# LECTURE 3 : FOURIER METHODS

## The Fourier Decomposition of a Time Series

The Fourier decomposition explains a time series entirely as a weighted sum of sinusoidal functions. Thus the generic element of the sample $y_0, \ldots, y_{T-1}$ is expressed

$$y_t = \sum_{j=0}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\}. \tag{2.26}$$

Assuming that $T = 2n$ is even, this sum comprises $T$ functions at frequencies

$$\omega_j = \frac{2\pi j}{T}, \quad j = 0, \ldots, n = \frac{T}{2} \tag{2.27}$$

which are at equally spaced points in the interval $[0, \pi]$.

Notice that

$$\begin{aligned} \sin(\omega_0 t) &= \sin(0) = 0, \\ \cos(\omega_0 t) &= \cos(0) = 1, \\ \sin(\omega_n t) &= \sin(\pi t) = 0, \\ \cos(\omega_n t) &= \cos(\pi t) = (-1)^t; \end{aligned} \tag{4.15}$$

so there are indeed $T$ nonzero tigonometrical functions and not $T + 2$ as, at first, there seem to be.

The highest velocity $\omega_n = \pi$ corresponds to the so-called Nyquist frequency. If $\omega \in (\pi, 2\pi)$ and if $\omega^* = 2\pi - \omega$, then

$$\begin{aligned} \cos(\omega t) &= \cos\{(2\pi - \omega^*)t\} \\ &= \cos(2\pi)\cos(\omega^* t) + \sin(2\pi)\sin(\omega^* t) \\ &= \cos(\omega^* t); \end{aligned} \tag{2.28}$$

which indicates that $\omega$ and $\omega^*$ are observationally indistinguishable. Thus, $\omega^* \in [0, \pi]$ is described as the alias of $\omega > \pi$.

**Calculation of the Fourier Coefficients**

Let $c_j = [c_{0j}, \ldots, c_{T-1,j}]'$ and $s_j = [s_{0,j}, \ldots, s_{T-1,j}]'$ represent vectors of $T$ values of the generic functions $\cos(\omega_j t)$ and $\sin(\omega_j t)$ respectively. Then there are the following orthogonality conditions:

$$\begin{aligned} c_i'c_j &= 0 \quad \text{if} \quad i \neq j, \\ s_i's_j &= 0 \quad \text{if} \quad i \neq j, \\ c_i's_j &= 0 \quad \text{for all} \quad i, j. \end{aligned} \tag{2.29}$$

In addition, there are the following sums of squares:

$$\begin{aligned} c_0'c_0 = c_n'c_n &= T, \\ s_0's_0 = s_n's_n &= 0, \\ c_j'c_j = s_j's_j &= \frac{T}{2}. \end{aligned} \tag{2.30}$$

The "regression" formulae for the Fourier coefficients are therefore

$$\alpha_0 = (i'i)^{-1}i'y = \frac{1}{T}\sum_t y_t = \bar{y}, \tag{2.31}$$

$$\alpha_j = (c_j'c_j)^{-1}c_j'y = \frac{2}{T}\sum_t y_t \cos\omega_i t, \tag{2.32}$$

$$\beta_j = (s_j's_j)^{-1}s_j'y = \frac{2}{T}\sum_t y_t \sin\omega_j t. \tag{2.33}$$

**The Fourier Decomposition and the Analysis of Variance**

The sum of squares of the elements of the vector $y$ is decomposed as

$$y'y = \alpha_0^2 i'i + \sum_j \alpha_j^2 c_j' c_j + \sum_j \beta_j^2 s_j' s_j. \tag{2.34}$$

Since $\alpha_0^2 i'i = \bar{y}^2 i'i = \bar{y}'\bar{y}$ where $\bar{y}' = [\bar{y}, \ldots, \bar{y}]$, it follows that $y'y - \alpha_0^2 i'i = y'y - \bar{y}'\bar{y} = (y - \bar{y})'(y - \bar{y})$. Therefore we can rewrite the equation as

$$(y - \bar{y})'(y - \bar{y}) = \frac{T}{2} \sum_j \left\{ \alpha_j^2 + \beta_j^2 \right\} = \frac{T}{2} \sum_j \rho_j^2, \tag{2.35}$$

and it follows that we can express the variance of the sample as

$$\begin{aligned} \frac{1}{T} \sum_{t=0}^{T-1} (y_t - \bar{y})^2 &= \frac{1}{2} \sum_{j=1}^{n} (\alpha_j^2 + \beta_j^2) \\ &= \frac{2}{T^2} \sum_j \left\{ \left( \sum_t y_t \cos \omega_j t \right)^2 + \left( \sum_t y_t \sin \omega_j t \right)^2 \right\}. \end{aligned} \tag{2.36}$$

