1 Overview

The purpose of this course is to provide insight into the process which is involved in creating mathematical models of dynamic systems. Such models are typically used for computer simulation to enable engineers to study the (dynamic) behaviour of such systems in order, for example, to assess design characteristics before a prototype is constructed or to design a suitable control system. A good model and an accurate simulation of a dynamic system can save much time and money and is often vital to assess a system’s behaviour.

This course concentrates more on the “how” of creating a model and simulating it in the Matlab/Simulink environment rather than on the “why” of the physics involved. The course will, by and large, not introduce many new concepts, but will instead focus on how things which have been covered previously in the degree programme can be used to build useful dynamic models. In a real engineering environment, very complicated models are often required. It is impossible to cover such detail in a one semester lecture course but as the course takes a reductionist approach, allowing complex problems to be partitioned into simpler ones, it is hoped that an understanding of how the techniques introduced here can be applied elsewhere will be conveyed.

Few pre-requisites are required for the course and it is suitable for any Leicester engineering undergraduate in their 3rd/4th year. The main courses which give useful background are the various mathematics modules (the topics of ordinary differential equations, vector and matrices are particularly relevant), the 1st year signals and systems module (the conceptual concepts of input-output behaviour, linearity and causality are of most interest rather than the mathematics) and the 2nd year dynamics module, although the latter is by no means necessary. The module does tend to suit certain companion modules and students also taking the State Variable Control module (EG3110) and the Dynamics of Engineering Systems (EG3160) may find there is some overlap, although it is emphasised that the focus of this module is different from both.

The module is taught by lectures (75 %) and computer classes (25 %) but it is vital that students keep up with the module by undertaking their own private study. The module is not particularly demanding but a certain amount of technical competence and care is required in order to construct the mathematical models. Moreover, practise is required in order to construct these models efficiently. The course does not stick entirely to one text book, but the first part in particular is loosely based in the first few chapters of Close et al. below. Good books to look at are the following:

1. **Modelling and Analysis of Dynamic Systems** Close, Frederick and Newell.
This course is based on several chapters from this book. Very readable and clearly written book. Main shortcoming is its lack of treatment of vector quantities.

2. **Advanced Engineering Dynamics** Harrison and Nettleton.
Rigorous book, useful for reference to vector quantities and for the latter parts of the course (e.g. Lagrangian dynamics). Not particularly easy to read and sometimes lacks detailed explanation.

Useful book for background reading but is more general than specific to this course. Quite easy to read but does not really emphasize the ‘systems’ approach which this course tries to convey.

A treatment of a variety of problems. Somewhat disorganised and a little too broad for the course.

5. **System Dynamics** Palm.
A book with a similar style to Close *et al.* above, but a little broader. It tends to start at an earlier point to Close (which may be useful) but is perhaps not quite a prescriptive in its methods.

The course aims to give an overview of modelling general dynamic systems which may be electrical, mechanical, fluidic, thermal etc. However, experience in previous years suggests that there is insufficient time to devote attention to all of the above systems. Therefore, in this course, attention is completely devoted to mechanical systems. Electrical engineers should not be put off though: the focus on the application of the techniques. The course roughly comprises four topics

- Translational mechanical systems
- Rotational mechanical systems
- Nonlinear systems and linearisation
- Lagrangian methods

with the themes of state-space modelling and system stability running through the course. Every fortnight a Matlab computer class will be held in place of a lecture. The aim of these is to enable students to familiarise themselves with the basics of the Matlab/Simulink package for creating dynamic models. The computer classes will tie in with lectures and enable the application of recently studied ideas.

It is emphasized that, while this set of notes does contain the skeleton of the course, it does not contain all the material and certainly **should not be relied upon entirely**. Attendance at lectures is normally a good idea.
2 Basics

2.1 Systems under consideration

This course will try to promote a systems way of looking at mechanical apparatus and, in particular, will often talk about inputs, outputs and states. An input is normally taken as an exogenous disturbance or command given to a system; an output is typically a variable which is observed; and a state is an internal variable which in some way governs its behaviour (this will be made clearer during the course). Our main concern is with dynamic systems. Recall the important properties of dynamic systems:

- They have MEMORY.
- That is, their output at a given time is dependent on past inputs.

If $G$ is a (continuous time) linear system we have the convolution relation:

$$y(t) = f(y(0), t) + \int_0^t G(t - \tau)u(\tau)d\tau$$

i.e the output at time $t$ depends on

- the output at time $t = 0$; plus
- the inputs over the interval $0 \leq \tau \leq t$ (a history of inputs)

Without exception we consider causal systems which have the property that

- Future events do not affect the present

We consider both linear and nonlinear systems in the course. Linear systems are easier to treat and have several useful properties and representations. Nonlinear systems are not as easy to handle but in certain cases can be handled conveniently by linear methods. This aspect will become a key topic in the course and so it is appropriate to delay a treatment of this until later.

2.2 Co-ordinates

When describing the motion of a mechanical system it is necessary to measure that motion with respect to a set of co-ordinates. Sometimes a set of co-ordinates is called frame of reference. It is important to understand the relation between different sets of co-ordinates because sometimes it may be convenient to model the various components of a system in one set of co-ordinates and then perform an analysis in another.
The motion of a particle \( P \) on a plane is depicted below. Two sets of co-ordinates are shown. In the \((\bar{x}, \bar{y})\) co-ordinates, the position of \( P \) is

\[
z_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}
\]

However, note that the \((\bar{x}, \bar{y})\) axes are displaced a distance

\[
z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
\]

from the origin of a second set of axes \((x, y)\). Thus, the position of \( P \) with respect to the second set of axes is

\[
z_2 = z_1 + z_0 = \begin{bmatrix} x_1 + x_0 \\ y_1 + y_0 \end{bmatrix}
\]

This relationship holds more generally and we can see that each position in \((\bar{x}, \bar{y})\) coordinates has a corresponding position in the \((x, y)\) co-ordinates. Furthermore the two are related through the relationship

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
\]

The above example assumes that the relationship between the first and second set of coordinates is constant over time. That is, there is no relative motion between the axes. In many situations (e.g. in the modelling of vehicles), there is relative motion between the two axes, i.e.

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} + \begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix}
\]

where the argument \((t)\) means that \((x_0(t), y_0(t))\) are functions of time.
2.2.1 Inertial frames of reference

The nature of \((x_0(t), y_0(t))\) effectively governs the type of reference frame a given set of coordinates fall into. If we assume that the \((x, y)\) coordinates are “fixed” (i.e. they do not move with time) then we define:

**Inertial Reference Frame**

- A special set of coordinates which is neither accelerating or rotating with respect to a fixed reference frame.
- Any frame of reference which is at a constant distance or velocity from another inertial reference frame

An inertial frame is special because standard Newtonian Laws of motion are applied in their simplest form. Typical example of Inertial reference frames would be the earth or the stars - for most engineering purposes they can be assumed fixed.

For the \((\bar{x}, \bar{y})\) coordinates to be *inertial* it follows that \(x_0(t)\) and \(y_0(t)\) must either be constant or moving with constant velocity (i.e. \(\ddot{x}_0(t)\) and \(\ddot{y}_0(t)\) must be constant, or equivalently \(\dot{x}_0(t)\) and \(\dot{y}_0(t)\) must be zero).

2.2.2 Non-inertial frames of reference

Quite simply a non-inertial frame of reference a set of coordinates which is not inertial! This means that

**Non-Inertial Reference Frame**

- A non-inertial reference frame is either accelerating or rotating with respect to an inertial reference frame.

