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Abstract

We report the results of an experiment we performed to test the matching probabilities for the Ellsberg paradox predicted by the quantum decision model of al-Nowaihi and Dhami (2016). We find that the theoretical predictions of that model are in conformity with our experimental results. This supports the thesis that violations of classical (Kolmogorov) probability theory may not be due to irrational behaviour but, rather, due to inadequacy of classical probability theory for the description of human behaviour. Unlike earlier quantum models of the Ellsberg paradox, our model makes essential use of quantum probability. It is also more parsimonious than earlier models.

Keywords: Quantum probability; the Ellsberg paradox; the law of total probability; the law of reciprocity; the Feynman rules; cognitive limitations.

JEL Classification: D03 (Behavioral microeconomics: Underlying principles).

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Highlights

- We test a simple quantum decision model of the Ellsberg paradox and find its predictions in agreement with the evidence.
- We show that the Ellsberg paradox reemerges if we combine the behavioral assumption of this paper with classical (non-quantum) probability theory. Hence, this paper makes essential use of quantum probability.
- We suggest that, far from being paradoxical, the behaviour of subjects in Ellsberg experiments is consistent with rational behaviour[1]. Rather, we suggest that the Ellsberg paradox illustrates the inadequacy of classical (Kolmogorov) probability theory for describing human behaviour.

1 Introduction

Situations of ambiguity are pervasive in decision making. The most successful approach is probably that of *source dependence* (Abdellaoui et al., 2011; Kothiyal et al., 2014; Dimmock et al., 2015). In this paper, we investigate the potential of *quantum decision theory* (QDT) to provide an alternative explanation. We concentrate on the canonical example of ambiguity, namely, the Ellsberg paradox (Keynes, 1921; Ellsberg 1961, 2001).[2] The Ellsberg paradox has proved to be a particularly useful vehicle for testing models of ambiguity.

Consider the following version of the Ellsberg experiment due to Dimmock et al. (2015). This involves two urns: The known urn (K) contains kn balls of n different colors and k balls of each color. The unknown urn (U) also contains kn balls of the same n colors as urn K but in unknown proportions. The subject is presented with the following bet. Suppose l of the n colors are chosen to be winning colors (hence, urn K contains kl balls of the winning colors). The subject wins a prize if a randomly drawn ball from an urn is of the winning color. Which of the two urns (K or U) should the subject choose?

[1] In the sense that subjects are not committing logical errors. Of course, humans do commit logical errors. However, our point is that this is not needed to explain the Ellsberg paradox.

[2] See the Introduction of al-Nowaihi and Dhami (2016) for a review of quantum approaches to the Ellsberg paradox and section 3 of that paper for a review of classical (non-quantum) approaches to ambiguity.

By the heuristic of *insufficient reason* (or *equal a-priori probabilities*)[3], the probability of drawing a ball of a winning color out of urn K is $p = \frac{kl}{kn} = \frac{l}{n}$. Although experimental subjects do not know the proportions of the different colors in urn U, they have no reason to favour one proportion over another. Hence, by the heuristic of insufficient reason, they should assign the same probability, $p = \frac{l}{n}$, to drawing a ball of a winning color from urn U. It follows that they should have no reason to prefer K to U or U to K on probabilistic grounds. They should be *ambiguity neutral*. However, what is observed in Dimmock et al. (2015) is the following. Subjects prefer U for low p but K for high p, i.e., they are *ambiguity seeking* for low probabilities but *ambiguity averse* for high probabilities.[4] We shall call this behavior the *Ellsberg paradox*.[5]

Consider subject i. Keep the contents of urn U fixed, but construct a new known urn, K_i, with a known number, M_i, of balls of the winning colors such that subject i is indifferent between urns K_i and U. Then $m_i\left(p\right) = \frac{M_i}{kn} = \frac{M_i}{k(l/p)}$ is the *matching probability* of p for subject i. Note that the definition of $m_i\left(p\right)$ is operational and does not depend on the particular decision theory assumed for the subjects. Let there be N subjects and let $m\left(p\right) = \frac{1}{N}\sum_{i=1}^{N} m_i\left(p\right)$ be the average of matching probabilities across all subjects. In their empirical exercise, Dimmock et al. (2015) report $m\left(0.1\right) = 0.22$, $m\left(0.5\right) = 0.40$, $m\left(0.9\right) = 0.69$. Thus, on average, subjects are ambiguity seeking for low probabilities ($m\left(0.1\right) > 0.1$) but ambiguity averse for medium and high probabilities ($m\left(0.5\right) < 0.5$, $m\left(0.9\right) < 0.9$).