The proportion of the variance which is attributable to the component at frequency $\omega_j$ is $(\alpha_j^2 + \beta_j^2)/2 = \rho_j^2/2$, where $\rho_j$ is the amplitude of the component. The graph of the function $I(\omega_j) = (T/2)(\alpha_j^2 + \beta_j^2)$ is know as the periodogram.

**The Periodogram and the Empirical Autocovariances**

The empirical autocovariance of lag $\tau$ is defined by the formula

$$c_\tau = \frac{1}{T} \sum_{t-\tau}^{T-1} (y_t - \bar{y})(y_{t-\tau} - \bar{y}). \tag{2.37}$$

The periodogram may be written as

$$\begin{aligned} I(\omega_j) &= \frac{2}{T} \left[ \left\{ \sum_{t=0}^{T-1} \cos(\omega_j t)(y_t - \bar{y}) \right\}^2 + \left\{ \sum_{t=0}^{T-1} \sin(\omega_j t)(y_t - \bar{y}) \right\}^2 \right] \\ &= \frac{2}{T} \left\{ \sum_t \sum_s \cos(\omega_j [t - s])(y_t - \bar{y})(y_s - \bar{y}) \right\}. \end{aligned} \tag{2.40}$$

On defining $\tau = t - s$ and using the definition of $c_\tau$ from (37), we can reduce the latter expression to

$$I(\omega_j) = 2 \sum_{\tau=1-T}^{T-1} \cos(\omega_j \tau) c_\tau, \tag{2.41}$$

which is a Fourier transform of the sequence of empirical autocovariances.

**Stationarity**

If $y(t) = \{y_t; t = 0, \pm 1, \pm 2, \ldots\}$ is weakly stationary process, then

$$\begin{aligned} E(y_t) &= \mu, \\ C(y_{t+i}, y_{t+j}) &= C(y_i, y_j) \\ &= \gamma_{|i-j|}. \end{aligned} \tag{4.2}$$

The autocovariance matrix of a stationary process corresponding to the $n$ elements $y_0, y_1, \ldots, y_{n-1}$ is given by

$$\Gamma = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \cdots & \gamma_0 \end{bmatrix}. \tag{4.5}$$

**The Filtering of White Noise**

A white-noise process is a sequence $\varepsilon(t)$ of uncorrelated random variables with mean zero and common variance $\sigma_\varepsilon^2$. Thus

$$\begin{aligned} E(\varepsilon_t) &= 0, \quad \text{for all } t \\ E(\varepsilon_{t+i}\varepsilon_{t+j}) &= \begin{cases} \sigma_\varepsilon^2, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases} \end{aligned} \tag{4.6}$$

By a process of linear filtering, a variety of time series may be constructed whose elements display complex interdependencies. If $\mu(L) = \mu_0 + \mu_1 L + \cdots + \mu_q L^q$, Then

$$\begin{aligned} y(t) &= \mu(L)\varepsilon(t) \\ &= \mu_0\varepsilon(t) + \mu_1\varepsilon(t-1) + \mu_2\varepsilon(t-2) + \cdots + \mu_q\varepsilon(t-q) \\ &= \sum_{i=0}^{q} \mu_i\varepsilon(t-i). \end{aligned} \tag{4.7}$$

An operator $\mu(L) = \{\mu_0 + \mu_1 L + \mu_2 L^2 + \cdots\}$ with an indefinite number of terms must obey the condition that

$$\sum_i |\mu_i| < \infty. \tag{4.8}$$

The autocovariances of the filtered sequence $y(t) = \mu(L)\varepsilon(t)$ may be determined by evaluating the expression

(4.9)
$$\begin{aligned}\gamma_\tau &= E(y_t y_{t-\tau}) \\ &= E\Big(\sum_i \mu_i\varepsilon_{t-i}\sum_j \mu_j\varepsilon_{t-\tau-j}\Big) \\ &= \sum_i\sum_j \mu_i\mu_j E(\varepsilon_{t-i}\varepsilon_{t-\tau-j}).\end{aligned}$$

