Exercise 1

1. A credit card company charges an annual interest rate of 15%, which is effective only if the interest on the outstanding debts is paid in monthly instalments. Otherwise, the interest charges are compounded with the borrowings. What would be the effective annual rate of interest if £Q are borrowed at the beginning of the year and repaid with interest at the end of the year?

Answer. Interest is charged each month on the outstanding loan at the rate of $15/12 = 1.25\%$. Therefore, after one year, the total amount that must be repaid is

$$\$Q(1 + 0.0125)^{12} = \$Q1.161,$$

which gives an effective annual rate of interest of 16%.

Using log tables, I compute $(1 + 0.0125)^{12}$ as

$$\begin{aligned}\text{Antilog}\{12 \times \log(1.0125)\} &= \text{Antilog}\{12 \times 0.0054\}\\ &= \text{Antilog}\{0.0640\} = 0.0159,\end{aligned}$$

which gives 16% approximately. (In fact, for this calculation, four-figure base-10 logarithms are insufficiently accurate.) The alternative is to enter 1.0125 in your calculator to do the multiplication twelve times over.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
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92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

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·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
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·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
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·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

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·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1 2 3	4 5 5	6 7 8
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·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1 2 3	4 5 6	7 8 9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1 2 3	4 5 6	7 8 9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1 2 3	4 5 6	7 8 9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1 2 3	4 5 6	7 8 9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1 2 3	4 5 6	7 9 10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1 2 3	4 5 7	8 9 10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1 2 3	4 6 7	8 9 10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1 2 3	5 6 7	8 9 10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1 2 4	5 6 7	8 9 11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1 2 4	5 6 7	8 10 11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1 2 4	5 6 7	9 10 11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1 3 4	5 6 8	9 10 11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1 3 4	5 6 8	9 10 12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1 3 4	5 7 8	9 10 12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1 3 4	5 7 8	9 11 12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1 3 4	5 7 8	10 11 12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1 3 4	6 7 8	10 11 13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1 3 4	6 7 9	10 11 13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1 3 4	6 7 9	10 12 13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2 3 5	6 8 9	11 12 14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2 3 5	6 8 9	11 12 14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2 3 5	6 8 9	11 13 14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2 3 5	6 8 10	11 13 15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2 3 5	7 8 10	12 13 15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2 3 5	7 8 10	12 13 15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2 3 5	7 9 10	12 14 16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2 4 5	7 9 11	12 14 16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2 4 5	7 9 11	13 14 16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2 4 6	7 9 11	13 15 17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2 4 6	8 9 11	13 15 17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2 4 6	8 10 12	14 15 17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2 4 6	8 10 12	14 16 18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2 4 6	8 10 12	14 16 18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2 4 6	8 10 12	15 17 19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2 4 6	8 11 13	15 17 19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2 4 7	9 11 13	15 17 20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2 4 7	9 11 13	16 18 20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2 5 7	9 11 14	16 18 20

2. How long will it take to double your investment if you receive an annual rate of interest of 5% and if the interest is compounded with the principal?

Answer. The equation to be solved is

$$(1+r)^n = 2$$

where $r = 0.05$ is the rate of interest. The solution, using log tables, is

$$n = \frac{\log 2}{\log(1+r)} = \frac{\log 2}{\log(1.05)} = \frac{0.3010}{0.0212} = 14.2,$$

which is confirmed by my computer. My calculator tells me that $1.05^{15} = 2.0789$

The natural number e. The number $e = \{2.7183\ldots\}$ is defined by

$$e = \lim(n \to \infty)\left(1 + \frac{1}{n}\right)^n.$$

The binomial expansion indicates that

$$\begin{aligned}(a+b)^n =& a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \\ & \frac{n(n-1)(n-2)}{3!}a^{n-3}b^2 + \cdots.\end{aligned}$$

Using this, we get

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n =& 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \\ & \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots.\end{aligned}$$

Taking limits as $n \to \infty$ of each term of the expansion gives

$$\lim(n \to \infty)\left(1+\frac{1}{n}\right)^n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$

The expansion of e^x. There is also

$$e^x = \lim(p \to \infty) \left(1 + \frac{1}{p}\right)^{px}$$
$$= \lim(n \to \infty) \left(1 + \frac{x}{n}\right)^{n} ; n = px.$$

Using the binomial expansion in the same way as before, it can be shown that

$$e^x = \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Also

$$e^{xt} = 1 + xt + \frac{x^2t^2}{2!} + \frac{x^3t^3}{3!} + \cdots.$$

3. With reference to the relevant Taylor-series expansions, demonstrate the following approximations

$$\ln(1 + x) \simeq x, \qquad e^x \simeq 1 + x,$$

which are valid when x is small. Calculate the value of n in question 2 using the relevant approximation in the denominator.