A simple quantum model of the Ellsberg paradox was introduced by al-Nowaihi and Dhami (2016). Their predicted matching probabilities, based on their quantum model, are $m\left(0.1\right) = 0.171$, $m\left(0.5\right) = 0.417$, $m\left(0.9\right) = 0.695$, which are close to those empirically observed by Dimmock et al. (2015). However, the mechanism used by Dimmock et al. (2015) is not incentive compatible. Specifically, Dimmock et al. (2015) constructed urn K_i as fol-

[3] *Insufficient reason* or *equal a-priori probabilities* is now commonly referred to as *indifference*. However, indifference has a well-established alternative meaning in economics. To avoid confusion, we shall use the older terminology.

[4] This terminology is in analogy to situations of risk, where a decision maker is risk averse (risk neutral, risk loving) if the certainty equivalent of a lottery is less (equal to, greater) than the expected value.

[5] Traditionally, the Ellsberg paradox is used to refer to ambiguity aversion only. Our usage is in conformity with Ellsberg's original usage (see Ellsberg, 2001) and recent scholarship (see Dimmock, et al., 2015).

lows. The ratio of the colors (whatever they are) in U were kept fixed. However, the ratio in K_i was varied until subject i declared indifference between K_i and U. It turns out that in this method of eliciting matching probabilities subjects have the incentive to declare a preference for U over K_i, even when the reverse is true. However, Dimmock et al. (2015) found no evidence in their data that this occurred.[6]

In this paper, we report the results of an experiment we performed using a new data set and the incentive compatible mechanism of Fox and Tversky (1995, study 2).[7] Our observed matching probabilities are in agreement with those predicted by al-Nowaihi and Dhami (2016).

If subjects do behave in Ellsberg experiments as predicted by quantum probability, then such behaviour is neither irrational nor paradoxical. Rather, it shows that classical (non-quantum) probability theory may be inadequate.

In *quantum decision theory* (QDT), unlike all other decision theories, events are not distributive, and this is the main difference between the two. Thus, in QDT the event "X *and* (Y *or* Z)" need not be equivalent to the event "(X *and* Y) *or* (X *and* Z)". On the other hand, in all other decision theories, these two events are equivalent. This non-distributive nature of QDT is the key to its success in explaining paradoxes of behaviour that other decision theories find difficult to explain. For example, *order effects*, the *Linda paradox*, the *disjunction fallacy* and the *conjunction fallacy*.[8] As a result of the non-distributive nature of QDT, the *law of total probability* does not generally hold. Instead, we use the *Feynman rules* and the *law of reciprocity*.[9]

We refer the reader to al-Nowaihi and Dhami (2016, section 4) for an introduction to the quantum concepts and tools needed for this paper. For an excellent book-length introduction to quantum decision theory, see Busemeyer and Bruza (2012). For papers examining the limits of standard quantum theory when applied to cognitive psychology, see Khrennikov et al. (2014)

[6] Dimmock et al. (2015, p26): "In chained questions, where answers to some questions determine subsequent questions, subjects may answer strategically (Harrison, 1986). In our experiment, this is unlikely. First, our subjects are less sophisticated than students. Second, it would primarily have happened in the end (only after discovery), at the 0.9 probability event, where it would increase ambiguity seeking. However, here we found strong ambiguity aversion".

[7] However, the Fox and Tversky (1995) method requires the elicitation of the subjects' utility functions (this is not required by the Dimmock et al., 2015, mechanism).

[8] See Busemeyer and Bruza (2012). In particular, sections 1.2, 4.1-4.3, 5.2 and 10.2.3.

[9] See Busemeyer and Bruza (2012), pp. 5, 13, 39.

and Basieva and Khrennikov (2015).

The rest of the paper is organized as follows. Section 2 gives the main stylized facts from Ellsberg experiments. Section 3 gives the main behavioral assumption of al-Nowaihi and Dhami (2016).[10] Proposition 1 of Section 4 shows that the Ellsberg paradox reemerges when this behavioral assumption is combined with classical (Kolmogorov) probability theory. Hence al-Nowaihi and Dhami (2016) makes essential use of quantum probability theory. Proposition 2 of Section 4 gives the main theoretical predictions of al-Nowaihi and Dhami (2016). Section 5 gives our experimental design. Sections 6 and 7 give our experimental results. Section 8 summarizes and concludes. Appendix A gives our experimental Instructions. Appendix B gives our post-experimental questionnaire. The proofs of our main theoretical result, Propositions 1 and 2, are given in Appendix C.

2 Stylized facts

The following are the main stylized facts of Ellsberg experiments.