Such reference frames are useful for representing the local motion of systems such as vehicles but it is important to remember that

- For Newton’s Laws to hold in non-inertial reference frames an extra *inertial force* corresponding to the reference frame’s acceleration must be included. This will be covered in more detail later.

In terms of our simple example, for the \((\bar{x}, \bar{y})\) coordinates to be *non-inertial*, it follows that \(x_0(t)\) and \(y_0(t)\) must be accelerating in some way; that is \(\ddot{x}_0(t)\) and \(\ddot{y}_0(t)\) must be non-zero.
This section is roughly based on the corresponding chapter in Close et al. The main idea of the section is to break up the modelling of mechanical systems into a small number of methodical steps. These steps can then be followed regardless of how complex a system is, enabling a mechanical system to be modelled in a systematic fashion.

### 3.1 Background and basics

#### 3.1.1 Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Variable</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>displacement</td>
<td>metres, $m$</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity</td>
<td>$ms^{-1}$</td>
</tr>
<tr>
<td>$a$</td>
<td>acceleration</td>
<td>$m.s^{-2}$</td>
</tr>
<tr>
<td>$f$</td>
<td>force</td>
<td>Newtons, $N$</td>
</tr>
<tr>
<td>$W$</td>
<td>energy or work</td>
<td>Joules (J) $[Nm]$</td>
</tr>
<tr>
<td>$P$</td>
<td>power</td>
<td>Watts (W) $[Nms^{-1}]$</td>
</tr>
</tbody>
</table>

- $x, v, a, f$ are all **functions of time**, although time dependence normally dropped (i.e. we write $x$ instead of $x(t)$ etc.)

- As normal

$$v = \frac{dx(t)}{dt} = \frac{dx}{dt}$$

$$a = \frac{dv(t)}{dt} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

In general $x, v, a, f$ are **vector quantities** but **often we shall take them to be scalar**. To indicate vectors, we use the bold case. In Cartesian Co-ordinates:

$$\mathbf{x} = \begin{bmatrix} x_x \\ x_y \\ x_z \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \frac{dx_x}{dt} \\ \frac{dx_y}{dt} \\ \frac{dx_z}{dt} \end{bmatrix}$$
3.1.2 Work

- Work can be defined in many ways. One interpretation: useful energy expended by a force.
- Generally: force times distance form.
- Work is a scalar quantity but can be either:
  - positive (work is begin done, energy is being dissipated)
  - negative (energy is being supplied)

- Generally
  \[ W = \int f \cdot dx \]
  where \( f \) is the force applied and \( dx \) is the displacement.

- For constant forces:
  \[ W = f \cdot x \]
  \[ W = |f||x|\cos \theta \]

  \( \theta \) is angle between the two vectors \( f \) and \( x \).

- This means that if \( f \) and \( x \) are in exactly the same direction i.e. \( \theta = 0^\circ \) maximum work is done.
- If the displacement vector is orthogonal to the force vector \( \theta = 90^\circ \), no work is done.

<table>
<thead>
<tr>
<th>force direction</th>
<th>displacement direction</th>
<th>theta</th>
<th>comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>→</td>
<td>→</td>
<td>0°</td>
<td>maximum positive work</td>
</tr>
<tr>
<td>→</td>
<td>←</td>
<td>180°</td>
<td>maximum negative work</td>
</tr>
<tr>
<td>→</td>
<td>↑</td>
<td>90°</td>
<td>no work done</td>
</tr>
</tbody>
</table>

⇒ a vertical force does no work in the horizontal direction.

3.1.3 Power

Power is, roughly, the work done per unit time (hence a scalar too):

\[ P = \frac{dW}{dt} \]

Using the chain rule we have

\[ dW = f \cdot dx = f \frac{dx}{dt} dt = f \cdot v dt \]

This implies that

\[ P = f \cdot v \]

Alternatively, power can actually define work:

- Work done between times \( t_0 \) and \( t_1 \):
  \[ W = \int_{t_0}^{t_1} P(t) dt \]
Work done up to time $t$:

$$W = w(t_0) + \int_{t_0}^{t} P(\lambda) d\lambda$$

($w(t_0)$ is work done up to $t_0$)

### 3.2 Element Laws

The first step in obtaining the model of a system is to write down a mathematical relationship that governs the behaviour of the components (“elements”) of that system. For many common “elements” of a system, these are well-known formulae which have been covered elsewhere. For completeness we will discuss common elements in detail.

#### 3.2.1 Mass

Newton’s second law:

The sum of forces acting on a body is equal to the rate of change of momentum

The element law is thus

$$\frac{d}{dt} (M \mathbf{v}) = \mathbf{f}$$

The kinetic energy of a mass, $M$:

$$W_k = \int \mathbf{f} \cdot d\mathbf{x}$$

$$= \int M \frac{d\mathbf{v}}{dt} \cdot d\mathbf{x}$$

$$= \int M \mathbf{v} \cdot d\mathbf{v}$$

$$= \frac{M}{2} \mathbf{v} \cdot \mathbf{v} + \text{constant}$$

Assuming zero-initial conditions the constant term is zero,

$$W_k = \frac{M}{2} \mathbf{v} \cdot \mathbf{v}$$

$$= \frac{M}{2} (v_x^2 + v_y^2 + v_z^2)$$

The potential energy due to gravity is:

$$W_p = Mgh$$

where $h$ is the height (i.e. in the direction $y$) above its reference position.
3.2.2 Viscous friction

Friction, in a variety of forms, is commonly encountered in mechanical systems. Depending on the nature of the friction involved, the mathematical model of a friction element may take a variety of forms. In this course, we mainly consider viscous friction and in this case a friction element is an element where there is an algebraic relationship between the relative velocities of two bodies and the force exerted:

\[ f = B \Delta v \]

The coefficient of friction, \( B \), has units \( N \text{m}^{-1} \) and \( \Delta v = v_2 - v_1 \) is the difference in velocities of the two bodies. Such a relationship is typically found in oil (or another viscous liquid) and dash-pot damping elements. For example

- A mass, \( M \), sliding on an oil film that has laminar flow is subject to viscous friction. The friction coefficient, \( B \) is:
  
  - proportional to the contact area and the viscosity of the oil
  - inversely proportional to the thickness of the film.

- Dampers found in, for example, car suspension systems and door closing devices. If the left terminal is stationary \( (v_2 = 0) \), then

\[ f = B v_1 \]

- Often in diagrams, wheels are used to denote negligible levels of friction.

3.2.3 Stiffness elements

Any mechanical element which undergoes a change in shape when subjected to a force, can be characterised by a stiffness element. The most common stiffness element is perhaps the spring and, in diagrams, springs are used to represent stiffness elements.
For a linear spring (or a spring operating in its linear regime) we have

\[ f = Kx \]

if one end of the spring is fixed (Hooke’s Law).

\[ f \]

\[ x_2 \]

\[ x_1 \]

\[ f \]

\[ d_0 + x_1 - x_2 \]

If both ends are free to move (as above) we have

\[
\begin{align*}
  f &= K\Delta x \\
  &= K(x_1 - x_2) \quad (2)
\end{align*}
\]

assuming \( x_1 > x_2 \).

