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The weak Borda effect and plurality–majority disagreement

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The weak Borda effect indicates a failure on the part of the plurality voting procedure to satisfy a criterion of reasonableness according to which none of the losers should be preferred to the winner by a majority of the voters. Contrary to the interpretation of a recent critic, it is not claimed that the weak Borda effect necessarily indicates plurality–majority disagreement. To minimize future misunderstanding, formal definitions of the plurality procedure, the weak Borda effect, and plurality–majority disagreement are given, and a simple method is derived for evaluating the exact probability of plurality–majority disagreement in small decision-making committees.

1. Introduction

Gillett (1984) recently criticized an article of mine (Colman, 1980) for proposing the likelihood of the weak Borda effect as an index of the propensity of the plurality voting procedure to produce an outcome that conflicts with the wishes of a majority of the voters. According to Gillett, the index is 'flawed and misleading' (p. 128). This criticism is evidently based on a misunderstanding; but since it is possible that other readers have similarly misunderstood my intentions, I shall endeavour in the following paragraphs to formalize and to clarify the issues at stake in an effort to minimize ambiguity. The article that Gillett criticized *does* contain an unfortunate blemish—an error that Gillett appears to have overlooked—and I welcome the opportunity to correct it in passing. Finally, I shall briefly extend my previous results by proposing a simple method of evaluating the probability of plurality–majority disagreement in small decision-making committees.

2. Preliminary formalization

It is assumed that there is a finite set of voters or individuals $I = \{1, 2, \ldots, i, \ldots, m\}$, $m \geqslant 3$, who seek to choose, through some voting procedure, an element or elements from a non-empty, finite set X of available options. Each voter $i \in I$ is assumed (see Fishburn, 1973, pp. 71–75) to have a linear preference ordering P_i on X; this means that

(1) for all $x \in X$, for all $i \in I$, not xP_ix (P is irreflexive);
(2) for all $x, y \in X$, $x \neq y$, for all $i \in I$, xP_iy or yP_ix (P is weakly connected); and
(3) for all $x, y, z \in X$, for all $i \in I$, xP_iy and yP_iz implies xP_iz (P is transitive).

The number of voters who prefer x to y is $n(x,y)$, that is, $n(x,y) = |\{i \in I\colon xP_iy\}|$. If $n(x,y) > n(y,x)$, then xMy (x is preferred to y by a majority of the voters) and vice versa.

The indexed set of voters' preference orderings on X is

$$D = \{P_1, \ldots, P_m\}.$$

A voting procedure, or more generally a social choice function, is a function whose domain is the set of ordered pairs (X, D) and which assigns to each such pair a non-empty subset $F(X, D)$ of X. In the context of voting, $F(X, D)$ may be interpreted as the set (possibly a singleton unless the vote is tied) of winners.

3. Plurality procedure and Condorcet criterion

The plurality (first-past-the-post) voting procedure, which I shall call F_p, selects from among the set of available options the element(s) that is (are) ranked top in the preference orderings of the greatest number of voters. (There is an assumption here that voting is sincere; this assumption is overlooked by Gillett, 1984, and by most social choice theorists, although in the real world insincere or strategic voting is not uncommon—see Colman, 1982*b*, chapter 11.) The plurality procedure is usually defined (e.g. Fishburn, 1973, p. 162; Richelson, 1975, p. 332) as follows:

$$F_p(X, D) = \{x\colon x \in X \text{ and } t(x, X, D) \geqslant t(y, X, D) \text{ for all } y \in X\},$$

where $t(x, X, D)$ is the number of voters who rank x at the top of their preference orderings: $t(x, X, D) = |\{i\colon xP_i y \text{ for all } y \in X - \{x\}\}|$.

As Gillett (1984) has reminded us, Condorcet (1785) proposed that if one of the available options is preferred to each of the others by a majority of the voters, then any reasonable voting procedure ought to choose this option as the unique winner. Formally, the Condorcet criterion says this:

$$x \in X \text{ and } xMy \text{ for all } y \in X - \{x\} \text{ implies } F(X, D) = \{x\}.$$

This criterion seems intuitively reasonable, but it is easy to show that the plurality procedure violates it. Consider a committee of $m = 7$ voters choosing from a set of three available options $X = \{a, b, c\}$. Assume the following set of voters' preference orderings:

$$D = \{(a>b>c), (a>b>c), (a>b>c), (b>a>c), (b>a>c), (c>b>a), (c>b>a)\},$$

where $x>y$ means $xP_i y$. Then bMa and bMc, but $F_p(X, D) = \{a\}$. That is, a majority of the voters prefer b to a, and a majority prefer b to c, but the unique plurality winner is a, not b. This is an example of plurality–majority disagreement, which Condorcet rightly deplored.

4. Borda effect and weak dominance

The example given above is also an instance of the weak Borda effect. I defined this effect informally in my earlier paper as a situation in which 'a majority of a committee or an electorate . . . prefer one of the defeated alternatives to the plurality winner' (Colman, 1980, p. 50). In the above example, the plurality winner is a, but b is preferred to a by a majority (four out of seven) of the voters. In the notation of the present paper, the weak Borda effect may be defined formally as follows:

$$F_p(X, D) = \{x\} \text{ and } yMx \text{ for some } y \in X.$$

I pointed out (pp. 52–53) that the weak Borda effect necessarily occurs when there is a unique plurality winner in a cyclic majority, that is, when $F_p(X, D) = \{a\}$ and $Y = \{a, b, \ldots, k\} \subseteq X$ and $aMb, bMc, \ldots, kMa$. But the weak Borda effect can occur without a cyclic majority, as the example in Section 3 above shows.