1. *Insensitivity*: Subjects are ambiguity averse for medium and high probabilities but ambiguity seeking for low probabilities; see Dimmock et al. (2015) for a recent survey of the literature as well as their own experimental results.

2. *Exchangeability*: Subjects are indifferent between colors. Subjects are indifferent between being asked to choose a color first or an urn first (Abdellaoui et al., 2011).[11]

[10] This is important. No mathematical structure on its own will yield empirically testable predictions. For example, in Newtonian mechanics we need Newton's second law of motion and law of gravity, initial conditions and simplifying assumptions. Calculus on its own will not yield empirically testable predictions. In quantum mechanics we need, for example, the momentum operator to be $p_x = -i\frac{h}{2\pi}\frac{\partial}{\partial x}$ and we need to specify a Hamiltonian for the system. Hilbert space on its own is insufficient.

[11] As an example, suppose that there are 100 balls each in urn K and urn U. There are two colors in Urn K, black and white (so 50 balls of each color). In contrast, in Urn U, the two colors (black and white) are in unknown proportion. The subject is told that if a ball of the color of his choice is drawn from an urn of his choice, then he wins a prize \$z. We may now proceed in one of two alternative ways. (1) First ask the subject to choose a color, then an urn. (2) First ask the subject to choose an urn, then a color. Exchangeability requires the answers in the two methods to be identical.

3. *No error*: Suppose that a subject prefers one urn (K or U) over the other. It is then explained to the subject that, according to classical probability theory, she should have been indifferent. She is offered the chance to revise her assessment. Subjects usually decline to change their assessment (Curley et al., 1986).

4. *Salience*: Ambiguity aversion is stronger when the two urns are presented together than when they are presented separately (Fox and Tversky, 1995; Chow and Sarin, 2001, 2002).[12]

5. *Anonymity* (or *fear of negative evaluation*): Ambiguity aversion does not occur if subjects are assured that their choice between urn U and urn K is anonymous (Curley et al., 1986; Trautmann et al., 2008).

In this paper, we show that the model of al-Nowaihi and Dhami (2016) is in accord with stylized fact 1, *insensitivity*, both qualitatively and quantitatively. It is also in accord with stylized facts 2 (*exchangeability*) and 3 (*No error*).

It may also be in accord with stylized facts 4 (*salience*) and 5 (*anonymity*). Suppose l of the n colors are winning colors. If a subject is presented with the two urns separately, or if the choice is made anonymously, then, maybe, that subject simply uses the heuristic of insufficient reason to conclude that the probability of drawing a winning ball is $\frac{l}{n}$, whether the subject is choosing from urn K or urn U. However, if the subject is presented with urns K and U together, and the choice is not under anonymity, then the subject may feel compelled to reason it through. However, the detailed development and testing of this is beyond the scope of this paper.

[12] Even more strikingly, Fox and Tversky (1995) found that for probability $\frac{1}{2}$, subjects exhibited ambiguity aversion with the value of urn U remaining approximately the same but urn K revalued upwards. Chow and Sarin (2001, 2002) did not find this result, but did find that ambiguity aversion is more pronounced when subjects are presented with K and U together.

3 Behavioral assumption: Construction of urn U in the mind of a subject

The framing of information is vital in choices. Subjects often simplify complex problems before solving them (Dhami, 2016)[13]. For Ellsberg experiments, subjects are typically told that urn U contains the same number of balls of the same colors as urn K, but in unknown proportions. However, the term "unknown proportions" is not defined any further, which begs the question of how subjects perceive this term. Pulford and Colman (2008) provide strong evidence that this is too cognitively challenging for subjects and that subjects do not consider all possible distributions of balls in urn U.

Recall, from the Introduction, that the known urn (K) contains kn balls of n different colors and k balls of each color. The unknown urn (U) also contains kn balls of the same n colors as urn K but in unknown proportions. The subject is asked to select one of the urns (K or U). A ball is drawn at random from the urn chosen by the subject. Suppose that l of the n colors are the winning colors (hence, urn K contains kl balls of the winning colors).

al-Nowaihi and Dhami (2016) conjecture that subjects model "unknown proportions" in a simple way, as described below.