**Energy in a spring.** The potential energy (PE) stored in a spring is calculated from

\[
W_p = \frac{1}{2}K(\Delta x)^2 \\
= \frac{1}{2}K(x_1 - x_2)^2 \quad (3)
\]

- PE is stored when the spring is **stretched or compressed**.
- When the spring is its natural length (i.e. no compression or stretching) no energy is stored.
- Under the assumption of linearity, the PE stored in a spring can be returned to the remainder of the mechanical system at some point in the future.
- For springs operating in nonlinear regimes, this may not be possible - the spring may absorb energy but not return it to the system e.g. creep.

### 3.2.4 Pulleys

Pulleys are often used in systems because they can **change the direction of motion in a translational system.** Frequently part of the system would move in a horizontal plane, and the other part in a vertical plane. An example is depicted below.

The pulley is a **nonlinear element.** Consider the diagram above: if a positive force \( f \) is applied horizontally (\( f \) is a pulling force) than an upward force \( f \) is applied to the mass at the other end of the pulley. However, if \( f \) is negative (a pushing force), the pulley rope will buckle and no force will appear at the other end. Thus force is only “transmitted” if the pulley is in tension.

Pulleys are often modelled as “ideal pulleys” in combination with other elements. An ideal pulley has the following properties:
It is also assumed that there is no slippage between cable and surface of cylinder - they both move with the same velocity.

3.3 Interconnection Laws

Having established the formulae which govern the motion of the individual elements of the system, the next step is to gather them together using interconnection laws which dictate how they interact.

3.3.1 D’Alembert’s Law

D’Alembert’s Law is essentially a re-statement of Newton’s 2nd Law in a more convenient form. For a constant mass we have:

\[ \sum_i (f_{\text{ext}})_i = M \frac{dv}{dt} \]

where \( \sum_i \) means summation over all external forces. Equivalently:

\[ \sum_i (f_{\text{ext}})_i - M \frac{dv}{dt} = 0 \]

This suggests that the mass in question can be considered in equilibrium if \( M \frac{dv}{dt} \) is thought of as an additional force. This 'additional force' is sometimes called the Inertial force or the D’Alembert force and including it allows us to write the force equations as one of equilibrium:

\[ \sum_I f_I = 0 \]

The above equation is known as D’Alembert’s law. The minus sign associated with \( M \frac{dv}{dt} \) indicates that the Inertial force acts in the opposite direction to the sum of the applied forces.

It is natural to wonder why we use D’Alembert’s law rather than Newton’s 2nd Law. Fundamentally there is no real difference, but it is often preferable because:

1. Less prone to sign error.
2. Produces equations having similar forms to other systems (makes analogies easier).
3. Handles non-inertial reference frames better.
4. Handles multi-body systems better.
5. Can be applied to any point in a system (i.e. a junction of two components etc.)

### 3.3.2 Law of Reaction forces

Law of Reaction forces is Newton’s Third Law of motion - often applied to junctions of elements:

> Accompanying any force of one element on another, there is a reaction force on the first element of equal magnitude and opposite direction

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**Example 1**

Consider mass $M$ connected to fixed wall using a spring of spring constant $K$.

![Diagram of Example 1](image)

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**Example 3**

Because the spring and damper are both connected between the wall and mass, both elements have the same displacement, $x$, and velocity, $v$.

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**Example 3**
As $B$ and $K$ have ends connected to the same elements, the elongation of both elements is $x_2 - x_1$ (assuming $x_2 > x_1$)

### 3.4 Deriving the system model

The mathematical model of the system under consideration is constructed by combining the formulae describing the behaviour of the various components in the systems (element laws) with the formulae describing how they interact (the interconnection laws). The following procedure is normally followed.

1. Draw appropriate **Free Body Diagrams**.

2. Apply **D’Alembert’s Law** to each mass/junction point in the system which moves with an unknown velocity.

3. Express all forces (except inputs) in terms of displacements velocities and accelerations.

4. Simplify the above to obtain the differential equations describing the system’s motion.

The above will become clearer with an example. Before proceeding however, it is useful to heed some cautionary notes:

- In dynamic systems the velocities and displacements are typically not *constants* but *functions of time*.

- It is often necessary to “pre-suppose” the sense (direction) of a displacement, velocity etc if it is unknown.
  - Pre-supposing certain directions can be conceptually more appealing than in others, although ultimately it will make no difference to the *form* of differential equations which are produced.
  - It is important to ensure consistency between the “pre-supposed” direction of motion and the Free Body Diagram, and also to the equations which are written down. If there is a lack of consistency, the equations will not make sense.

### Example 1 - Simple mass-spring-damper system

Consider the horizontal motion of a mass (i.e. we consider only scalar quantities):
Horizontal forces which are included:

- $f_K$: force exerted by the spring
- $f_B$: force exerted by the damper
- $f_a$: applied force
- $f_I$: inertial force

There is only one mass, so the Free-Body Diagram is:

```
    f_B
    M
    f_K
    . . . f_I
    f_a
```

Some remarks are in order.

- In the above diagram, we have assumed that the mass will move to the right (corresponding to $f_a$ being positive) but this assumption is *arbitrary*. An equally valid, but perhaps less conceptually appealing choice, would have been to assume that the mass moves to the left. **Remember:** $f_a$ is generally a function of time, not a constant.

- However, given the above choice of direction, it is then vital that there is agreement between this choice and our free-body diagram. If the movement of the mass is to the right, then the springs and dampers both become extended and thus the force they exert on the mass (to oppose the motion) is to the left. Note also that, as we have assumed motion to the right, the inertial force, $f_I$, acts in the opposite direction i.e. to the left.

The next step is then to apply D’Alembert’s Law to the mass and express the individual forces using the Element Laws. This yields

$$ f_a(t) - f_I(t) - f_B(t) - f_K(t) = 0 $$

\[ \Leftrightarrow \quad f_a - M \ddot{v} - Bv - Kx = 0 \]

\[ \Leftrightarrow \quad f_a - M \ddot{x} - B\ddot{x} - Kx = 0 \]

\[ \Leftrightarrow \quad f_a = M \ddot{x} + B\ddot{x} + Kx \]

which is the equation of motion describing the system’s behaviour.

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### 3.5 Relative Motion

As mentioned earlier, it is often more convenient to measure motion with respect to some moving reference frame rather than a fixed set of coordinates. If this moving reference frame is either accelerating or rotating, it is not an *inertial* reference frame and some care is required in deriving accurate differential equations - recall an extra *inertial force* is required to account for the acceleration of the reference frame.
Example 2 - Relative motion

- \( x \) - position of \( M_1 \) with respect to fixed reference.
- \( z \) - position of \( M_2 \) with respect to \( M_1 \).

Assume neither spring is compressed or stretched when \( x = z = 0 \).

Consider free body diagram associated with \( M_2 \) first.

\[
\begin{align*}
B_2 \dot{z} &\quad \rightarrow f_a(t) \\
K_2(x + z) &\quad \rightarrow -M_2(\ddot{x} + \ddot{z})
\end{align*}
\]

Note that

- The damper \( B_2 \) opposes the motion of \( M_2 \) with respect to \( M_1 \): as \( z \) is taken to be positive to the right, then the force exerted by \( B_2 \) is therefore to the left.
- Note the Inertial force: \( f_{I,M2} = M_2(\ddot{x} + \ddot{z}) \) This is not “standard” Newtonian motion as we are assuming \( \dot{x}(t) \) is varying.
- If \( \dot{x}(t) \) was constant (i.e. \( z \) was measured with respect to an Inertial reference frame), then \( M_2(\ddot{x} + \ddot{z}) = M_2 \ddot{z} \) as \( \ddot{x} = 0 \).
- D’Alembert’s Law is easier to apply to non-Inertial reference frames. (it just appears as an extra force).

