Lecture 2

- Convolution in time is equivalent to multiplication in frequency
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- Sampling and the sampling theorem
- Restoring a continuous signal from its samples (in frequency domain)
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- Describing systems with difference equations – more examples
- z-transfer function, H(z) = z-transform of the unit impulse response, h(n)
- The fundamental theorem of z-transforms
- The z-transfer function of a LTI system – from difference equation to H(z)

We have shown in the last lecture that for all LTI systems the output can be obtained by the convolution of the input signal and the impulse response of the system. Today we will further explore the world of discrete signals, discussing their relationship with continuous signals. We shall start by showing that convolution in the time domain is equivalent to multiplication in the frequency domain and vice versa. Then we hall use that result to explain the sampling theorem.

The equivalence of convolution in the time domain to multiplication in the frequency domain

Let the input signal, x(t), to a LTI system (filter) be

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{-j2\pi ft}\, df$$

and the impulse response, h(t), of the LTI system (filter) be

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{-j2\pi ft}\, df$$

Now consider the output signal y(t) as the convolution of h(t) and x(t)

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Form the Fourier transform of both sides

$$\int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right] e^{-j2\pi ft}\, dt$$

which is equivalent to (assuming the order of integration can be changed)

$$\int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} h(\tau) \left[\int_{-\infty}^{\infty} x(t-\tau)\, e^{-j2\pi ft}\, dt \right] d\tau$$

Changing variables: $t' = t-\tau$, $t = t'+\tau$, $dt = dt'$

$$\int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} h(\tau) \left[\int_{-\infty}^{\infty} x(t')\, e^{-j2\pi ft'} e^{-j2\pi f\tau} dt' \right] d\tau$$

$$\int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau} \left[\int_{-\infty}^{\infty} x(t')\, e^{-j2\pi ft'} dt' \right] d\tau$$

$$\int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, X(f)\, d\tau = X(f) \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$Y(f) = H(f)\, X(f)$$

X(f): Fourier transform of the input signal

H(f): Fourier transform of impulse response (= frequency response function)

Y(f): Fourier transform of the output (filtered) signal

The above derivation was done using the continuous time domain functions h(t) and w(t), but it is equally valid for the discrete-time series h(nT) and w(nT).

The equivalence of multiplication in the time domain (windowing) to convolution in the frequency domain

Let the inverse Fourier transform of a function h(t) be

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{+j2\pi ft}\, df$$

and the Fourier transform of the function w(t) be

$$W(f) = \int_{-\infty}^{\infty} w(t)\, e^{-j2\pi ft}\, dt$$

Then the Fourier transform of the product is

$$H_w(f) = \int_{-\infty}^{\infty} w(t)\, h(t)\, e^{-j2\pi ft} dt$$

And can be manipulated as follows

$$\begin{aligned}
H_w(f) &= \int_{-\infty}^{\infty} w(t) \left[\int_{-\infty}^{\infty} H(f') e^{+j2\pi f' t}\, df'\right] e^{-j2\pi ft}\, dt = \\
&= \int_{-\infty}^{\infty} H(f') \left[\int_{-\infty}^{\infty} w(t)\, e^{+j(f'-f)2\pi t}\, dt\right] df' = \\
&= \int_{-\infty}^{\infty} H(f') \left[\int_{-\infty}^{\infty} w(t)\, e^{-j(f-f')2\pi t}\, dt\right] df' = \\
&= \int_{-\infty}^{\infty} H(f')\, W(f-f')\, df'
\end{aligned}$$

That is,

$$H_w(f) = H(f) * W(f)$$

- $W(f)$: Fourier transform of the function w(t)
- $H(f)$: Fourier transform of the function h(t)
- $H_w(f)$: Fourier transform of the product w(t)h(t)

The above derivation was done using the continuous time domain functions h(t) and w(t), but it is equally valid for the discrete-time series h(nT) and w(nT).

Sampling and the sampling theorem

The ideal sampler is a system that multiplies the input signal *x(t)* by a discrete train of equally spaced unit impulses $\delta(t)$.