From equation (6), it follows that

(4.10)
$$\gamma_\tau = \sigma_\varepsilon^2\sum_j \mu_j\mu_{j+\tau};$$

and so the variance of the filtered sequence is

(4.11)
$$\gamma_0 = \sigma_\varepsilon^2\sum_j \mu_j^2.$$

**The $z$-transform**

The $z$-transform of the infinite sequence $y(t) = \{y_t; t = 0, \pm1, \pm2, \ldots\}$ is defined by

(4.12)
$$y(z) = \sum_{\tau=-\infty}^{\infty} y_t z^t.$$

If $y(t) = \mu(L)\varepsilon(t)$ is a moving-average process, then the $z$-transform is given by $y(z) = \mu(z)\varepsilon(z)$ where $\mu(z) = \{\mu_0 + \mu_1 z + \mu_2 z^2 + \cdots\}$ has the same form as the operator $\mu(L)$, and where $\varepsilon(z)$ is the $z$-transform of the white-noise sequence.

The $z$-transform of a sequence of autocovariances is called the autocovariance generating function. For the moving-average process, this is given by

(4.13)
$$\begin{aligned}\gamma(z) &= \sigma_\varepsilon^2\mu(z)\mu(z^{-1}) \\ &= \sigma_\varepsilon^2\sum_i \mu_i z^i\sum_j \mu_j z^{-j} \\ &= \sigma_\varepsilon^2\sum_i\sum_j \mu_i\mu_j z^{i-j} \\ &= \sum_\tau\Big\{\sigma_\varepsilon^2\sum_j \mu_j\mu_{j+\tau}\Big\}z^\tau \quad ; \quad \tau = i - j \\ &= \sum_{\tau=-\infty}^{\infty}\gamma_\tau z^\tau.\end{aligned}$$

The final equality is by virtue of equation (10).

**The Spectral Representation of a Stationary Process**

By writing $\alpha_j = dA(\omega_j)$, $\beta_j = dB(\omega_j)$ where $A(\omega)$, $B(\omega)$ are step functions with discontinuities at the points $\{\omega_j; j = 0, \ldots, n\}$, the expression for the Fourier representation of a finite sequence can be written as

$$y_t = \sum_j \Big\{ \cos(\omega_j t) dA(\omega_j) + \sin(\omega_j t) dB(\omega_j) \Big\}. \tag{4.19}$$

In the limit, as $n \rightarrow \infty$, the summation is replaced by an integral to give the expression

$$y(t) = \int_0^\pi \Big\{ \cos(\omega t) dA(\omega) + \sin(\omega t) dB(\omega) \Big\}. \tag{4.20}$$

In order to derive a statistical theory for the process that generates $y(t)$, one must make some assumptions concerning the functions $A(\omega)$ and $B(\omega)$.

First, it is assumed that $A(\omega)$ and $B(\omega)$ represent a pair of stochastic processes of zero mean which are indexed on the continuous parameter $\omega$. Thus

$$E\big\{dA(\omega)\big\} = E\big\{dB(\omega)\big\} = 0. \tag{4.21}$$

Next, it is assumed that $A(\omega)$ and $B(\omega)$ mutually uncorrelated and that non-overlapping increments within each process are uncorrelated. Thus

$$\begin{aligned} E\big\{dA(\omega)dB(\lambda)\big\} &= 0 \quad \text{for all} \quad \omega, \lambda, \\ E\big\{dA(\omega)dA(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda, \\ E\big\{dB(\omega)dB(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda. \end{aligned} \tag{4.22}$$

The finally it is assumed that the variance of the increments is given by a

$$\begin{aligned} V\big\{dA(\omega)\big\} = V\big\{dB(\omega)\big\} &= 2dF(\omega) \\ &= 2f(\omega)d\omega. \end{aligned} \tag{4.23}$$

We can see that, unlike $A(\omega)$ and $B(\omega)$, $F(\omega)$ is a continuous differentiable function. The function $F(\omega)$ and its derivative $f(\omega)$ are the spectral distribution function and the spectral density function, respectively.