Using D’Alembert’s Law we get the equations of motion for \( M_2 \):

\[
\begin{align*}
f_a - M_2(\ddot{x} + \ddot{z}) - K_2(x + z) - B_2 \dot{z} &= 0 \\
M_2 \ddot{z} + B_2 \dot{z} + K_2 z + M_2 \ddot{x} + K_2 x &= f_a
\end{align*}
\]

Using the Law of Reaction forces, free-body diagram for \( M_1 \) is

\[
\begin{align*}
B_1 \dot{x} &\quad \rightarrow B_2 \dot{z} \\
B_3 \dot{x} &\quad \rightarrow -M_1 \ddot{x} \quad K_1 x
\end{align*}
\]
and the equations of motion for $M_1$ become

\[
B_2 \ddot{z} - K_1 x - M_1 \ddot{x} - B_1 \dot{x} - B_3 \dot{x} = 0 \\
M_1 \ddot{x} + (B_1 + B_3) \dot{x} + K_1 x - B_2 \ddot{z} = 0
\]

In this example it appears easier to consider the mass $M_2$ first and then use the Law of Reaction forces to help obtain the Free Body Diagram for $M_1$. However, it is just as valid to consider $M_1$ first instead.

### Example 2 - Non-zero steady-state spring compression

When modelling a system containing springs, it is often convenient to measure displacements with respect to coordinates which are such that the values of the displacement variables are zero when the springs are neither stretched nor compressed. Sometimes this may not be convenient. Such a situation is shown below.

There are two ways in which we could proceed.

- First, let us assume the displacement $x$ is measured with respect to the spring being neither stretched or compressed (in essence this would be the case when the force $f_a$ acted to oppose gravity). The free body diagram then becomes

  ![Free Body Diagram](image)

  D’Alembert’s law then gives the differential equations

  \[
  M \ddot{x} + B \dot{x} + Kx - f_a(t) - Mg = 0 \\
  M \ddot{x} + B \dot{x} + Kx = Mg + f_a(t)
  \]

  Note that these differential equations feature the gravitational force $Mg$.

- Another way of proceeding is to measure the motion with respect to static equilibrium (steady state). To this end, suppose that $f_a(t) = 0$ and the mass is not moving. In this case we have that $x = x_0$ where $x_0$ is the constant displacement caused by gravity. As $x_0$ is constant, this implies that

  \[
  \dot{x} = 0, \quad \ddot{x} = 0
  \]
Thus equation (4) reduces to

\[ Kx_0 = Mg \]

So at static equilibrium the extension of the spring is \( x_0 = Mg/K \).

Next let us assume \( f_a(t) \neq 0 \) and the mass is moving. Now let the displacement \( x \) be given by

\[ x = x_0 + z \]

i.e. the displacement at rest, plus the additional displacement caused by the force \( f_a(t) \). Using \( x = x_0 + z \) in equation (4) we get

\[ M(\ddot{x}_0 + \dot{z}) + B(\dot{x}_0 + \dot{z}) + K(x + z_0) = Mg + f_a(t) \]

However, as \( \dot{x}_0 = 0 \) and \( \ddot{x}_0 = 0 \) this reduces to

\[ M\ddot{z} + B\dot{z} + K(x_0 + z) = Mg + f_a(t) \]

But as \( Kx_0 = Mg \) (from before) this further reduces to

\[ M\ddot{z} + B\dot{z} + Kz = f_a(t) \]  

Comparing equations (4) and 5) we see that we can **ignore** the effect of the gravitational force when drawing the free-body diagram providing that

- The displacement is defined to be the displacement from the static position when there are no inputs other than gravity;
- The masses are suspended vertically by linear springs (If the springs are nonlinear, the conclusion is not valid);

\[ f_K(t) = K(x_0 + z) = Kx_0 + Kz \]

The same is true in general if constant forces are acting on a system; through judicious choice of coordinates, one can derive differential equations in which these forces are absent, providing only linear springs are present.
This section is based on the corresponding chapter in Close et al. and it follows a similar development to that in the previous section but concentrates on rotational systems and, more specifically, systems which rotate in the plane - that is they only have one axis of rotation. This substantially simplifies the development of the results, but it should be understood that, when one considers general three-dimensional rotation, things become considerably more complicated. This is covered in more detail in the EG3160 course.

4.1 Background and Basics

4.1.1 Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Variable</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>angular displacement</td>
<td>radians, $rad$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>angular velocity</td>
<td>$rads^{-1}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>angular acceleration</td>
<td>$rads^{-2}$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>torque</td>
<td>Newton-metres, $Nm$</td>
</tr>
</tbody>
</table>

$\theta$, $\omega$, $\alpha$, $\tau$ are all functions of time, although time dependence is normally dropped ($\theta$ instead of $\theta(t)$ etc.)

\[
\omega = \frac{d\theta(t)}{dt} = \frac{d\theta}{dt} \\
\alpha = \frac{d\omega(t)}{dt} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}
\]

In general $\theta$, $\omega$, $\alpha$, $\tau$ are vector quantities, but normally we shall take them to be scalar

4.1.2 Work and power

Work done by a torque is defined as

\[
W = \int \tau d\theta
\]

Power is the rate at which work is done, viz:

\[
P = \frac{dW}{dt}
\]
Hence, power can be calculated as

\[ P = \tau \omega \]

The energy supplied (up until time \( t \)) is obtained as

\[ W(t) = W(t_0) + \int_{t_0}^{t} P(\lambda) d\lambda \]

4.2 Element Laws

4.2.1 Moment of Inertia

Newton’s Second Law for rotational bodies:

\[ \frac{d}{dt} (J\omega) = \tau \]

where

- \( J\omega \) is angular momentum of the body
- \( \tau \) is the net torque applied about the fixed axis of rotation
- \( J \) is the moment of Inertia

The Moment of Inertia for a body whose mass, \( M \), can be considered concentrated at a point is:

\[ J = ML^2 \]

where \( L \) is the distance from the point to axis of rotation.

\[ \theta \]

**NB** - Generally, expression for calculating Moment of Inertia involves an integral.

If we have an axis which does not pass through the centre of mass, we use the parallel axis theorem

- \( J_0 \) = the moment of inertia about the parallel axis that passes through the centre of mass
- \( d \) = distance between the two axes.
• Then $J = J_0 + Md^2$

Example 1 - Parallel Axis Theorem

In the figure, axis of rotation is at one end of the slender bar. But the centre of mass is in the middle of the bar $(d = (L/2))$ so we have

$$J = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

For constant moment of Inertia, Newton’s Law is:

$$\frac{d}{dt}(J\omega) = \tau \Rightarrow J\omega = \tau$$

Note that this is a stronger requirement than constant mass because it requires both constant mass and constant distance from centre of mass.

Kinetic energy (KE) for a rotational body:

$$KE = \frac{1}{2}J\omega^2$$

Potential energy (PE) for a rotational body

$$PE = Mgh$$

Note PE is same as for translational systems

4.2.2 Viscous friction

Similar to translational systems, a rotational viscous friction element describes an algebraic relationship between the torque and relative angular velocities between the two surfaces. Typical instances in which viscous friction would occur would be when two rotating bodies are separate by a film of oil (see below), or when rotational damping elements are employed.
Angular velocities of the two rotating cylinders: $\omega_1$ and $\omega_2$.