The sampling theorem

If a continuous, ***band limited*** signal *x(t)* is sampled to produce a sequence *x(nT)* at equally spaced intervals of T seconds, then the original continuous signal can be obtained exactly from its samples as

$$x(t) = T\sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\pi(t-nT)/T\right]}{\pi(t-nT)}$$

provided that the signal was ***sampled fast enough***, that is, if

1. Saying that the signal is band limited means saying that

$$X(f) = 0 \qquad \text{for } f > f_{\max}$$

and

2. Saying that we sampled fast enough means saying that

$$f_{\max} < \frac{1}{2T}$$

$$T < \frac{1}{2f_{\max}}\,, \quad \text{but since } T = \frac{1}{f_{sam}}$$

$$\frac{1}{f_{sam}} < \frac{1}{2f_{\max}}\,, \text{ or}$$

$$f_{sam} > 2f_{\max} \text{ (Nyquist criterion)}$$

Then

$$x(t) = T\sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\pi(t-nT)/T\right]}{\pi(t-nT)}$$

This is easier to see in the frequency domain

$$x(t) \leftrightarrow X(f)$$

$$\delta(t) \leftrightarrow \Delta(f)$$

$$x(t)\delta(t) \leftrightarrow X(f) * \Delta(f)$$

and to reconstitute $X(f)$ from $X(f) * \Delta(f)$ one has to multiply that by a rectangular function in the frequency domain, *Q(f)*, that is

$$X(f) = \left[X(f) * \Delta(f)\right] Q(f)$$

This corresponds to applying a low-pass filter with cut-off frequency above f_{max}. Normally one chooses the cut-off frequency to be at $f_{sam}/2$.

This is equivalent to the following operation in the time domain

$$x(t) = \left[x(t)\delta(t)\right] * q(t)$$

and since $x(t)\delta(t) = x(nT)$

$$x(t) = x(nT) * q(t)$$

that is, to obtain the original signal back one has to convolve the sequence of samples with the inverse Fourier transform of the rectangular function *Q(f).* This happens to be a sinc function.

Describing systems with difference equations – more examples

Let's consider a couple of difference equations

Let's say you need to find the moving average (also called the running average) of a series

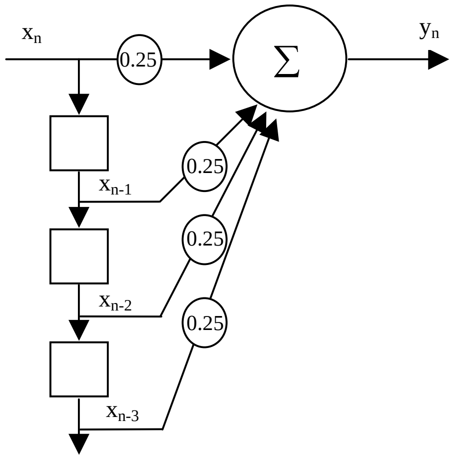
$$y(n) = \frac{1}{4}\left[x(n) + x(n-1) + x(n-2) + x(n-3)\right]$$

The counter is sometimes written as a subscript, as follows

$$y_n = \frac{1}{4}\left[x_n + x_{n-1} + x_{n-2} + x_{n-3}\right]$$

The expression is called a difference equation. In this example we form the average of the last 4 measurements.

A useful diagram to represent the implementation of that difference equation is



where the little squares represent the operator ‘unit sample time delay’.

Now say we have the following sequence and want to filter it:

x(n)	0	0	0	4	2	6	6	2	2	0	0	0	0	0
y(n)	?	?	?	?	?	?	?	?	?	?	?	?	?	?

we have done this kind of thing yesterday and the solution is:

x(n):	0	0	0	4	2	6	6	2	2	0	0	0	0	0
y(n):	?	?	?	1	1.5	3	4.5	4	4	2.5	1	0.5	0	0

Another example

Let’s now start with a continuous-time system described by its differential equation

$$x(t) = y(t) + \frac{dy(t)}{dt}$$

Let’s try and derive an equivalent discrete-time filter

If we say $\frac{dy(y)}{dt} \cong \frac{y_n - y_{n-1}}{T}$

then the discrete version of the filter becomes

$$x_n = y_n + \frac{y_n - y_{n-1}}{T}$$

$$x_n = \frac{Ty_n}{T} + \frac{y_n - y_{n-1}}{T}$$

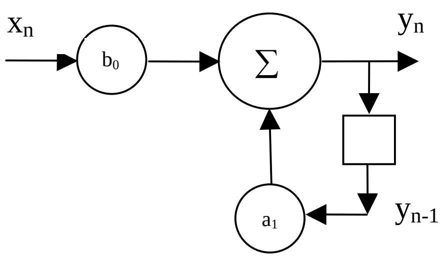
$$x_n = y_n \frac{T+1}{T} - y_{n-1} \frac{1}{T}$$

$$x_n \frac{T}{T+1} = y_n - y_{n-1} \frac{1}{T+1}$$

$$y_n = x_n \frac{T}{T+1} + y_{n-1} \frac{1}{T+1}$$ **, or**

$$y_n = b_0 x_n + a_1 y_{n-1}$$

Again, a useful diagram to represent the implementation of that difference equation is



Now we have 2 ways of describing discrete systems:

i. Impulse response and convolution sum;

ii. Difference equation or associated diagram.

