BINOMIAL OPTION PRICING MODEL

A One-Step Binomial Model The Binomial Option Pricing Model is a simple device that is used for determining the price $c_{\tau|0}$ that should be attributed initially to a call option that gives the right to purchase an asset at time τ at a strike price of $K_{\tau|0}$.

The model supposes a portfolio where the assets are N units of stock, with a spot price of S_0 per unit, and the liability is one call option. The initial value of the portfolio at time $t = 0$ will be

$$V_0 = NS_0 - c_{\tau|0}. \tag{1}$$

There will be an direct relationship between the ensuing movements of the stock price and the option price. When the value of the stock, which is the asset, rises, that of the option, which is a liability, will also rise. It is possible to devise a portfolio in which such movements are exactly offsetting.

We may begin by envisaging two eventualities affecting the spot price of the stock. Either it increases to become $S_\tau^u = S_0U$ at time $t = \tau$, where $U > 1$, or else it decreases to become $S_\tau^d = S_0D$, where $D < 1$. In the hands of its owner, the value of the call option will be $S_\tau - K_{\tau|0}$, if $S_\tau > K_{\tau|0}$, in which case the option will be exercised, or it will be worthless, if $S_\tau \leq K_{\tau|0}$. Thus

$$c_{\tau|\tau} = \max(S_\tau - K_{\tau|0}, 0). \tag{2}$$

Let the values of the option corresponding to S_τ^u and S_τ^d be c_τ^u and c_τ^d, respectively. Then, the value of the portfolio at time $t = \tau$ will be

$$V_\tau = \begin{cases} NS_0U - c_\tau^u, & \text{if } S_\tau = S_\tau^u = S_0U; \\ NS_0D - c_\tau^d, & \text{if } S_\tau = S_\tau^d = S_0D. \end{cases} \tag{3}$$

The number of units of the asset can be chosen so that the values of the portfolio are the same in these two cases. Then, the portfolio will be riskless; and, according to the argument that there should be no arbitrage opportunities, it should earn the same as the sum V_0 invested for τ periods at the riskless rate of interest. Thus

$$V_\tau = V_0 e^{r\tau} \quad \text{or, equally,} \quad V_0 = V_\tau e^{-r\tau}. \tag{4}$$

There are therefore two equations that cover the two eventualities:

$$\begin{aligned} V_\tau &= S_0UN - c_\tau^u &\iff&& c_\tau^u &= S_0UN - V_\tau, \\ V_\tau &= S_0DN - c_\tau^d &\iff&& c_\tau^d &= S_0DN - V_\tau. \end{aligned} \tag{5}$$

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Their solutions for N and V_τ are as follows:

$$N = \frac{c_\tau^u - c_\tau^d}{S_0(U - D)} \qquad \text{and} \qquad V_\tau = \frac{c_\tau^u D - c_\tau^d U}{U - D}. \tag{6}$$

From (1), it follows that the initial value of the option is

$$\begin{aligned} c_{\tau|0} &= NS_0 - V_0 \\ &= NS_0 - V_\tau e^{-r\tau}, \end{aligned} \tag{7}$$

where the second equality follows from (4). Putting the values for N and V_τ from (6) into this equation gives

$$\begin{aligned} c_{\tau|0} &= \left\{ \frac{c_\tau^u - c_\tau^d}{S_0(U - D)} \right\} S_0 - \left\{ \frac{c_\tau^u D - c_\tau^d U}{U - D} \right\} e^{-r\tau} \\ &= e^{-r\tau} \left\{ \frac{e^{r\tau}(c_\tau^u - c_\tau^d) - (c_\tau^u D - c_\tau^d U)}{U - D} \right\} \\ &= e^{-r\tau} \left\{ \left(\frac{e^{r\tau} - D}{U - D} \right) c_\tau^u + \left(\frac{U - e^{r\tau}}{U - D} \right) c_\tau^d \right\}. \end{aligned} \tag{8}$$

The latter, which is the value of the call option at time $t = 0$, can be written as

$$c_{\tau|0} = e^{-r\tau} \left\{ c_\tau^u p + c_\tau^d (1 - p) \right\}, \tag{9}$$

where

$$p = \frac{e^{r\tau} - D}{U - D} \qquad \text{and} \qquad 1 - p = \frac{U - e^{r\tau}}{U - D} \tag{10}$$

can be regarded as the probabilities of c_τ^u and c_τ^d, respectively.