$\Rightarrow$ relative angular velocity: $\Delta \omega = \omega_2 - \omega_1$

- Torque exerted: $\tau = B \Delta \omega$
- Torque exerted on each cylinder in the directions that tend to reduce the relative angular velocity.
- Hence
  - Counter-clockwise on inner cylinder
  - Clockwise on outer cylinder
- Friction co-efficient has units: $Nms$

### 4.2.3 Rotational Stiffness

Rotational stiffness is usually associated with a torsional spring (mainspring of a clock), or with a relatively thin, flexible shaft. It is an element which satisfies:

\[
\tau = K \Delta \theta 
\]

where $\Delta \theta = \theta_2 - \theta_1$ and $K$ is the stiffness constant with units $Nm$. We assume that the moment of inertia of the stiffness element is either:

- Negligible
- Represented by separate element

The torques exerted on the two ends of the stiffness element are equal in magnitude but opposite in direction (see diagram below)

![Diagram of rotation]

For the system:

![Diagram of system]

- Applied torque passes through the first shaft and is exerted directly on the body of moment of inertia $J$ (because we assume that the shaft itself has no Moment of Inertia.)

PE is stored in a twisted stiffness element and can affect the system at later times:

\[
PE = w_p = \frac{1}{2} K (\Delta \theta)^2
\]
4.2.4 The Lever

An ideal lever is assumed to be a rigid bar having:

- No mass
- No friction
- No momentum
- No stored energy

Normally (but not always) assume that the pivot-point is fixed.

If magnitude of angle of rotation is small (less than 0.25 radians) the motion of the ends of the lever can be considered strictly translational, and confined to one direction. Note that we have:

\[ x_1 = d_1 \sin \theta \]
\[ x_2 = d_2 \sin \theta \]

So, for small \( \theta \) we have:

\[ x_1 \approx d_1 \theta \]
\[ x_2 \approx d_2 \theta \]

Eliminating \( \theta \) (which is assumed small): \( x_2 = \frac{d_2}{d_1} x_1 \)

Differentiating we also get that:

\[ \dot{x}_2 = v_2 = \frac{d_2}{d_1} v_1 \]

As the sum of the moments about the pivot point vanishes (absence of mass) we get that:

\[ f_2 d_2 - f_1 d_1 = 0 \]
\[ f_2 = \frac{d_1}{d_2} f_1 \] \hspace{1cm} (8)

The pivot does exert a downward force of \( f_1 + f_2 \) on the lever ...but this does not exert moment about pivot.

**NB1** - Algebraic relationships for the lever involve pairs of the same variables (displacements, forces etc) - unlike other elements.

**NB2** - If lever’s mass cannot be neglected, must include its moment of inertia when summing moments about its pivot points and (8) is no longer valid.

4.2.5 Gears

Ideal gears have
Any inertia or friction in an actual pair of gears can be represented by separate lumped elements in the free-body diagrams.

Relative size of two gears result in a proportionality constant for

- the angular displacements
- the angular velocities
- the transmitted torques

of the respected shafts. It is convenient to visualise the pair of ideal gears as two circles:

Circles are tangent at contact point and rotate without slipping. Spacing of teeth must be equal for each gear in a pair, so the radii of the gears are proportional to the number of teeth

\[
N = \frac{r_2}{r_1} = \frac{n_2}{n_1}
\]  \hspace{1cm} (9)

where \(N\) is the gear ratio (this follows by considering that the circumference of the gears is \(2\pi r_i\) and the number of teeth is the circumference divided by the gear teeth spacing)

**Torque relationship** At the point where the teeth mesh: \(f_1 = f_2\), so torques are

\(r_1 = f_1 r_1\) (in anti-clockwise direction)

\(r_2 = f_2 r_2\) (also in anti-clockwise direction)

So the torque transmitted by the gears is

\[
\tau_1 = \frac{r_1}{r_2} \tau_2
\]

and is the same sense as the the applied torque.

### 4.3 Interconnection Laws

#### 4.3.1 D’Alembert’s Law

As with translational systems, D’Alembert’s Law for rotational systems is essentially a re-statement of Newton’s 2nd Law but this time for rotating bodies. For a constant moment of Inertia we have

\[
J \ddot{\omega} = \sum_i (\tau_{ext})_i
\]
where $\sum_i (\tau_{ext})_i$ = ‘sum of external torques’ acting on body. D’Alembert’s Law re-writes this as:

$$\sum_i (\tau_{ext})_i - J\dot{\omega} = 0$$

The term $J\dot{\omega}$ is the Inertial Torque. When the Inertial Torque is included with other torques acting on the body:

$$\sum_i \tau_i = 0$$

which is D’Alembert’s Law for rotating bodies. Note that Inertial Torque acts in opposite sense to that of $\omega, \theta, \alpha$.

Also, D’Alembert’s Law can also be applied to a junction point which has no moment of Inertia.

### 4.3.2 Law of Reaction Torques

Essentially Newton’s 3rd Law for rotational systems:

> For two bodies rotating about the same axis, any torque exerted by one element on another is accompanied by a reaction torque of equal magnitude and opposite direction.

Example:

A “counterclockwise” torque $K\theta$ exerted by the shaft $K$ on the disk $J$...is accompanied by a “clockwise” torque exerted by the disk $J$ on the shaft $K$.

For bodies not rotating about the same axis, the magnitude of the two torques is not necessarily equal.

- For gears, forces at point of contact are equal and opposite.
- But as the gears have different radii, the torque exerted by the first gear has a different magnitude to the torque exerted on the second.

### 4.3.3 Geometric considerations

**Displacement.**

If two components of a system are joined, rigidly, along the axis of rotation, the parts of these components which are joined will have the same angular motion.
The bar is assumed rigid (infinitely stiff), so \( \theta_1 = \theta_2 \).

**Translational/rotational effects.**

A point on a body which moves with angular displacement of \( \theta \) *rads* at a distance \( r \) metres from the axis of rotation will move the equivalent of \( x = \theta r \) metres when interpreted from a linear perspective.

### 4.4 Obtaining the system model

The process of obtaining a system model for rotational systems is essentially the same as with translational systems except that the rotational interconnection laws are used instead. The process is perhaps best illustrated using simple examples.

---

**Example 1 - Rotational Mass spring damper**

![Rotational Mass spring damper diagram](image)

**Problem**

Given:

- Input, \( \tau_a(t) \).
- Outputs:
  - Angular velocity of the disk (\( \omega \)).
  - Counter clock-wise torque exerted by disc on flexible shaft.

Derive the state variable model of the system.

**Solution**

1. Draw Free-body diagram:

![Free-body diagram](image)
Note that all torques other than the applied torque oppose the direction of motion.

2. Apply D’Alembert’s Law

\[ J\ddot{\omega} + B\omega + K\theta - \tau_a(t) = 0 \]
\[ J\ddot{\theta} + B\dot{\theta} + K\theta - \tau_a(t) = 0 \]

3. Define state variables: Let \( \theta \) and \( \dot{\theta} = \omega \) be state variables. So we have:

\[ \dot{\theta} = \omega \]
\[ J\dot{\omega} = -B\omega - K\theta + \tau_a(t) \]

In state-variable form:

\[ \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{J} & -\frac{B}{J} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \tau_a(t) \]

or

\[ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ K & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \]

The outputs are \( \omega \) and the force exerted by the disc on the flexible shaft, \( K\theta \).