Answer. Observe that, in this case, we must work with natural logarithms as opposed to base-10 logarithms. The calculation is as follows:

$$n = \frac{\ln 2}{\ln(1 + r)} \simeq \frac{\ln 2}{r} = 13.86.$$

The Expansion of $(1-x)^{-1}$

The simplest of all power series expansions is that of the geometric progression:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots.$$

There are various way of achieving this expansion, including by the use of Taylor's theorem. Another way is by the method of detached coefficients.

We assume that $(1-x)^{-1} = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots\}$, and we rewrite this equation as

$$1 = \{1-x\}\{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots\}.$$

Then, by performing the multiplication on the RHS, and by equating the coefficients of the same powers of x on the two sides of the equation, we find that

$$\begin{aligned}
1 &= \alpha_0, & \alpha_0 &= 1,\\
0 &= \alpha_1 - \alpha_0, & \alpha_1 &= \alpha_0 = 1,\\
0 &= \alpha_2 - \alpha_1, & \alpha_2 &= \alpha_1 = 1,\\
&\vdots & &\vdots\\
0 &= \alpha_n - \alpha_{n-1}, & \alpha_n &= \alpha_{n-1} = 1.
\end{aligned}$$

Another way of achieving the expansion is by long division:

$$
\begin{array}{r|l}
 & 1 + x + x^2 + \cdots \\
\hline
1 - x & 1 \\
 & \underline{1 - x} \\
 & \quad x \\
 & \quad \underline{x - x^2} \\
 & \qquad x^2 \\
 & \qquad x^2 - x^3
\end{array}
$$

The Sum of a Geometric Progression

We can also proceed in the opposite direction. That is to say, we can evaluate $S = \{1 + x + x^2 + \cdots\}$ to show that $S = (1 - x)^{-1}$. The calculation is as follows:

$$
\begin{aligned}
S &= 1 + x + x^2 + \cdots \\
xS &= \phantom{1 + {}} x + x^2 + \cdots \\
\hline
S - xS &= 1.
\end{aligned}
$$

Then $S(1 - x) = 1$ immediately implies that $S = 1/(1 - x)$.

The Partial Sum of a Geometric Progression

There is also a formula for the partial sum of the first n terms of the series, which is $S_n = 1 + x + \cdots + x^{n-1}$. Consider the following subtraction:

$$\begin{array}{rl} S = & 1 + x + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots \\ x^n S = & \qquad\qquad\qquad\qquad x^n + x^{n+1} + \cdots \\ \hline S - x^n S = & 1 + x + \cdots + x^{n-1}. \end{array}$$

This shows that $S(1 - x^n) = S_n$, whence

$$S_n = \frac{1 - x^n}{1 - x}.$$

Example. An annuity is a sequence of regular payments, made once a year, until the end of the nth year. Usually, such an annuity may be sold to another holder; and, almost invariably, its outstanding value can be redeemed from the institution which has contracted to make the payments. There is clearly a need to determine the present value of the annuity if it is to be sold or redeemed. The principle which is applied for this purpose is that of discounting.

Imagine that a sum of $\pounds A$ is invested for one year at an annual rate of interest of $r \times 100\%$. At the end the year, the principal sum is returned together with the interest via a payment of $\pounds(1+r)A$. A straightforward conclusion is that $\pounds(1+r)A$ to be paid one year hence has the value of $\pounds A$ paid today. By the same token, $\pounds A$ to be paid one year hence has a present value of

$$V = \frac{A}{1+r} = A\delta, \qquad \text{where} \quad \delta = \frac{1}{1+r} \quad \text{is the discount rate.}$$

It follows that $\pounds A$ to be paid two years hence has a present value of $\pounds A\delta^2$. More generally, if the sum of $\pounds A$ is to be paid n years hence, then it is worth $\pounds A\delta^n$ today.

The present value of an annuity of $\pounds rA$ to be paid for the next n years is therefore

$$\begin{aligned} V_n &= Ar(\delta + \delta^2 + \cdots + \delta^n) = Ar\delta\big(1 + \delta + \cdots + \delta^{n-1}\big) \\ &= Ar\delta\frac{1-\delta^n}{1-\delta} = A(1-\delta^n), \qquad \text{since} \quad \frac{\delta}{1-\delta} = r. \end{aligned}$$

If the principal sum is to be repaid at the end of the nth year, then the present value of the contract will be

$$A(1-\delta^n) + A\delta^n = A,$$

which is precisely equal to the value of the sum that is to be invested.

4. Using the expression $x = e^y = 10^z$, find a formula that will enable you to convert between $\log_e x$ and $\log_{10} x$, commonly denoted by $\log x$ and $\ln x$, respectively—the latter being described as a natural logarithm or a Naperian logarithm.

