n(n-1)}{2}p^2q^{n-2} + npq^{n-1} + p^n.\end{aligned}$$

In a tidier notation, this becomes

$$(p+q)^n = \sum_{x=0}^{n} \frac{n!}{(n-x)!x!}p^x q^{n-x}.$$

We can find the coefficient of the binomial expansions of successive degrees by the simple device known as Pascal's triangle:

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}$$

The numbers in each row but the first are obtained by adding two adjacent numbers in the row above. The rule is true even for the units that border the triangle if we suppose that there are some invisible zeros extending indefinitely on either side of each row.

Instead of relying merely upon observation to establish the formula for the binomial expansion, we should prefer to derive the formula by algebraic methods. Before we do so, we must reaffirm some notions concerning permutations and combinations that are essential to a proper derivation.

Permutations. Let us consider a set of three letters $\{a, b, c\}$ and let us find the number of ways in which they can be can arranged in a distinct order. We may pick any one of the three to put in the first position. Either of the two remaining letters may be placed in the second position. The third position must be filled by the unused letter. With three ways of filling the first place, two of filling the second and only one way of filling the third, there are altogether $3 \times 2 \times 1 = 6$ different arrangements. These arrangements or permutations are

$$abc, \quad cab, \quad bca, \quad cba, \quad bac \quad acb.$$

Now let us consider an unordered set of n objects denoted by

$$\{x_i; i = 1, \ldots, n\},$$

and let us ascertain how many different permutations arise in this case. The answer can be found through a litany of questions and answers which we may denote by $[(Q_i, A_i); i = 1, \ldots, n]$:

Q_1:	In how may ways can the first place be filled?	A_1: n
Q_2:	In how may ways can the second place be filled?	A_2: $n-1$ ways,
Q_3:	In how may ways can the third place be filled?	A_3: $n-2$ ways,
$\vdots$		
Q_r:	In how may ways can the rth place be filled?	A_r: $n-r+1$ ways,
$\vdots$		
Q_n:	In how may ways can the nth place be filled?	A_n: 1 way.

If the concern is to distinguish all possible orderings, then we can recognise

$$n(n-1)(n-2)\cdots 3.2.1 = n!$$

different permutations of the objects. We call this number n-factorial, which is written as $n!$, and it represents the number of permutations of n objects taken n at a time. We also denote this by

$${}^nP_n = n!.$$

We shall use the same question-and-answer approach in deriving several other important results concerning permutations. First we shall ask

Q: How many ordered sets can we recognise if r of the n objects are so alike as to be indistinguishable?

A reasoned answer is as follows:

A: Within any permutation there are r objects which are indistinguishable. We can permute the r objects amongst themselves without noticing any differences. Thus the suggestion that there might be $n!$ permutations would overestimate the number of distinguishable permutations by a factor of $r!$; and, therefore, there are only

$$\frac{n!}{r!} \quad \text{recognizably distinct permutations.}$$

Q: How many distinct permutations can we recognise if the n objects are divided into two sets of r and $n-r$ objects—eg. red billiard balls and white billiard balls—where two objects in the same set are indistinguishable?

A: By extending the previous argument, we should find that the answer is

$$\frac{n!}{(n-r)!r!}.$$

Q: How many ways can we construct a permutation of r objects selected from a set of n objects?

A: There are two ways of reaching the answer to this question. The first is to use the litany of questions and answers which enabled us to discover the total number of permutations of n objects. This time, however, we proceed no further than the question Q_r, for the reason that there are no more than r places to fill. The answer we seek is the number of way of filling the first r places. The second way is to consider the number of distinct permutation of n objects when $n-r$ of them are indistinguishable. We fail to distinguish amongst these objects because they all share the same characteristic which is that they have been omitted from the selection. Either way, we conclude that the number is

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Combinations. Combinations are selections of objects in which no attention is payed to the ordering. The essential result is found in answer to just one question:

Q: How many ways can we construct an unordered set of r objects selected from amongst n objects?

A: Consider the total number of permutations of r objects selected from amongst n. This is ${}^nP_r$. But each of the permutations distinguishes the order of the r objects it comprises. There are $r!$ different orderings or permutations of r objects; and, to find the number of different selections or combinations when no attention is paid to the ordering, we must deflate the number ${}^nP_r$ by a factor of $r!$. Thus the total number of combinations is

$$\begin{aligned} {}^nC_r &= \frac{1}{r!}{}^nP_r = \frac{n!}{(n-r)!r!} \\ &= {}^nC_{n-r}. \end{aligned}$$

Notice that we have already derived precisely this number in answer to a seemingly different question concerning the number of recognizably distinct permutations of a set of n objects of which r were in one category and $n-r$ in another. In the present case, there are also two such categories: the category of those objects which are included in the selection and the category of those which are excluded from it.