**The Complex Exponential Form of the Spectral Representation**

In order to express equation (20) in terms of complex exponentials, we may define a pair of conjugate complex stochastic processes:

$$\begin{aligned} dZ(\omega) &= \frac{1}{2}\big\{dA(\omega) - idB(\omega)\big\}, \\ dZ^*(\omega) &= \frac{1}{2}\big\{dA(\omega) + idB(\omega)\big\}. \end{aligned} \tag{4.24}$$

Also, we may extend the domain of the functions $A(\omega)$, $B(\omega)$ from $[0, \pi]$ to $[-\pi, \pi]$ by regarding $A(\omega)$ as an even function such that $A(-\omega) = A(\omega)$ and by regarding $B(\omega)$ as an odd function such that $B(-\omega) = -B(\omega)$. Then we have

$$dZ^*(\omega) = dZ(-\omega). \tag{4.25}$$

From conditions under (22), it follows that

$$\begin{aligned} &E\big\{dZ(\omega)dZ^*(\lambda)\big\} = 0 \quad \text{if} \quad \omega \neq \lambda, \\ &E\{dZ(\omega)dZ^*(\omega)\} = f(\omega)d\omega. \end{aligned} \tag{4.26}$$

These results may be used to reexpress equation (20) as

$$\begin{aligned} y(t) &= \int_0^\pi \left\{ \frac{(e^{i\omega t} + e^{-i\omega t})}{2} dA(\omega) - i\frac{(e^{i\omega t} - e^{-i\omega t})}{2} dB(\omega) \right\} \\ &= \int_0^\pi \left\{ e^{i\omega t} \frac{\{dA(\omega) - idB(\omega)\}}{2} + e^{-i\omega t} \frac{\{dA(\omega) + idB(\omega)\}}{2} \right\} \\ &= \int_0^\pi \left\{ e^{i\omega t} dZ(\omega) + e^{-i\omega t} dZ^*(\omega) \right\}. \end{aligned} \tag{4.27}$$

When the integral is extended over the range $[-\pi, \pi]$, this becomes

$$y(t) = \int_{-\pi}^{\pi} e^{i\omega t} dZ_y(\omega). \tag{4.28}$$

This is commonly described as the spectral representation of the process $y(t)$.

**The Autocovariances and the Spectral Density Function**

The sequence of the autocovariances of the process $y(t)$ may be expressed in terms of the spectrum of the process. From equation (28), it follows that the autocovariance $y_t$ at lag $\tau = t - k$ is given by

$$\begin{aligned}\gamma_\tau = C(y_t, y_k) &= E\Big\{\int_\omega e^{i\omega t} dZ_y(\omega)\int_\lambda e^{-i\lambda k} dZ_y(-\lambda)\Big\}\\ &= \int_\omega\int_\lambda e^{i\omega t}e^{-i\lambda k}E\{dZ_y(\omega)dZ_y^*(\lambda)\}\\ &= \int_\omega e^{i\omega\tau}E\{dZ_y(\omega)dZ_y^*(\omega)\}\\ &= \int_\omega e^{i\omega\tau}f_y(\omega)d\omega.\end{aligned} \tag{4.29}$$

Here the final equalities are derived by using the results (25) and (26). This equation indicates that the Fourier transform of the spectrum is the autocovariance function.

The inverse mapping from the autocovariances to the spectrum is given by

$$\begin{aligned}f(\omega) &= \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\gamma_\tau e^{-i\omega\tau}\\ &= \frac{1}{2\pi}\Big\{\gamma_0 + 2\sum_{\tau=1}^{\infty}\gamma_\tau\cos(\omega\tau)\Big\}.\end{aligned} \tag{4.30}$$

This function is directly comparable to the periodogram of a data sequence which is defined under (2.41).

To demonstrate the relationship which exists between equations (29) and (30), we may substitute the latter into the former to give

$$\begin{aligned}\gamma_\tau &= \int_{-\pi}^{\pi} e^{i\omega\tau}\Big\{\frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\gamma_\tau e^{-i\omega\tau}\Big\}d\omega\\ &= \frac{1}{2\pi}\sum_{\kappa=-\infty}^{\infty}\gamma_\kappa\int_{-\pi}^{\pi}e^{i\omega(\tau-\kappa)}d\omega.\end{aligned} \tag{4.31}$$

From the fact that

$$\int_{-\pi}^{\pi} e^{i\omega(\tau-\kappa)}d\omega = \begin{cases} 2\pi, & \text{if } \kappa = \tau;\\ 0, & \text{if } \kappa \neq \tau,\end{cases} \tag{4.32}$$

it can be seen that the RHS of the equation reduces to $\gamma_\tau$. This serves to show that equations (29) and (30) do indeed represent a Fourier transform and its inverse.