The aims of my 1980 paper were to draw attention to the existence of this relatively neglected voting paradox and to derive some results concerning its likelihood of occurrence in small decision-making committees under various assumptions about the voters' preference patterns. According to Gillett (1984), 'Colman (1980) states his intention of investigating the discrepancy which can arise between the respective outcomes of the plurality and majority voting procedures' (p. 128). Gillett pointed out that when there is a plurality winner in a cyclic majority, although the weak Borda effect occurs, there is no majority winner and therefore no plurality–majority disagreement. Hence, he argues, 'the likelihood of the weak Borda effect is a very poor indicator of the likelihood of plurality/majority disagreement' (p. 128).

Gillett's (1984) critique is misdirected, because I did not at any point state an intention of investigating the discrepancy to which he refers, nor did I propose the likelihood of the weak Borda effect as an indicator of the likelihood of plurality–majority disagreement, nor have I stated any such intention or made any such proposal in my other writings on the Borda effect (e.g. Colman, 1982*a*, chapter 4, 1982*b*, chapter 10; Colman & Poutney, 1975, 1978). No; the likelihood of the weak Borda effect is an indicator—an exact measure, in fact—of the propensity of the plurality voting procedure to violate the following more subtle criterion, which some social choice theorists have proposed for judging the fairness and reasonableness of voting procedures:

If $x \in F(X, D)$ and $y \in X - F(X, D)$, then not yMx.

Schwartz (1972) calls this criterion *weak dominance*; Richelson (1978*a*, *b*, 1981) discusses it among various other *extended Condorcet* criteria; both consider it a desirable criterion for reasonable voting systems to satisfy. The plurality voting procedure violates this criterion precisely when the weak Borda effect occurs.

If a cyclic majority involving all of the available options exists, for example if $X = \{a, b, c\}$ and aMb, bMc, cMa, then the only acceptable outcome according to weak dominance, and the only one that satisfies Schwartz (1972), for example, is a tied vote between all the options, that is, $F(X, D) = X$. Gillett (1984) suggests that one of the virtues of the plurality procedure is that it can produce a unique winner when there is no majority winner. Whether this is a virtue is unclear, since the plurality procedure achieves its supposed advantage only by *concealing* the fact that there is no majority winner! One could use a similar argument to show the 'distinct advantage' of the following procedure over the plurality procedure:

$F(X, D) = \{x\colon x \text{ is the Prime Minister's favourite option}\}$.

This procedure always selects a unique winner, even when the plurality procedure fails to do so (that is, when there is a plurality tie), but whether this makes it preferable to plurality voting is debatable.

5. Probability of plurality–majority disagreement

In my 1980 article I examined in detail the case $X = \{a, b, c\}$, $m = 7$. For a *completely uniform culture*, where all individual preference orderings are equiprobable, I gave an expression for the probability of the weak Borda effect in which a is the plurality winner and b is preferred to a by a majority of the voters. The expression (at the top of p. 52) contains a misprint which I ought to have spotted at the proofreading stage. The correct version, using the notation of the present paper, is

$$P(F_p(X, D) = \{a\}, bMa) = \frac{7!}{3!2!2!}\left(\frac{1}{3}\right)^5\left(\frac{1}{6}\right)^2 = \frac{35}{1458}.$$

Remaining with the case $X = \{a, b, c\}$, $m = 7$, and the completely uniform culture, it is worth examining the consequences of the fact that plurality–majority disagreement occurs whenever there is a weak Borda effect in the absence of a cyclic majority. What is the probability of a cyclic majority in conjunction with a weak

Borda effect in this culture? First, note that a Borda effect is implied by a cyclic majority if and only if there is a unique plurality winner, so the quantity that we seek is the probability of a cyclic majority in conjunction with a unique plurality winner. This can occur only if there is a 3–2–2 plurality split, that is, if one of the options tops the preference ordering of three of the voters, another tops and preference ordering of two voters, and the third tops the preference ordering of the remaining two voters. Such a plurality split can occur in three ways, with *a* or *b* or *c* as the plurality winner.

Let us assume first that *a* is the unique plurality winner. Then there are just two sets of preference orderings that yield cyclic majorities:

$$D = \{(a>b>c), (a>b>c), (a>b>c), (b>c>a), (b>c>a), (c>a>b), (c>a>b)\};$$

$$D' = \{(a>c>b), (a>c>b), (a>c>b), (b>a>c), (b>a>c), (c>b>a), (c>b>a)\}.$$

There are $3! = 6$ different ways of ordering three things, therefore in a completely uniform culture each ordering occurs (by definition) with a probability of 1/6. The probability of D above is therefore equivalent to the probability of choosing three red, two white and two blue balls at random from an urn containing balls of six colours (including these) in equal proportions. This is given by the multinomial

$$p(D) = [7!/(3!2!2!)](1/6)^7 = 35/46\,656.$$

By symmetry, the probability of D' is also equal to 35/46 656, so the probability of a cyclic majority in conjunction with the weak Borda effect in which *a* is the unique plurality winner is 35/23 328. The probability of a cyclic majority in conjunction with a weak Borda effect with *a* or *b* or *c* as the plurality winner is evidently three times as great:

$$p(F_p(X, D) = \{x\}, xMyMzMx) = 35/7776.$$

As shown in Colman (1980), the probability of a weak Borda effect in this culture is

$$p(F_p(X, D) = \{x\}, yMx) = 735/5832.$$

Subtracting the first quantity from the second, we arrive at the exact probability of plurality–majority disagreement in this culture:

$$p(F_p(X, D) = \{x\}, x \neq y \in X, yMz \text{ for all } z \in X) = 35/288 = 0{\cdot}122.$$

This method of evaluating the exact probability of plurality–majority disagreement can be generalized straightforwardly to other well-defined cultures.

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