1. We replace colors by numerals (this is justified by stylized fact 2). Furthermore, we consider only two numerals: 1 and 2. The known urn K contains kn balls, kl of which are labeled "1" and $kn-kl$ are labeled "2". We shall adopt the heuristic of insufficient reason. Thus, ball 1 is drawn from K with probability $p = \frac{kl}{kn} = \frac{l}{n}$ and ball 2 is drawn from K with probability $1 - p = \frac{kn-kl}{kn} = \frac{n-l}{n}$.[14]

2. Point 1 allows us to consider urn K as simply having two balls– one of the balls, the winning ball labeled "1", is drawn with probability $p = \frac{l}{n}$. The only other remaining ball, labeled "2", is drawn with probability $1 - p = \frac{n-l}{n}$. To compare with the evidence reported in Dimmock et al. (2015), we are interested in $p = 0.1$, 0.5 and 0.9. Likewise urn U will also have two balls labeled 1 and 2 but the proportions will be unknown, as the following construction shows.

[13] Examples and analysis are provided throughout Dhami (2016); see, for instance, Part 7.

[14] This transformation is only for analytic convenience. In our experiments subjects are always presented with colored balls whose ratios match the probabilities.

3. A subject is presented with two urns, K and U. Urn K has two balls, labeled 1 and 2, while urn U is initially empty. We conjecture that in the mind of a subject urn U is constructed as follows. In two successive and independent rounds, a ball is drawn at random from urn K and placed in urn U without revealing the labels, 1 or 2, to the subject. At the end of each of the two rounds, the ball that was drawn from urn K is replaced with an identically labeled ball. At the end of the two rounds, urn U contains two balls. The possibilities are that both could be labeled 1, both could be labeled 2, or one could be labeled 1 and the other labeled 2.

4. A ball is drawn at random from whichever urn the subject chooses (K or U). The subject wins a monetary prize $v > 0$ if ball 1 is drawn but wins nothing if ball 2 is drawn.

5. Since we have two balls and two states, we work in a 4-dimensional space.

Our behavioral assumption about how a subject mentally constructs urn U, outlined above (in particular, point 3), will play an essential role in explaining the Ellsberg paradox. The question then arises whether this behavioral assumption can also explain the Ellsberg paradox when combined with classical (Kolmogorov) probability theory. Proposition 1, below, establishes that this is not the case (see section 9.3 for the proof).

Proposition 1 : *If the probability of drawing ball* 1 *from the known urn* K *is* p*, then the classical probability of drawing ball* 1 *from the unknown urn* U *is also* p*.*

Thus, even with our behavioral assumption, the classical treatment gives the same probability, p, of winning whether a subject chooses urn K or urn U. Hence, if a subject strictly prefers K to U (or U to K), then this subject is violating classical theory.

This is in contrast to other quantum explanations of the Ellsberg paradox (see the Introduction of al-Nowaihi and Dhami, 2016, for a review). These other explanations introduce auxiliary assumptions[15] that when combined with classical (non-quantum) probability theory can also explain the Ellsberg paradox.[16]

[15] All theories need auxiliary assumptions to produce empirically testable predictions.

[16] Busemeyer and Bruza (2012, section 9.1.2) conclude "In short, quantum models of

4 A quantum decision model of the Ellsberg paradox

We now give the result of the quantum model of al-Nowaihi and Dhami (2016), section 5.

Proposition 2 *(al-Nowaihi and Dhami, 2016): If the probability of drawing ball* 1 *from the known urn* K *is* p*, then the quantum probability of drawing ball* 1 *from the unknown urn* U *is*

$$Q(p) = \frac{5p^3 - 8p^2 + 4p}{2 - p}. \tag{1}$$

For the proof, see section 9.3, below. Suppose the contents of the unknown urn U are kept fixed but a new known urn, K_i, is constructed so that the probability of drawing ball 1 from urn K_i is now $Q(p)$. In section 4.1, below, we shall prove that subject i should be indifferent between U and K_i, i.e., $Q(p)$ is the *matching probability* of p.

From (1), we get

$$Q(0.1) = 0.17105,\ Q(0.5) = 0.41667,\ Q(0.9) = 0.69545, \tag{2}$$

in close agreement with the evidence given by Dimmock et al. (2015) and our own evidence given later in this paper.

The following results are easily established.

$$Q(0) = 0,\ Q(1) = 1.$$

$$Q(p) + Q(1-p) < 1 \text{ for all } 0 < p < 1.$$

$$\lim_{p\to 0} Q(p) = 0,\ \lim_{p\to 0} \frac{Q(p)}{p} = 2,\ \lim_{p\to 1} \frac{Q(p)}{p} = 1.$$

decision making can accommodate the Allais and Ellsberg paradoxes. But so can non-additive weighted utility models, and so these paradoxes do not point to any unique advantage for the quantum model". Furthermore, there is considerable arbitrariness in the choice of weights in weighted utility models. Hence they introduce flexibility at the cost of lower predictive power. In our model, we replace weights with quantum probabilities which are parameter-free. Thus, our explanation of the Ellsberg paradox is more parsimonious than all the other explanations.