**Example 1 - Lever**

Even though levers are rotational elements, providing the displacement of either end is sufficiently small, they can be treated as translational elements. The following example from Close *et al.* illustrates this quite nicely.

![Lever Diagram](image)

**Problem Given**

- Input: \( x_4(t) \).
- Output: position of mass, \( M(x_1(t)) \).
Draw Free-body diagrams, derive equations of motion and compute state-space matrices.

Assumptions

- $\theta$ is small (only horizontal motion considered).

Solution

1. Assume $x_4 > x_3$. Draw free-body diagrams:

\[ B \ddot{x}_1 \quad M \quad K_1(x_1 + x_2) \quad K_2(x_4 - x_3) \]

\[ M \ddot{v}_1 \quad F_r \quad K_1(x_1 + x_2) \]

$F_r$ is the reaction force exerted by the wall on the pivot.

2. Apply D’Alembert’s Law. For the mass:

\[ M \ddot{x}_1 + B \ddot{x}_1 + K_1(x_1 + x_2) = 0 \]

As $F_r$ acts through the center of rotation, it makes no contribution to the sum of moments, so summing moments about the pivot:

\[ K_2(x_4 - x_3)d_1 = K_1(x_1 + x_2)d_2 \quad (10) \]

**NB:** We have not needed (or wanted) to use $F_r$.

3. Express $x_2, x_3$ in terms of system states and inputs. As $\theta$ is small:

\[ x_3 = d_1 \sin \theta \approx d_1 \theta \]
\[ x_2 = d_2 \sin \theta \approx d_2 \theta \]

Eliminating $\theta$ gives

\[ x_3 = \frac{d_1}{d_2} x_2 \]

Using this in (10):

\[ K_2(x_4 - \frac{d_1}{d_2} x_2)d_1 = K_1(x_1 + x_2)d_2 \quad (11) \]
\[ K_2d_1x_4 - K_1d_2x_1 = (K_1d_2 + K_2d_1^2/d_2)x_2 \]

which after simplification becomes:

\[ x_2 = \frac{d_2(K_2d_1x_4 - K_1d_2x_1)}{K_1d_2^2 + K_2d_1^2} \quad (12) \]
4. (12) can be used in equation (2) to obtain (after some algebra):

\[ M_1 \ddot{v}_1 + Bv_1 + \beta d_1 x_1 + \beta d_2 x_4 = 0 \]

where

\[ \beta := \frac{K_2 d_1}{K_1 d_2^2 + K_2 d_1^2} \]

5. Defining the states of the system to be \( x_1 \) and \( v_1 = \dot{x}_1 \) we obtain:

\[
\begin{align*}
\dot{v}_1 &= \frac{1}{M} (-Bv_1 - \beta d_1 x_1 - \beta d_2 x_4) \\
\dot{x}_1 &= v_1
\end{align*}
\]

(13)

---

**Example 2 - Gears**

![Gears Diagram]

**Problem**

Given

- Inputs: \( \tau_{a1}(t), \tau_{a2}(t) \)
- Output \( \theta_1 \)

Derive equations of motion for system (and state-space matrices)

**Cautionary note**

- Initial thought: use \( \theta_1, \theta_2, \omega_1, \omega_2 \) as state variables.
- Cannot do this! \( \theta_1 \) and \( \theta_2 \) are related by the gear ratio (as are \( \omega_1 \) and \( \omega_2 \)).
- State variables must be linearly independent (cannot express one in terms of linear combinations of another)...

**Solution**

1. Free-body diagrams:
2. Using D’Alembert’s Law, equations of motion:

\[ J_1 \ddot{\omega}_1 + B_1 \omega_1 + K_1 \theta_1 + r_1 f_c = \tau_{a1}(t) \]  
(14)

\[ J_2 \ddot{\omega}_2 + B_2 \omega_2 + K_2 \theta_2 - r_2 f_c = \tau_{a2}(t) \]  
(15)

From geometry of gears (\(N = \text{gear ratio}\)):

\[ \theta_1 = N \theta_2 \]  
(16)

\[ \omega_1 = N \omega_2 \]  
(17)

3. Choose \( \omega_2, \theta_2 \) as state variables. Equation (14) becomes:

\[ J_1 N \dot{\omega}_2 + B_1 N \omega_2 + K_1 N \theta_2 + r_1 f_c = \tau_{a1}(t) \]  
(18)

From equation (15),

\[ f_c = \frac{1}{r_2} (J_2 \ddot{\omega}_2 + B_2 \omega_2 + K_2 \theta_2 - \tau_{a2}(t)) \]  
(19)

Using this in equation (18):

\[ J_1 N \dot{\omega}_2 + B_1 N \omega_2 + K_1 N \theta_2 + \frac{r_1}{r_2} (J_2 \ddot{\omega}_2 + B_2 \omega_2 + K_2 \theta_2 - \tau_{a2}(t)) = \tau_{a1}(t) \]  
(20)

But \( r_1/r_2 = N^{-1} \), so we get

\[ J_1 N \dot{\omega}_2 + B_1 N \omega_2 + K_1 N \theta_2 + J_2 N^{-1} \dot{\omega}_2 + B_2 N^{-1} \omega_2 + K_2 N^{-1} \theta_2 - N^{-1} \tau_{a2}(t) = \tau_{a1}(t) \]

Multiplying through be \(N\) and defining:

\[ \tilde{J} := J_1 N^2 + J_2 \]  
(21)

\[ \tilde{B} := B_1 N^2 + B_2 \]  
(22)

\[ \tilde{K} := K_1 N^2 + K_2 \]  
(23)

we have

\[ \dot{\theta}_2 = \omega_2 \]  
(24)

\[ \dot{\omega}_2 = \frac{1}{\tilde{J}} [-\tilde{B} \omega_2 - \tilde{K} \theta_2 + \tau_{a2}(t) + N \tau_{a1}(t)] \]  
(25)

4. In state-space, the state equation becomes:

\[
\begin{bmatrix}
\dot{\theta}_2 \\
\dot{\omega}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\tilde{K}/\tilde{J} & -\tilde{B}/\tilde{J}
\end{bmatrix} \begin{bmatrix}
\theta_2 \\
\omega_2
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
N/\tilde{J} & 1/\tilde{J}
\end{bmatrix} \begin{bmatrix}
\tau_{a1}(t) \\
\tau_{a2}(t)
\end{bmatrix}
\]
(26)

\[ y = \begin{bmatrix}
N & 0
\end{bmatrix} \begin{bmatrix}
\theta_2 \\
\omega_2
\end{bmatrix} \]  
(27)
So far we have largely dealt with *linear systems*. For the systems we’re dealing with here, linear systems are those systems which are described by linear differential equations, or equivalently those systems which have a linear state-space representation, i.e.

\[
\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\
y = Cx + Du \quad y \in \mathbb{R}^p
\]

Note that both the state and output equations are *linear* in the state \((x)\) and the input \((u)\) vectors. It is normally very convenient to use linear systems because there are many powerful and well-developed techniques which can be applied to such systems.

Unfortunately, the real world is *nonlinear* and all *real* systems are nonlinear to a certain degree. It is often convenient to express nonlinear systems in a similar state-variable form to linear systems. In particular nonlinear (time-invariant) systems are described by the state-space equations

\[
\dot{x} = f(x, u) \quad x, f(\cdot) \in \mathbb{R}^n, u \in \mathbb{R}^m \\
y = g(x, u) \quad g(\cdot), y \in \mathbb{R}^p
\]

Note that the state and output equations are not necessarily linear in the state \((x)\) and the input \((u)\) vectors. Two examples of nonlinear systems are

\[
\dot{x} = x^2 + u \\
y = x \\
\dot{x} = -x + u^2 \\
y = \sin(x)
\]

Notice that we cannot associate a set of \((A, B, C, D)\) matrices with either of these systems - hence neither system is linear.

