Exercise 2

1. A stock price is currently $S_0 = 40$. At the end of the month, it will be either $S_1^u = 42$ or $S_1^d = 38$. The risk-free rate of continuously compounded interest is 8% *per annum.* What is the value $c_{1|0}$ of a one-month European call option with a strike price of \$39?

Answer. The value of a portfolio depends upon the price S_1 of stock, which will be either $S_1^u = 42$ or $S_1^d = 38$ at time $t = 1$, which is at the end of the month.

If $S_1 = S_1^u = 42$, then the call option will be worth $c_{1|0}^u = 42 - 39 = 3$ to whoever is holding it. If $S_1 = S_1^d = 38$, then the call option will be worth $c_{1|0}^d = 0$. The values of the portfolio in either case are

$$V_1 = \begin{cases} S_1^u N - c_{1|0}^u = 42N - 3, & \text{if } S_1 = S_1^u = 42; \\ S_1^d N - c_{1|0}^d = 38N, & \text{if } S_1 = S_1^d = 38. \end{cases}$$

For the risk-free portfolio that is used to value the stock option, these two values must be equal:

$$V_1 = 42N - 3 = 38N \quad \Longrightarrow \quad N = 0.75.$$

But, we have two equations here in two unknowns, N and V_1, so we can also find

$$V_1 = \begin{cases} (42 \times 0.75) - 3 = 28.52; \\ 38 \times 0.75 = 28.5. \end{cases}$$

When this is discounted to its present value, it must be equal to the value of the portfolio at time $t = 0$:

$$V_0 = S_0 N - c_{1|0} = V_1 e^{-r/12}.$$

Therefore, using $\exp\{-8/(12 \times 100)\} = 0.993356$, we get

$$\begin{aligned} c_{1|0} &= S_0 N - V_1 e^{-r/12} \\ &= (40 \times 0.75) - \left(28.5 \times e^{-8/\{12\times 100\}}\right) = 1.69 \end{aligned}$$

For an alternative solution, recall that the probability that the stock price will increase is

$$\begin{aligned} p = \frac{e^{r\tau} - D}{U - D} &= \frac{S_0(e^{r\tau} - D)}{S_0(U - D)} = \frac{S_0 e^{r\tau} - S_1^d}{S_1^u - S_1^d} \\ &= \frac{\left(40 \times e^{8/\{12\times 100\}}\right) - 38}{42 - 38} = 0.5669, \end{aligned}$$

where we have used $\exp\{8/(12 \times 100)\} = 1.00669$. It follows that

$$\begin{aligned} c_{\tau|0} &= e^{-r\tau}\left\{c_\tau^u p + c_\tau^d (1 - p)\right\} \\ &= e^{-8/\{12\times 100\}}\left\{(3 \times 0.5699) + (0 \times [1 - 0.5699])\right\} = 1.69. \end{aligned}$$

Observe that the condition defining p can also be written as

$$S_0 e^{r\tau} = p S_1^u + (1 - p) S_1^d.$$

2. A stock price is currently 50. At the end of six months, it will be either 45 or 55. The risk free-rate of interest continuously compounded is 10% per annum. What is the value of a six-month European put option with a strike price of 50?

Answer. The put option, written at time $t = 0$, obliges the writer to buy a unit of stock at time $t = 1$, which is in six months time, for a price of $K_{1|0} = 50$ if the holder so wishes. This will happen if the actual price is $S_1^d = 45$, in which case $p_{1|0}^d = 5$. If the price is $S_1^u = 55$, then the put option will not be exercised as it will have no value ($p_{1|0}^u = 0$). A portfolio consisting of N units of stock as the assets and one put option as the liability will have the following values at time $t = 1$:

$$V_1 = \begin{cases} S_1^u N - p_{1|0}^u = 55N, & \text{if } S_1 = S_1^u = 55; \\ S_1^d N - p_{1|0}^d = 45N - 5, & \text{if } S_1 = S_1^d = 45. \end{cases}$$

For the risk-free portfolio that is used to value the stock option, these two values must be equal:

$$V_1 = 55N = 45N - 5 \quad \Longrightarrow \quad N = -0.5.$$

Having a negative quantity of stock might mean that you have borrowed stock and short sold it, and that you must return it to the owner by buying it back on the market.

The solution for the portfolio value is

$$V_1 = \begin{cases} 55 \times -0.5 = -27.5; \\ (45 \times -0.5) = -27.5, \end{cases}$$

which is regardless of which is the actual value of the stock at time $t = 1$.

When this is discounted to its present value, it must be equal to the value of the portfolio at time $t = 0$:

$$V_0 = S_0 N - p_{1|0} = V_1 e^{-(r \times 0.5)}.$$

Therefore,

$$\begin{aligned} p_{1|0} &= S_0 N - V_1 e^{-r} \\ &= (50 \times -0.5) - \left(27.5 \times e^{-0.1 \times 0.5}\right) = 1.16. \end{aligned}$$

3. Let the annual rate of interest be r and let the price of a share at the present time of $t = 0$ be $S_0 = 100$. Suppose that, after one year, when $t = 1$, the price will be either $S_1^u = 200$ or $S_1^d = 50$. A call option to buy the share at time $t = 1$ at a price of $K_{1|0} = 150$ can be purchased at time $t = 0$ for $c_{1|0}$.

Show that, unless $c_{1|0} = \{100 - 50(1 + r)^{-1}\}/3$, there will always exist a combination of x shares and y options that will yield a profit. (Here, x is negative, if you are selling shares at time $t = 0$, and postive, if your are purchasing them, and likewise for the number of options purchased or sold.)

Answer. The value of the portfolio at time $t = 1$ depends upon the price of stock:

$$V_1 = \begin{cases} S_1^u x + c_{1|0}^u y = 200x + 50y, & \text{if } S_1 = S_1^u = 200; \\ S_1^d x = 50x, & \text{if } S_1 = S_1^d = 50. \end{cases}$$

Here, $c_{1|0}^u = S_1^u - K_{1|0} = 200 - 150 = 50$ is the worth of the option at time $t = 1$, if the price of stock is 200, since the option allows it to be purchased for 150. If the price of the stock is 50 at time $t = 1$, then option is worthless ($c_1^d = 0$).

Now choose x and y to ensure that the portfolio has the same value regardless of the outcome of the share price. Then

$$V_1 = 200x + 50y = 50x \quad \Longrightarrow \quad y = -3x,$$

which implies that

$$V_1 = 50x.$$

The gain from the portfolio is the value of these returns less the cost of the loan that services the transactions:

$$\begin{aligned}\text{Gain} &= V_1 - (S_0x + c_{1|0}y)(1+r)\\ &= 50x - (100x + c_{1|0}y)(1+r)\\ &= 50x - (100 + c_{1|0}3)x(1+r).\end{aligned}$$

Setting the gain to zero allows us to eliminate x. Then, solving for $c_{1|0}$ gives

$$c_{1|0} = \{100 - 50(1+r)^{-1}\}/3.$$