$$p < 0.4 \Rightarrow Q(p) > p,\ p = 0.4 \Rightarrow Q(p) = p,\ p > 0.4 \Rightarrow Q(p) < p. \quad (3)$$

Note that (1) is parameter free. By contrast, the probably most successful approach to ambiguity, *source dependent theory* (Abdellaoui et al., 2011; Kothiyal et al., 2014; Dimmock et al., 2015), requires the specification of two probability weighting functions, $w_K(p)$ and $w_U(p)$, one for urn K and one for urn U. These require the estimation of at least two parameters. For example, using Prelec (1998) probability weighting functions, $w_K(p) = e^{-\beta_K(-\ln p)^{\alpha_K}}$ and $w_U(p) = e^{-\beta_U(-\ln p)^{\alpha_U}}$, requires estimating two parameters: $\alpha = \frac{\ln \beta_U - \ln \beta_K}{\alpha_K}$, $\beta = \frac{\alpha_U}{\alpha_K}$.[17]

4.1 Quantum probabilities are matching probabilities

If p is the probability of drawing ball 1 from the known urn K, then $Q(p)$, given by (1), is the quantum probability of drawing ball 1 from the unknown urn U. Let u_i be the utility function of a subject, i, participating in the Ellsberg experiment as perceived by the subject (recall section 3). Normalize u_i so that $u_i(0) = 0$. The subject wins the sum of money, $v > 0$, if ball 1 is drawn from the unknown urn U, but zero if ball 2 is drawn from that same urn. Hence, her projective expected utility (in the sense of La Mura, 2009) is

$$Q(p)\, u_i(v)\,. \quad (4)$$

Now construct a new known urn K_i from which ball 1 is drawn with probability $Q(p)$. Her projective expected utility is

$$Q(p)\, u_i(v)\,. \quad (5)$$

Hence, from (4) and (5), $Q(p)$ is the matching probability for p. Thus, subject i is ambiguity *averse*, *neutral* or *seeking* according to $Q(p)$ being *less than*, *equal to* or *greater than* p.

From (3), it then follows that:

$$\begin{aligned} p < 0.4 \Rightarrow Q(p) > p &: \text{ ambiguity seeking,} \\ p = 0.4 \Rightarrow Q(p) = p &: \text{ ambiguity neutral,} \\ p > 0.4 \Rightarrow Q(p) < p &: \text{ ambiguity averse.} \end{aligned}$$

[17] Let $u_i(v)$ be the utility of subject i, normalized so that $u_i(0) = 0$. From the definition of matching probabilities, we have $w_K(m(p))\, u_i(v) = w_U(p)\, u_i(v)$. This gives $\ln(-\ln m(p)) = \alpha + \beta \ln(-\ln p)$.

Thus, our model is in agreement with stylized fact 1 (*insensitivity*).

5 Experimental design

Our subjects were 295 undergraduate students from Qingdao Agricultural University in China. They attended 8 sessions; no one participated in more than one session. The experimental instructions are given in the Appendix A.

Our treatment was a paper-based classroom experiment. There were three tasks, *Task 1*, *Task 2* and *Task 3*, that were, respectively, designed to implement the three cases $p = 0.5$, $p = 0.1$, $p = 0.9$ (see (2)). Each task required two tables to be completed. The materials for each task were handed out at the beginning of that task and collected before the next task started.

In each task, there is one known urn (Box K) and one unknown urn (Box U). The composition of the 100 colored balls of k different colors in Box K is known; varying this composition gives us the three cases $p = 0.5$, 0.1, 0.9. Box U contains 100 colored balls of the same colors as in Box K, but in unknown proportions. The composition of Box U is randomly decided at the end of the experiment in the following way. Each ball is equally likely to be drawn. The random draw follows the uniform distribution. For example, in Task 2, there are in total 10 different colors. A priori, each color is equally likely to be drawn. Thus, at each stage of the construction of Box U, each color has a probability 0.1 of being the color of the next ball to be placed in Box U. There can be from 0 to 100 balls of any particular color but subject to the restriction that the total number of balls in Box U is 100 balls. The prize for drawing a winning-color ball is 10 *Yuan* whether it is drawn from Box K or Box U. We now explain the three tasks.