A Multi-Step Binomial Model To produce a more practical model, we need to value the option under the assumption that there are many revaluations of the stock between the time $t = 0$, when the option is written, and the time $t = \tau$, when it expires. This will given rise to a wide range of possible eventual prices.

To begin the generalisation to a multistep model, we may consider a model with two steps. In the first step, the spot price of the stock moves either up to $S^u = S_0 U$ or down to $S^d = S_0 D$. In the next step, the price can move up or down from these values to give three possible prices $S^{uu} = S_0 U^2$, $S^{ud} = S_0 UD$ and $S^{dd} = S_0 D^2$. The following table displays these outcomes:

		$S^{uu} = S_0 U^2$
	$S^u = S_0 U$	
S_0		$S^{ud} = S_0 UD$
	$S^d = S_0 D$	
		$S^{dd} = S_0 D^2$

The method of pricing the call option is to start at the time of expiry and to work backwards so as to derive prices for the option at the intermediate nodes of the binomial tree. From these, one can derive the price $c_{\tau|0}$ of the option at the base of the tree.

Let the values of the option corresponding to the outcomes S^{uu}, S^{ud} and S^{dd} be denoted by c^{uu}, c^{ud} and c^{dd}, respectively, and let those corresponding to the intermediate outcomes S^u, S^d be c^u, c^d. (Here, we are omitting the temporal subscripts for ease of notation.) Then,

$$c^u = e^{-r\tau/2}\left\{c^{uu}p + c^{ud}(1-p)\right\}, \quad c^d = e^{-r\tau/2}\left\{c^{ud}p + c^{dd}(1-p)\right\}. \tag{11}$$

From these, we derive

$$\begin{aligned} c_{\tau|0} &= e^{-r\tau/2}\left\{c^u p + c^d(1-p)\right\} \\ &= e^{-r\tau}\left\{\Big[c^{uu}p + c^{ud}(1-p)\Big]p + \Big[c^{ud}p + c^{dd}(1-p)\Big](1-p)\right\} \\ &= e^{-r\tau}\left\{c^{uu}p^2 + 2c^{ud}p(1-p) + c^{dd}(1-p)^2\right\}. \end{aligned} \tag{12}$$

The generalisation to n sub periods is as follows:

$$\begin{aligned} c_{\tau|0} &= e^{-r\tau}\left\{\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} c^{uj,d(n-j)}\right\} \\ &= e^{-r\tau}E(c_{\tau|\tau}). \end{aligned} \tag{13}$$

Here, $c_{\tau|\tau} = c^{uj,d(n-j)}$ is the value of the option after the price has reached the value of $S_\tau = S_0U^jD^{n-j}$ by moving up j times and down $n-j$ times. This is given by

$$\begin{aligned} c^{uj,d(n-j)} &= \max(S_\tau - K_{\tau|0}, 0) \\ &= \max(S_0U^jD^{n-j} - K_{\tau|0}, 0). \end{aligned} \tag{14}$$

By subdividing the period $[0,\tau]$ into n sub periods, we succeed in generating a range of possible outcomes for the value of S_τ, which are $n+1$ in number. In fact, as $n \to \infty$, the trajectory price tends to that of a geometric Brownian motion.

Convergence of the Binomial to the Black–Scholes Model The Black–Scholes formula for the price of a European call option is

$$c_{\tau|0} = S_0\Phi(d_1) - K_{\tau|0}e^{-r\tau}\Phi(d_2), \tag{15}$$

where $\Phi(d)$ denotes the value of the cumulative Normal distribution function that is the probability that $z \leq d$ when $z \sim N(0,1)$ is a standard normal variate, and where

$$
\begin{aligned}
d_1 &= \frac{\ln(S_0/K_{\tau|0}) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad \text{and} \\
d_2 &= \frac{\ln(S_0/K_{\tau|0}) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}.
\end{aligned} \tag{16}
$$

We can show that, as the number n of the subintervals of the finite period $[0,\tau]$ increases indefinitely, the binomial formula for the value $c_{\tau|0}$ of the call option converges on the Black–Scholes formula.

