LECTURE 5 : IDENTIFICATION OF ARMA MODELS

Autocorrelation function (ACF). Given a sample $y_0, y_1, \ldots, y_{T-1}$ of T observations, we define the sample autocorrelation function to be the sequence of values

$$r_\tau = c_\tau / c_0, \quad \tau = 0, 1, \ldots, T-1, \tag{1}$$

wherein

$$c_\tau = \frac{1}{T} \sum_{t=\tau}^{T-1} (y_t - \bar{y})(y_{t-\tau} - \bar{y}) \tag{2}$$

is the empirical autocovariance at lag τ and c_0 is the sample variance.

As a guide to determining whether the parent autocorrelations are in fact zero after lag q, we may use the result that, for a sample of size T, the standard deviation of r_τ is approximately

$$\frac{1}{\sqrt{T}} \left\{ 1 + 2(r_1^2 + r_1^2 + \cdots + r_q^2) \right\}^{1/2} \quad \text{for} \quad \tau > q. \tag{4}$$

A simpler measure of the significance of the autocorrelations is provided by the limits of $\pm 1.96/\sqrt{T}$ which are the approximate 95% confidence bounds for the autocorrelations of a white-noise sequence.

Partial autocorrelation function (PACF). The sample partial autocorrelation p_τ at lag τ is simply the correlation between the two sets of residuals obtained from regressing the elements y_t and $y_{t-\tau}$ on the set of intervening values $y_1, y_2, \ldots, y_{t-\tau+1}$. The partial autocorrelation measures the dependence between y_t and $y_{t-\tau}$ after the effect of the intervening values has been removed.

The significance of the values of the partial autocorrelations is judged by the fact that, for a pth order process, their standard deviations for all lags greater that p are approximated by $1/\sqrt{T}$. Thus the bounds of $\pm 1.96/\sqrt{T}$ are also plotted on the graph of the partial autocorrelation function.

For the ARMA model $y(t) = \{\mu(L)/\alpha(L)\}\varepsilon(t)$, the autocovariance generating function is given by

$$\gamma(z) = \sigma_\varepsilon^2 \frac{\mu(z)\mu(z^{-1})}{\alpha(z)\alpha(z^{-1})} = \sigma_\varepsilon^2 \left| \frac{\mu(z)}{\alpha(z)} \right|^2 . \tag{9}$$

Consider therefore the rational function

$$\frac{\mu(z^{-1})}{\alpha(z^{-1})} = z^{p-q} \frac{\mu_0 z^q + \mu_1 z^{q-1} + \cdots + \mu_q}{\alpha_0 z^p + \alpha_1 z^{p-1} + \cdots + \alpha_p} \tag{10}$$

which we shall describe as the transfer function of the ARMA model. The numerator and denominator may be factorised to give

$$\frac{\mu(z^{-1})}{\alpha(z^{-1})} = z^{p-q} \frac{\mu_0}{\alpha_0} \frac{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_q)}{(z-\kappa_1)(z-\kappa_2)\cdots(z-\kappa_p)}, \tag{11}$$

The modulus of the function evaluated on the unit circle is

$$\begin{aligned} \left| \frac{\mu(e^{-i\omega})}{\alpha(e^{-i\omega})} \right| &= \frac{\mu_0}{\alpha_0} \frac{|e^{i\omega} - \lambda_1|\ |e^{i\omega} - \lambda_2| \cdots |e^{i\omega} - \lambda_q|}{|e^{i\omega} - \kappa_1|\ |e^{i\omega} - \kappa_2| \cdots |e^{i\omega} - \kappa_p|} \\ &= \frac{\mu_0}{\alpha_0} \frac{\prod \rho_j(\omega)}{\prod \lambda_j(\omega)} \end{aligned} . \tag{14}$$