1. In Task 1, there are 50 purple balls and 50 yellow balls in Box K, and *purple* is the winning color ($p = 0.5$).[18] The decision maker is shown two tables. In Table 1, the choices are to express a preference to receive a monetary amount x for sure, express indifference between x or betting that a purple ball will be drawn from Box K, or express a preference for betting that a purple ball will be drawn from Box K. The monetary amount is varied from $x = 0$ to $x = 10$ and subjects have to state a

[18] These are the same colors as chosen by Dimmock et al. (2015).

choice in each case.[19] Box U has 100 balls that are either purple or yellow but the proportions are unknown; as explained above. Table 2 replaces Box K in Table 1 with Box U but it is otherwise identical. At the end of the experiment, one of the choices from Task 1 is picked at random to be played for real.

2. In Task 2, there are 10 different colors (including purple) in Box K, and *purple* is the winning color ($p = 0.1$). Box U has 100 balls of the same 10 colors but in unknown proportions. The remaining procedure is as described in Task 1.

3. In Task 3, there are 10 different colors (including purple) in Box K, and the winning color is any ball that is not purple ($p = 0.9$). Box U has 100 balls of the same 10 colors but in unknown proportions. The remaining procedure is as described in Task 1.

6 Experimental results

Consider a sample of N subjects. Choose a probability, p, for drawing a winning ball from urn K. For each of these N subjects, we elicit their matching probability. Find the matching probability, $m_i(p)$, for each subject, i, $i = 1, 2, ..., N$, the sample average, $m(p) = \frac{1}{N}\sum_{i=1}^{N} m_i(p)$ and the sample variance, $s^2 = \frac{1}{N}\sum_{i=1}^{N}(m_i(p) - m(p))^2$. The *t-statistic* is $t = \frac{m(p)-Q(p)}{s/\sqrt{N}}$, where $Q(p)$ is the quantum prediction.

It might not be surprising to see much unsystematic variability in the matching probabilities, $m_i(p)$, across the sample.[20] However, if our quantum model is correct, then, for large N, this unsystematic variability should be largely cancelled out in aggregate. Hence, we would expect t to be approximately normally distributed with mean 0 and variance 1.[21] For ease of reference, we give the critical values for each of the conventional significance levels (10%, 5%, 1%) for a two-tailed test for the standard normal distribution in Table 1, below.

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[19] The experiments were conducted in China, so the monetary amount is in units of Chinese Yuan.

[20] Sufficient conditions for this are that $m_i(p) = m(p) + \epsilon_i$, where $E(\epsilon_i) = 0$, $i = 1, 2, ..., N$ and ϵ_i and ϵ_j are identically and independently for $i \neq j$.

[21] See, for example, Chapter 5 of Wooldridge (2015).

Table 1: Significance levels and the corresponding critical values.

Significance level	Critical value
10%	± 1.64
5%	± 1.96
1%	± 2.58

We collected in total 19470 (= $11 \times 2 \times 3 \times 295$) data points.[22] There were 259, 262 and 263 consistent decision makers for the $p = 0.1$, $p = 0.5$ and $p = 0.9$ cases, respectively[23]. We estimated the cash equivalents for the decisions in the two tables in Appendix A in the following way. If there is one unique tick in the "Indifference" column in the table, then the cash equivalent is the corresponding amount of money s/he receives for sure (x); On the other hand, if there is no tick in the "Indifference" column, then the cash equivalent is estimated by the midpoint between the lowest amount of money that is preferred to the uncertain bet, and the highest amount of money for which the bet was preferred; we are following the methodology in study 2 of Fox and Tversky (1995).

To find the matching probability with the cash equivalents that we obtained, it is necessary to assume a form for the utility function.[24] We use the power function[25] for the utility of player i,

$$u_i(x) = x^{\sigma_i}, x \geq 0, \sigma_i > 0. \tag{6}$$

Let v be the monetary payment to a subject if a winning ball is drawn. Let p be the probability of selecting a winning ball from the known urn (K).

[22]This indicates 11 data points (for the 11 rows of Tables 1 and 2; see Appendix A); 2 Tables corresponding to the known and unknown urns (Tables 1 and 2 in Appendix A); 3 tasks (Task 1, Task 2, and Task 3); and 295 subjects in the experiment.

[23]We discarded the inconsistent decision makers from the analysis as follows. We discarded data with the following two patterns: firstly, choosing more than once in the "Indifference" column in the table; secondly, choosing back and forth in any two or three columns. For the first case, we cannot identify the unique cash equivalent; while, it seems that the subjects with the second behavioral pattern don't show a clear ambiguity attitude. This left us with over 250 subjects.

[24]This is necessary in the methodology in study 2 of Fox and Tversky (1995); which is incentive compatible. It is not necessary in the methodology of Dimmock et al. (2015). However, the latter is not incentive compatible.

[25]The power form of the utility function is a popular choice (Kahneman and Tversky, 2000). For an axiomatization, see al-Nowaihi et al. (2008).