Nonlinearities normally arise in systems in, broadly, one of two ways.

1. **Nonlinear Elements.** In the foregoing sections we have assumed most elements are linear (except the pulley) and hence their element laws have been linear. In reality the element laws are only linear over a certain range of operation. A good example is a spring which is only linear for “small” extensions and compressions. It is well known that if a spring is stretched beyond a certain point Hooke’s law ceases to hold and elastic or plastic deformation occurs.

Nonlinear elements are very common and can sometimes be modelled in block diagram form as a combination of a purely linear block plus a nonlinear block. Many of these can be found in the Simulink library and some common ones are

- Saturations (modelling limits)
- Dead-zones (modelling nonlinear meshing of gears etc)
- Backlash (modelling various types of nonlinear friction)
- Sign functions (modelling relays etc)
2. System level nonlinearities

Quite often, the way in which elements are connected gives rise to nonlinear systems. A good example is the simple pendulum.

The differential equation describing the motion is

\[ M L^2 \ddot{\theta} + B \dot{\theta} + M g L \sin \theta = \tau_a(t) \]

Using state-variables \( \theta, \omega = \dot{\theta} \), equations of motion are

\[
\begin{align*}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\frac{g}{L} \sin \theta - \frac{B}{M L^2} \omega + \frac{1}{M L^2} \tau_a(t)
\end{align*}
\]

The term \( \sin \theta \) causes these differential equations to be nonlinear - hence cannot find state-space matrices of system.

Typically, nonlinear systems are rather more difficult to work with than linear systems. At a conceptual level, it is not as easy to develop the same level of intuition for nonlinear systems and, moreover, the mathematics tends to be less developed, more difficult to use and conservative. Part of the reason for this is that nonlinear systems are infinite in their variety, meaning that methods useful for one class of nonlinear system may be of little use for other classes. Furthermore, nonlinear systems exhibit a much richer types of asymptotic behaviour than linear systems. Whereas linear systems are either convergent (stable), divergent (unstable) or oscillatory (periodic), nonlinear systems may also exhibit chaotic behaviour (bounded and ordered, but seemingly random) and quasi periodic behaviour (“almost” periodic). Oscillatory behaviour in a nonlinear system normally takes the form of a limit cycle which is a closed path in the state-space - in turn these limit cycles may be stable or unstable as well.

### 5.1 Linearisation

Hopefully the above section demonstrates some of the problems in treating nonlinear systems. One natural question to ask when considering nonlinear systems therefore is: how useful are linear techniques? It transpires that in many instances linear techniques are very useful for nonlinear systems and it is therefore of interest to study the idea of linearisation which essentially means the conversion of a nonlinear system into a linear one. Normally around the point of linearisation (sometimes called an operating point or an equilibrium point) the linear system is a good approximation of the nonlinear one if certain conditions are satisfied. A nonlinear system can be linearised in several ways.
1. Small value approximation
   e.g. For pendulum, can approximate $\sin \theta$ as $\theta$ for small values of $\theta$. Hence pendulum equations become

\[
\begin{align*}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\frac{g}{L} \theta - \frac{B}{ML^2} \omega + \frac{1}{ML^2} \tau_a(t)
\end{align*}
\]

Cannot be applied generally, particularly when $f(x, u)$ is an even function.

2. Taylor Series truncation
   Nonlinear equations can be considered at a given point and expanded. Taylor series is then truncated at linear term. (more about this later)

3. Change of variables
   Infinite number of state-space realisations!
   Hence if state-equations are written e.g. as

\[
\dot{x} = f(x)
\]

Find (nonlinear) invertible transformation such that $z = T(x)$. and such that

\[
\dot{z} = \frac{\partial T}{\partial x} f \circ T^{-1}(z)
\]

and $\frac{\partial T}{\partial x} f \circ T^{-1}(.)$ is linear.

...Not always possible, can be difficult to find suitable transformation $T(x)$.

5.2 Taylor Series Expansion

Perhaps the most useful and widely applicable linearisation method is the Taylor Series Expansion. There are roughly three parts to this method:

1. Find an equilibrium point i.e. $x_e, u_e$

2. Expand $f(x, u)$ around equilibrium point and discard all terms of order higher than linear

3. Make change of variables

In detail the steps can be described as

1. Finding an equilibrium point

Before equations can be linearised, must find an equilibrium point - known as ‘trimming’ the equations. Trimming can be done algebraically or numerically. For large systems sometimes the algebraic approach is difficult. Simulink can ‘trim’ nonlinear equations.
Trimming: find point \((x, u) = (x_e, u_e)\) such that
\[
\dot{x} = f(x, u)|_{(x_e, u_e)} = 0
\] (32)

\(x_e, u_e\) is to be chosen, but difficulties arise because:

- \(f(x, u)\) may have multiple equilibria; or
- \(f(x, u)\) may not have equilibrium at specified point \((x, u) = (x_e, u_e)\).

e.g. \(\dot{x} = f(x, u) = x^2 + u\) has multiple equilibria at 
\[-u = x^2\]
but no equilibria for any \(u > 0\).

Numerical linearisation involves minimising a cost function \(J(x, u)\), such that \(f(x, u)\) is driven close to zero. Precise workings of this outside scope of course (related to numerical optimisation).

2. Suppose that an equilibrium point \((x_e, u_e)\) has been found. Then
\[
\dot{x}_e = f(x_e, u_e) = 0
\]

Now let us consider \(x, u\) as a perturbation about that equilibrium
\[
x = x_e + \Delta x
\]
\[
u = u_e + \Delta u
\]
Expanding \(f(x, u)\) about the point \((x_e, u_e)\) using the Taylor Series gives
\[
\dot{x} = f(x_e + \Delta x, u_e + \Delta u) = f(x_e, u_e) + \frac{\partial f}{\partial x}(x_e, u_e)\Delta x + \frac{\partial f}{\partial u}(x_e, u_e)\Delta u + \text{H.O.T}
\]
where
\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}_{x=x_e,u=u_e}
\begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\
\vdots & & \vdots \\
\frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m}
\end{bmatrix}_{x=x_e,u=u_e}
\] (33)

But we know that \(f(x_e, u_e) = 0\), so we have
\[
\dot{x} = \frac{\partial f}{\partial x}(x_e, u_e)\Delta x + \frac{\partial f}{\partial u}(x_e, u_e)\Delta u
\]

3. So far, we are not very close to our familiar linear state-space form. However note that
\[
\dot{x} = \frac{d}{dt}(x_e + \Delta x) = \frac{d}{dt}\Delta x
\]

Thus, defining a new state-vector \(\tilde{x} = \Delta x\) and input \(\tilde{u} = \Delta u\) we get
\[
\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}
\]
where
\[
A := \frac{\partial f}{\partial x}(x_e, u_e) \quad B := \frac{\partial f}{\partial u}(x_e, u_e)
\]
are constant matrices (as they are evaluated at the fixed equilibrium point \((x_e, u_e)\). In a similar manner, we can expand

\[ y = g(x, u) \]

in its Taylor series to get

\[ y = g(x_e, u_e) + \frac{\partial g}{\partial x}(x_e, u_e)\Delta x + \frac{\partial g}{\partial u}(x_e, u_e)\Delta u + \text{H.O.T.} \] (34)

Thus if we define

\[ C := \frac{\partial g}{\partial x}(x_e, u_e) \quad D := \frac{\partial g}{\partial u}(x_e, u_e) \]

in terms of our new state and input vectors we get

\[ y = g(x_e, u_e) + C\tilde{x} + D\tilde{u} \] (35)

If we now let \( y_e = g(x_e, u_e) \) we can define

\[ \tilde{y} = y - y_e \]

and obtain

\[ \tilde{y} = C\tilde{x} + D\tilde{u} \] (36)

Notes on Taylor Series expansion

- Linearised system is in different co-ordinates to nonlinear system. The linear co-ordinates are defined with respect to the equilibrium point of the nonlinear system
- Linearisation only a good approximation around equilibrium point.
- This method assumes that \( f(x, u) \) is continuously differentiable with respect to the state \( x \) and the input \( u \), i.e. \( f(\cdot) \) is smooth.