We may begin by simplifying the binomial formula. Observe that, for some outcomes, there is $\max(S_0U^jD^{n-j} - K_{\tau|0}, 0) = 0$. Let a be the smallest number of upward movements of the underlying stock price that will ensure that the call option has a positive value, which is to say that it finishes in the money. Then, $S_0U^aD^{n-a} \simeq K_{\tau|0}$; and only the binomial paths from $j = a$ onwards need be taken into account. Therefore, equation (13) can be rewritten as

$$
\begin{aligned}
c_{\tau|0} &= e^{-r\tau}\left\{\sum_{j=a}^{n}\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}[S_0U^jD^{n-j} - K_{\tau|0}]\right\} \\
&= S_0\left\{e^{-r\tau}\sum_{j=a}^{n}\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}U^j, D^{n-j}\right\} \\
&\quad - K_{\tau|0}e^{-r\tau}\left\{\sum_{j=a}^{n}\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}\right\}.
\end{aligned} \tag{17}
$$

To demonstrate that this converges to equation (15) as $n \to \infty$, it must be shown that the terms in braces, associated with S_0 and $K_{\tau|0}e^{-r\tau}$, converge to $\Phi(d_1)$ and $\Phi(d_2)$respectively.

The term associated with $K_{\tau|0}e^{-r\tau}$ is a simple binomial sum; and, in the limit as $n \to \infty$, it converges to the partial integral of a standard normal distribution. The term associated with S_0 can be simplified so that it too becomes a binomial sum that converges to a normal integral.

Define the growth factor R by the equation $R^n = e^{-r\tau}$. Then, in reference to (10), it can be seen that, within the context of the n-period binomial model, there is

$$
p = \frac{R - D}{U - D} \quad \text{and} \quad 1 - p = \frac{U - R}{U - D}. \tag{18}
$$

Now define

$$
p_* = \frac{U}{R}p \quad \text{and} \quad 1 - p_* = \frac{D}{R}(1 - p). \tag{19}
$$

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Then the term associated with S_0 can be written as

$$\sum_{j=a}^{n} \frac{n!}{(n-j)!j!} p_*^j (1-p_*)^{n-j}. \tag{20}$$

The task is now to replace the binomial sums, as $n \to \infty$, by the corresponding partial integrals of the standard normal distribution.

First, observe that the condition $S_0 U^a D^{n-a} \simeq K_{\tau|0}$ can be solved to give

$$a = \frac{\ln(K_{\tau|0}/S_0) - n\ln D}{\ln(U/D)} + O(n^{-1/2}). \tag{21}$$

Next, let $S_\tau = S_0 U^j D^{n-j}$ be the stock price on expiry. This gives

$$\ln(S_\tau/S_0) = j\ln(U/D) + n\ln D, \tag{22}$$

from which

$$\begin{aligned} E\{\ln(S_\tau/S_0)\} &= E(j)\ln(U/D) + n\ln D \quad \text{and} \\ V\{\ln(S_\tau/S_0)\} &= V(j)\{\ln(U/D)\}^2. \end{aligned} \tag{23}$$

The latter are solved to give

$$E(j) = \frac{E\{\ln(S_\tau/S_0)\} - n\ln D}{\ln(U/D)} \quad \text{and} \quad V(j) = \frac{V\{\ln(S_\tau/S_0)\}}{\{\ln(U/D)\}^2}. \tag{24}$$

Now the value a, which marks the first term in each of the binomial sums, must to converted to a value that will serve as the limit of the corresponding integrals of the standard normal distribution.

The standardised value in question is $d = -\{a - E(j)\}/\sqrt{V(j)}$, to which a negative sign has been applied to ensure that the integral is over the interval $(-\infty, d]$, which accords with the usual tabulation of the cumulative normal distribution, instead of the interval to $(-d, \infty]$, which would correspond more directly to the binomial summation from a to n.