Answer. Here, there are $y = \log_e(x)$ and $z = \log_{10}(x)$. By taking natural logarithms of the equation $x = e^y = 10^z$, we get

$$y = \log_e(x) = z \times \log_e(10) = \log_{10}(x) \times \log_e(10)$$

Also, observe that

$$10 = e^{\log_e(10)} \qquad \text{implies} \qquad \log_{10}(10) = 1 = \log_e(10) \times \log_{10}(e).$$

Therefore,

$$\log_e(x) = \frac{\log_{10}(x)}{\log_{10}(e)} \qquad \text{and} \qquad \log_{10}(x) = \frac{\log_e(x)}{\log_e(10)}.$$

ARTIST: Sam Cooke, TITLE: Wonderful World (Don't Know Much)

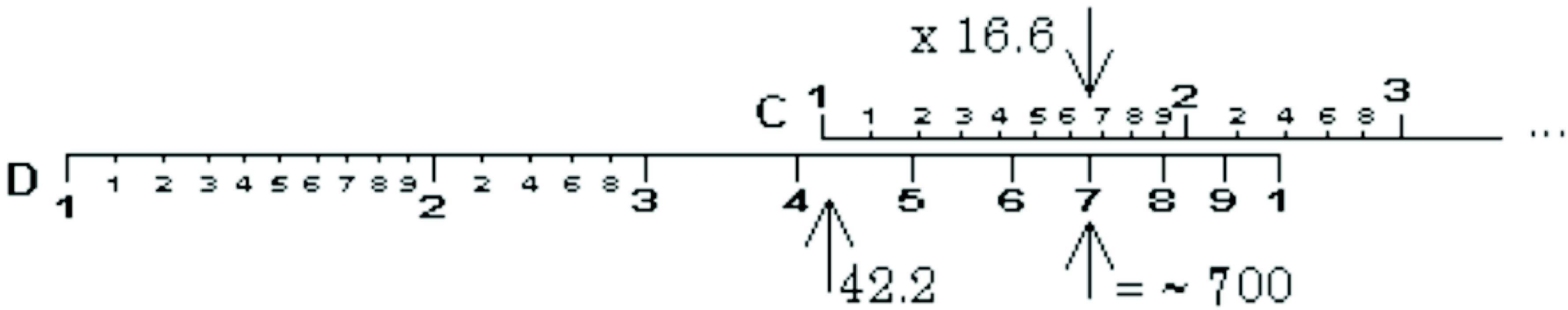
Don't know much about history
Don't know much biology
Don't know much about a science book
Don't know much about the French I took
But I do know that I love you
And I know that if you love me too
What a wonderful world this would be

Don't know much about geography
Don't know much trigonometry
Don't know much about algebra
Don't know what a slide rule is for
But I know that one and one is two
And if this one could be with you
What a wonderful world this would be

Now, I don't claim to be an "A" student
But I'm trying to be
For maybe by being an "A" student baby
I can win your love for me

Basic Slide Rule Instructions

To multiply two numbers on a typical slide rule, the user marks one of the factors on the upper C scale. (In this case, it is 16.6). The second factor is marked on the lower D scale (In this case, it is 42.2). Then, the upper scale is slid forwards until its starting value of unity is aligned with the point on the lower scale marking the second factor. The point which is reached on the lower scale by the mark on the upper scale corresponds to the product of the two factors (700). By these means, the user effectively adds the logs (lengths) of the two numbers and finds the antilog of the sum.



A calculation on a slide rule showing that 42.2 x 16.6 = 700

5. Let $S_n = 1 + r + r^2 + \cdots + r^{n-1}$, which is a partial sum of n terms of an geometric progression. Show that $S_n + r^n = 1 + rS_n$ and thence derive an expression of S_n in terms of r.

Answer. By rearranging the given expression, we get $S_n(1 - r) = 1 - r^n$, whence

$$S_n = \frac{1 - r^n}{1 - r}.$$

6. Annual payments of £M must be made over a period of n years to redeem a mortgage. The present value of this stream of payments is

$$M(\delta + \delta^2 + \cdots + \delta^n).$$

where $\delta = (1+r)^{-1}$ is the rate of discount and r is the rate of interest. The present value of the stream of payments must equate to the value L of the loan.

If the loan was for £150,000 and the rate of interest was fixed at 5% for the entire period, what should be the size of the annual payment in order to redeem the loan in 20 year's time?

Answer. The present value of the payments is

$$M\delta(1 + \delta + \delta^2 + \cdots + \delta^{n-1}) = M\frac{\delta(1-\delta^n)}{1-\delta} = L,$$

whence

$$M = L\frac{1-\delta}{\delta(1-\delta^n)} = L\frac{\gamma^n(\gamma-1)}{\gamma^n - 1} \quad \text{where} \quad \gamma = \delta^{-1} = 1+r.$$

with $L = 150,000$, $n = 20$ and $1 + r = 1.05$, we find that $M = 12,036$.