Example. Given a set of n objects, we may define a so-called power set which is the set of all sets derived by making selections or combinations of these objects. It is straightforward to deduce that there are exactly 2^n objects in the power set. This is demonstrated by setting $p = q = 1$ in the binomial expansion above to reach the conclusion that

$$2^n = \sum_x \binom{n}{x} = \sum_x \frac{n!}{x!(n-x)!}.$$

Each element of this sum is the number of ways of selcting x objects from amongst n, and the sum is for all values of $x = 0, 1, \ldots, n$.

The Binomial Theorem. Now we are in a position to derive our conclusion regarding the binomial theorem without, this time, having recourse to empirical induction. Our object is to determine the coefficient associated with the generic term $p^x q^{n-x}$ in the expansion of

$$(p+q)^n = (p+q)(p+q)\cdots(p+q),$$

where the RHS displays the n factors that are to be multiplied together. The coefficients of the various elements of the expansion are as follows:

p^n The coefficient is unity, since there is only one way of choosing n of the p's from amongst the n factors.

$p^{n-1}q$ This term is the product of q selected from one of the factors and $n-1$ p's provided by the remaining factors. There are $n = {}^nC_1$ ways of selecting the single q.

$p^{n-2}q^2$ The coefficent associated with this term is the number of ways of selecting two q's from n factors which is ${}^nC_2 = n(n-1)/2$.

$\vdots$

$p^{n-r}q^r$ The coefficent associated with this terms is the number of ways of selecting r q's from n factors which is ${}^nC_r = n!/\{(n-r)!r!\}$.

From such reasoning, it follows that

$$\begin{aligned}(p+q)^n =& p^n + {}^nC_1p^{n-1}q + {}^nC_2p^{n-2}q^2 + \\ &\cdots + {}^nC_rp^{n-r}q^r + \cdots \\ &+ {}^nC_{n-2}p^2q^{n-2} + {}^nC_{n-1}pq^{n-1} + q^n \\ =& \sum_{x=0}^{n} \frac{n!}{(n-x)!x!}p^xq^{n-x}.\end{aligned}$$

The Binomial Probability Distribution. We wish to find, for example, the number of ways of getting a total of x heads in n tosses of a coin. First we consider a single toss of the coin. Let us take the ith toss, and let us denote the outcome by $x_i = 1$ if it is heads and by $x_i = 0$ if it is tails. Heads might be described as a sucess, whence the probability of a sucess will be $P(x_i = 1) = p$. Tails might be described as a failure and the corresponding probability of this event is $P(x_i = 0) = 1 - p$. For a fair coin, we should have $p = 1 - p = \frac{1}{2}$, of course.

We are now able to define a probabilty function for the outcome of the ith trial. This is

$$f(x_i) = p^{x_i}(1-p)^{1-x_i} \quad \text{with} \quad x_i \in \{0, 1\}.$$

The experiment of tossing a coin once is called a Bernoulli trial, in common with any other experiment with a random dichotomous outcome. The corresponding probability function is called a point binomial.

If the coin is tossed n times, then the probability of any particular sequence of heads and tails, denoted by $(x_1, x_2, \ldots, x_n)$ is given by

$$\begin{aligned}P(x_1, x_2, \ldots, x_n) &= f(x_1)f(x_2)\cdots f(x_n) \\ &= p^{\sum x_i}(1-p)^{n-\sum x_i} \\ &= p^x(1-p)^{n-x},\end{aligned}$$

where we have defined $x = \sum x_i$. This result follows from the independence of thed Bernouilli trials whereby $P(x_i, x_j) = P(x_i)P(x_j)$ is the probability of the occurrence of x_i and x_j together in the sequence.

Altogether there are

$$\binom{n}{x} = \frac{n!}{(n-x)!x!} = nC_x$$

different sequences $(x_1, x_2, \ldots, x_n)$ which contain x heads; so the probability of the event of x heads in n tosses is given by

$$b(x; n, p) = \frac{n!}{(n-x)!x!} p^x q^{n-x}.$$

This is called the binomial probability mass function.