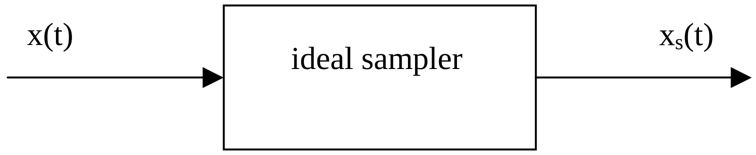
We saw (via the examples above) that difference equations are easy things to manipulate and we saw that one can easily obtain the output of a discrete-time system from its input by applying the difference equation sample by sample. It is easy but time consuming.

In the continuous-time domain we saw that the Laplace transform allowed us to move from convolution integrals in the time domain to products in the frequency domain. What we want now is something similar to the Laplace transform to help us deal with discrete-time systems and do multiplications in the frequency domain rather than convolutions in the time domain. This is the z-transform.

This would also change a differential equation into an algebraic equation and allow us to know the frequency response of discrete-time LTI systems.

The z-transform of a discrete signal

Consider the ideal sampler



where

$x_s(t) = x(t)d(t)$, and

$d(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$,

where δ(t) is the Dirac delta function, or, in 'modern' terminology, d(t) is simply a train of unit impulse samples and $x_s(t)$ can be written as

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Now let's apply the Laplace transform to the sampled signal $x_s(t)$.

$$\int_0^{\infty} x_s(t)\, e^{-st} dt = \int_0^{\infty} x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)\, e^{-st} dt$$

$$X_s(s) = \sum_{n=-\infty}^{\infty} x(nT) e^{-snT}$$

Define $e^{sT} = z$

$$X_s(s) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

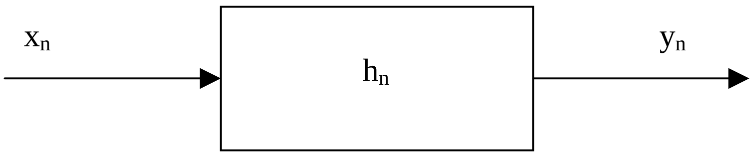
Since this is now a function of z and not of s, we will call it *X(z)* and if there are no samples for negative time, or n<0, the summation can start at zero

$$X_s(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

This is the Laplace transform of the sampled signal.

OK, now, does it do us any good?

To answer that we have to look at the effect of z-transformation on the convolution sum and difference equations.



where h_n is the impulse response of the DSP system, or the response to a unit sample.

We saw that the output is given by

$$y(n)=\sum_{k=-\infty}^{\infty}x(k)h(n-k),$$ or, using the ‘lazy’ notation

$$y_n=\sum_{k=-\infty}^{\infty}x_k h_{n-k}$$

Let’s z-transform this

$$\sum_{n=-\infty}^{\infty}y_n z^{-n}=\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}x_k h_{n-k}z^{-n}$$

Let’s change variables: write *n-k=m*, *n=m+k* and leave all in function of *m* and *k*,

$$Y(z)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}x_k h_m z^{-(m+k)}$$

$$Y(z)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}x_k h_m z^{-m-k}$$

$$Y(z)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}x_k h_m z^{-m}z^{-k}$$

$$Y(z)=\sum_{k=-\infty}^{\infty}x_k z^{-k}\sum_{m=-\infty}^{\infty}h_m z^{-m}$$

$$Y(z)=X(z)H(z)$$

H(z) is called the z-transfer function of the system whose unit impulse response is h(n) and is equal to the z-transform of the unit impulse response.

The fundamental theorem of z-transforms

The fundamental theorem of z-transforms is that the if the z-transform of x_n is X(z) then the z-transform of the same series delayed by a number of samples, say x_{n-L} is given by $z^{-L}X(z)$.

$$ZT\left\{x_{n-k_0}\right\} = z^{-k_0} X(z)$$

This is analogous to the fundamental theorem of Laplace transforms that stated that

$$LT\left\{\frac{dx(t)}{dt}\right\} = sX(s)$$

OK, so what? What is the usefulness of this and why is it called 'fundamental' theorem?

The z-transfer function of a LTI system – from difference equation to H(z)

Using the fundamental theorem is very easy:

Say we have

$$y_n = y_{n-1} + x_n - x_{n-5}$$

And we want to find the z-transfer function, i.e., $H(z)$.

We know $H(z) = \frac{Y(z)}{X(z)}$

So let's apply the z-transform to the difference equation

$$Y(z) = Y(z)z^{-1} + X(z) - X(z)z^{-5}$$

$$X(z)\left[1 - z^{-5}\right] = Y(z)\left[1 - z^{-1}\right], \quad \text{and}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-5}}{1 - z^{-1}}$$

This is great because once we know the z-transfer function of a system we know its frequency response. We shall deal with this in the next few lectures.