Substituting the expressions from (21) and (24) into the expression for d gives

$$d = \frac{-\{a - E(j)\}}{\sqrt{V(j)}} = \frac{\ln(S_0/K_{\tau|0}) + E\{\ln(S_\tau/S_0)\}}{\sqrt{V\{\ln(S_\tau/S_0)\}}} - O(n^{-1/2}). \tag{25}$$

As $n \to \infty$, the term of order $n^{-1/2}$ vanishes. Also, the trajectory of the stock price converges to a geometric Brownian motion; and, from the note on continuous stochastic processes, we can gather the result that $V\{\ln(S_\tau/S_0)\} =$

$\sigma^2\tau$. This is regardless of the size of the drift parameter μ, which will vary with the values of p and p_*. Therefore, in the limit, there is

$$d = \frac{\ln(S_0/K_{\tau|0}) + E\{\ln(S_\tau/S_0)\}}{\sigma\sqrt{\tau}}. \tag{26}$$

It remains to show that

$$E\{\ln(S_\tau/S_0)\} = \begin{cases} (r - \sigma^2/2)\tau & \text{if the probability of } U \text{ is } p, \\ (r + \sigma^2/2)\tau & \text{if the probability of } U \text{ is } p_*. \end{cases} \tag{27}$$

First, we consider $S_\tau/S_0 = \prod_{i=1}^n (S_i/S_{i-1})$, where S_n is synonymous with S_τ. Since this is a product of a sequence of independent and identically distributed random variables, there is

$$E(S_\tau/S_0) = \prod_{i=1}^n E(S_i/S_{i-1}) = \{E(S_i/S_{i-1})\}^n. \tag{28}$$

Moreover, since $S_i/S_{i-1} = U$ with probability p and $S_i/S_{i-1} = D$ with probability $1 - p$, the expected value of this ratio is

$$\begin{aligned} E(S_i/S_{i-1}) &= pU + (1-p)D \\ &= R, \end{aligned} \tag{29}$$

where the second equality follows in view of the definitions of (18). Putting this back into (28) gives

$$E(S_\tau/S_0) = R^n \quad \text{and} \quad \ln\{E(S_\tau/S_0)\} = n\ln R. \tag{30}$$

It follows from a property of the log-normal distribution that

$$\ln\{E(S_\tau/S_0)\} = E\{\ln(S_\tau/S_0)\} + \frac{1}{2}V\{\ln(S_\tau/S_0)\}. \tag{31}$$

This is rearranged to give

$$\begin{aligned} E\{\ln(S_\tau/S_0)\} &= \ln\{E(S_\tau/S_0)\} - \frac{1}{2}V\{\ln(S_\tau/S_0)\} \\ &= (r - \sigma^2/2)\tau. \end{aligned} \tag{32}$$

The final equality follows on recalling the definitions that $R^n = e^{r\tau}$ and that $V\{\ln(S_0/S_\tau)\} = \tau\sigma^2$. This provides the first equality of (27).

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Now, in pursuit of the second equality of (27), we must consider $S_0/S_\tau = \prod_{i=1}^{n}(S_{i-1}/S_i)$, which is the inverse of the ratio in question. In the manner of (28), there is

$$E(S_0/S_\tau) = \prod_{i=1}^{n} E(S_{i-1}/S_i) = \{E(S_{i-1}/S_i)\}^n. \tag{33}$$

However, the expected value of the inverse ratio is

$$\begin{aligned} E(S_{i-1}/S_i) &= p_* U^{-1} + (1-p_*)D^{-1} \\ &= R^{-1}, \end{aligned} \tag{34}$$

which follows in view of the definitions of p_* and $1-p_*$ of (19). Putting this back into (33) gives

$$E(S_0/S_\tau) = R^{-n} \quad \text{whence} \quad \ln\{E(S_0/S_\tau)\} = -n \ln R. \tag{35}$$

Now the object is to find $E\{\ln(S_\tau/S_0)\}$ from $\ln\{E(S_0/S_\tau)\}$. The property of the log-normal distribution that gave (31) now gives

$$\begin{aligned} \ln\{E(S_0/S_\tau)\} &= E\{\ln(S_0/S_\tau)\} + \frac{1}{2}V\{\ln(S_0/S_\tau)\} \\ &= -E\{\ln(S_\tau/S_0)\} + \frac{1}{2}V\{\ln(S_\tau/S_0)\}. \end{aligned} \tag{36}$$

Here, the second equality follows from the inversion of the ratio. This induces a change of sign of its logarithm, which affects the expected value on the RHS but not the variance. Rearranging the expression and using the result from (35) gives

$$\begin{aligned} E\{\ln(S_\tau/S_0)\} &= n \ln R + \frac{1}{2}V\{\ln(S_\tau/S_0)\} \\ &= (r + \sigma^2/2)\tau. \end{aligned} \tag{37}$$

This provides the second equality of (27).