Technical Notes on Taylor Series Expansion

- We have have assumed that \( f(x, u) \) is continuously differentiable or, equivalently, smooth. We denote this as

\[ f(\cdot) \in C^1 \] (37)

where \( C^1 \) is the class of functions whose first derivatives are continuous.

- However there are many functions which are not smooth everywhere. They essentially fall into two categories
  - Non-smooth functions: functions themselves are continuous, but their first derivatives are not. Sometimes denoted \( f(\cdot) \in C^0 \).
  - Discontinuous functions: functions themselves contain a discontinuity at some point. \( f(\cdot) \notin C^0 \).

These types of functions cause problems for the Taylor Series and care has to be exercised in applying it.
6 Lagrangian Dynamics

The final section in the course is devoted to modelling systems using Lagrange’s equations. As with the earlier parts of the book, the goal of the course is not to give a thorough expose of the derivation and philosophy of Lagrange’s equations, but to use the method as a tool for the derivation of dynamics models. More detail on the derivation of Lagrange’s equation is given in the 3rd/4th year dynamics course.

Some motivation for the use of Lagrange’s equations is needed however. The main problem with use of the Newton/D’Alembert approach to modelling systems is that it can become unwieldy for large complex systems. In particular accounting for each force and it’s direction can become difficult and it can be time consuming to work out all forces/torques acting on a body. An alternative to the force/torque based approach of D’Alembert is the essentially energy-based method of Lagrange.

Lagrange’s equations are the following

\[
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = F_{q_i}
\]

where the Lagrangian $L$ is given by

\[
L = K - U
\]

where $K$ is the system’s kinetic energy and $U$ is the system’s potential energy. It is also important to note that

$\begin{align*}
q_i & \quad \text{generalised coordinates of the system} \\
i & \quad \{1, 2, \ldots, n\} \\
F_{q_i} & \quad \text{generalised nonconservative forces acting on system}
\end{align*}$

Thus Lagrange’s equations are effectively $n$ separate equations. The meaning of generalised coordinates and forces will become clear with an example.

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**Example 1 - pendulum (classic example)**

The pendulum is only free to rotate in the plane, so it has one degree of freedom; there are no constraints. Thus $i = 1$ and $q_i = \theta$, and hence there will be only one Lagrange equation. In order to form the Lagrangian we must calculate the kinetic and potential energy.
Rotational kinetic energy
\[ K = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} Ml^2 \dot{\theta}^2 \]

- Potential energy - height of bob above ground:
\[ U = Mg h = Mg(h_0 - l \cos \theta) \]

Thus the Lagrangian \( L \) is
\[ L = K - U = \frac{1}{2} Ml^2 \dot{\theta}^2 - Mg(h_0 - l \cos \theta) \]

We can now carry out the differentiation required for Lagrange’s equations

1. \[ \frac{\partial L}{\partial \dot{\theta}} = Ml^2 \dot{\theta} \]
   and hence
   \[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = Ml^2 \ddot{\theta} \]

2. \[ \frac{\partial L}{\partial \theta} = \frac{d}{\partial \theta} Mg \cos \theta = -Mgl \sin \theta \]

3. The generalised force \( F_{qi} \) is the torque \( \tau \), so Lagrange’s equation becomes
\[ Ml^2 \ddot{\theta} - [-Mgl \sin \theta] = \tau \]

Re-arranging slightly we thus get the equations of motion as
\[ \ddot{\theta} = -\frac{g}{l} \sin \theta + \frac{1}{Ml^2} \tau \]

Although in this example Lagrange’s equations seem a more complicated way of deriving the equations of motion, for larger examples they cope somewhat better.

### 6.1 Lagrange’s Equations and non-conservative forces

One of the key assumptions when using Lagrange’s equations in the spirit introduced above is that the generalised forces involved are conservative, that is they conserve mechanical energy. However in many instances we encounter dissipative forces which dissipate mechanical energy. A particularly important member of the latter class is the force arising due to viscous friction which converts mechanical energy into heat. Thus in many situations, it is necessary to find some way of including these into Lagrange’s equations. The example below shows an easy way of accounting for these non-conservative forces.
Example 2 - Mass-spring damper with Lagrange

We consider the standard mass-spring damper example used earlier on. Note that the systems can only move horizontally so there is only one degree of freedom ($i = 1$). Unlike the previous Lagrange example, the presence of the damper introduces a non-conservative force. To account for this, Lagrange’s equations are altered to.

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{x}} \right] - \frac{\partial L}{\partial x} = f_a - B\dot{x}$$

As before $L = K - U$ and it is easy to see that

$$K = \frac{1}{2} M \dot{x}^2 \quad U = \frac{1}{2} K x^2$$

(note slightly strange notation: $K$ represent kinetic energy and spring constant) and hence

$$L = \frac{1}{2} (M \dot{x}^2 - K x^2)$$

Thus it follows that

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{x}} \right] = M\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = -Kx$$

Putting these together we thus have

$$M\ddot{x} - (-Kx) = f_a - B\dot{x}$$

or, more familiarly

$$M\ddot{x} + B\dot{x} + Kx = f_a(t)$$

In general, free-body diagrams can be useful in determining the direction of the dissipative force.

### 6.2 Lagrange’s Equations and Rayleigh’s Dissipation Function

Rayleigh’s Dissipation Function is a rather grand name for a modification that can be made to the form of Lagrange’s equations in order to account for dissipative forces in mechanical systems. Recall that in the foregoing section we “fudged” Lagrange’s equations by adding a dissipative term to account for the damping force, viz

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{x}} \right] - \frac{\partial L}{\partial x} = F - B\dot{x}$$

Note that

$$B\dot{x} = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2} B\dot{x}^2 \right]$$

So defining Rayleigh’s Dissipation Function, $R$, as

$$R := \frac{1}{2} B \dot{x}^2$$

our modified Lagrange equations become

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{x}} \right] - \frac{\partial L}{\partial x} + \frac{\partial R}{\partial \dot{x}} = F$$

It transpires that this holds more generally and thus, when dampers are present in a system we can use the following form of Lagrange’s equations

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_i} \right] - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = F_{q_i}$$

where Rayleigh’s Dissipation Function, $R$, accounts for the damping forces present in the system. Its general form is

$$R := \frac{1}{2} \sum_{j=1}^{n_d} B_j (\dot{q}_j - \dot{q}_{j-1})$$

where $q_j$ are the generalised velocities of points connected to dampers and $n_d$ is the number of dampers present.