Let $m_i(p)$ be the matching probability, for subject i, of selecting a winning ball from the unknown urn (U). Additionally, the monetary valuation of the known urn (K) to subject i is denoted by v_{iK}, while the monetary valuation of the unknown urn (U) to subject i is denoted by v_{iU}. v_{iK} and v_{iU} are respectively the cash equivalents in the corresponding tables (recall the cash equivalents explained above).

Firstly, for the known urn (K), we have

$$(v_{iK})^{\sigma_i} = p(v)^{\sigma_i}. \tag{7}$$

Solve it for σ_i, to get

$$\sigma_i = \frac{-\ln p}{\ln v - \ln v_{iK}}, \tag{8}$$

where all quantities on the right hand side are known. Therefore, σ_i can be calculated using known quantities. Specifically, $v = 10$ *Yuan*; $p = 0.1$, $p = 0.5$ or $p = 0.9$ in the three cases; v_{iK} is the cash equivalent that we determine from the experiment.[26]

Similarly, for the unknown urn (U), we have

$$(v_{iU})^{\sigma_i} = m_i(p)(v)^{\sigma_i}. \tag{9}$$

Solve for $m_i(p)$, to get

$$m_i(p) = (\frac{v_{iU}}{v})^{\sigma_i}. \tag{10}$$

Substitute from (8) into (10) to get

$$m_i(p) = (\frac{v_{iU}}{v})^{\frac{-\ln p}{\ln v - \ln v_{iK}}}. \tag{11}$$

Since all quantities on the right hand side of (11) are known, the matching probability can be found (recall v_{iU} is the cash equivalent). Following this approach, we find the mean matching probabilities, $m(p)$, and standard deviations, which are listed in Table 2, below. The fifth column of Table 2 shows the theoretical predictions for the three matching probabilities, $Q(0.1)$, $Q(0.5)$, and $Q(0.9)$, respectively.[27]

[26] One subject chose $v_{iK} = v = 10$, for $p = 0.9$. Since the denominator in (8) would then be zero for these values, we discarded this observation.

[27] The theoretically predicted values are found by substituting the values of p, 0.1, 0.5, and 0.9, respectively, into (1).

Table 2: t-test for the means.

Matching probability	Mean	Standard deviation	Sample size	Quantum probability Q	t-stat
$m\,(0.1)$	0.1864	0.1708	259	0.1711	1.4437
$m\,(0.5)$	0.4038	0.1416	262	0.4167	-1.4758
$m\,(0.9)$	0.7258	0.2056	263	0.6955	2.3906

Table 2, below, shows that the theoretically predicted matching probabilities are quite close to the mean values we obtained from our experiments.

Our null and alternative hypotheses are: $H_0 : m\,(p) = Q\,(p)$ and $H_1 : m\,(p) \neq Q\,(p)$. From Tables 1 and 2, 1.4758, 1.4437, 2.3906 are all less than 2.58. Thus, our experimental results fail to reject our quantum model at the 1% level of significance. Since $m\,(0.1) > 0.1$, $m\,(0.5) < 0.5$, $m\,(0.9) < 0.9$, we find ambiguity seeking for the low probability but ambiguity aversion for the medium and high probabilities.

7 Demographic results

In their answers to question 8 on the post-experimental questionnaire (Appendix B), only 4 out of the 295 subjects reported that color affected their decisions. In their answers to question 6, almost none reported prior experience with similar experiments in the past. In their answers to question 4, Degree of study, all students simply gave "undergraduate", thus giving us no useful information. From the answers to question 3 (Field of study), we obtained the data for economics/non-economics. Not surprisingly, we found high colinearity between year of study and age, so we have not reported the latter.

7.1 Mann-Whitney U tests

We used two-sided Mann-Whitney U test (nonparametric test) to examine if the demographic characteristics in Appendix B affected the subjects' reported matching probabilities for $p = 0.1$, $p = 0.5$ and $p = 0.9$ in our treatment. The results are shown in Table 3. At the 1% level, no significant differences were found between any of the two groups (male/female;

Table 3: Mann-Whitney U test results.

Group	Matching probability	MWU p-value	Sig diff
	$m\,(0.1)$	0.9533	No
Male vs. Female	$m\,(0.5)$	0.2825	No
	$m\,(0.9)$	0.5205	No
	$m\,(0.1)$	0.8941	No
Econ vs. Non-econ	$m\,(0.5)$	0.7529	No
	$m\,(0.9)$	0.1230	No
	$m\,(0.1)$	0.0496	No*
Stats vs. Non-stats	$m\,(0.5)$	0.2053	No
	$m\,(0.9)$	0.7413	No
	$m\,(0.1)$	0.2944	No
Year 1 vs. Year 2	$m\,(0.5)$	0.3981	No
	$m\,(0.9)$	0.0546	No**
	$m\,(0.1)$	0.6826	No
Year 2 vs. Year 3	$m\,(0.5)$	0.8746	No
	$m\,(0.9)$	0.0245	No*
	$m\,(0.1)$	0.0998	No**
Year 1 vs. Year 3	$m\,(0.5)$	0.2693	No
	$m\,(0.9)$	0.4442	No

economics/non-economics students; statistics/non-statistics students).

Note: In Table 3, "No" denotes no significant difference at 1%; "No*" denotes difference significant at 5% but not at 1%; "No**" denotes difference significant at 10% but not at 1% nor 5%.

7.2 t-tests

For each demographic group, we also performed a t-test to see if the average reported matching probability, $m\,(p)$, differed significantly from the predicted value of the quantum probability, $Q\,(p)$. We report the results in Table 4, below. The only group that showed a significant difference was the group of students with prior training in statistics.

Table 4: t-test results.

Group	Matching probability	Mean	Standard deviation	Sample size	t-stat	Sig diff
	$m\,(0.1)$	0.1786	0.1296	23	0.2789	No
Econ	$m\,(0.5)$	0.4080	0.1706	23	-0.2449	No
	$m\,(0.9)$	0.7603	0.2315	20	1.2531	No
	$m\,(0.1)$	0.1871	0.1429	236	1.7275	No
Non-econ	$m\,(0.5)$	0.4126	0.1712	239	-0.3662	No
	$m\,(0.9)$	0.7227	0.2036	243	2.0881	No
	$m\,(0.1)$	0.1829	0.1362	116	0.9371	No
Male	$m\,(0.5)$	0.4239	0.1738	120	0.4557	No
	$m\,(0.9)$	0.7319	0.2046	124	1.9838	No
	$m\,(0.1)$	0.1892	0.1462	143	1.4846	No
Female	$m\,(0.5)$	0.4023	0.1682	142	-1.0181	No
	$m\,(0.9)$	0.7199	0.2071	139	1.3919	No
	$m\,(0.1)$	0.1773	0.1465	171	0.5579	No
Year 1	$m\,(0.5)$	0.4204	0.1729	173	0.2838	No
	$m\,(0.9)$	0.7195	0.2136	172	1.4763	No
	$m\,(0.1)$	0.1817	0.0939	27	0.5893	No
Year 2	$m\,(0.5)$	0.4070	0.1357	29	-0.3837	No
	$m\,(0.9)$	0.7820	0.1799	28	2.5457	No
	$m\,(0.1)$	0.2150	0.1576	60	2.1601	No
Year 3	$m\,(0.5)$	0.3909	0.1816	59	-1.0900	No
	$m\,(0.9)$	0.7134	0.1930	62	0.7323	No
	$m\,(0.1)$	0.2076	0.1555	126	2.638	Yes (1%)
Stat	$m\,(0.5)$	0.4043	0.1653	128	-0.847	No
	$m\,(0.9)$	0.7398	0.1710	130	2.957	Yes (1%)
	$m\,(0.1)$	0.1663	0.1243	133	-0.441	No
Non-stat	$m\,(0.5)$	0.4198	0.1762	134	0.206	No
	$m\,(0.9)$	0.7117	0.2344	133	0.800	No

Table 5: Comparison with classical probabilities.

Matching probability	Mean	Standard deviation	Sample size	Classical probability	t-stat	Sig diff
$m(0.1)$	0.2076	0.1555	126	0.1	7.7672	Yes
$m(0.5)$	0.4043	0.1653	128	0.5	-6.55	Yes
$m(0.9)$	0.7398	0.1710	130	0.9	-10.682	Yes

To keep things in perspective, we report in Table 5 how well the classical prediction fairs against the evidence.

Since the absolute values of the t-statistics in Table 5 are large relative to the critical values (Table 1), it follows that the classical prediction is strongly rejected for students trained in statistics.

8 Summary and conclusions

In this paper, we reported the results of our tests of the matching probabilities predicted by the quantum model of al-Nowaihi and Dhami (2016). These predicted matching probabilities agreed with those we observed (section 6). According to our Mann-Whitney U-tests, none of the demographic characteristics were significant. The only demographic characteristic we found to be significant according to our t-tests was a prior training in statistics (section 7). However, even for these students, the quantum prediction is far closer to the evidence than the classical prediction.

We showed (Proposition 1) that the Ellsberg paradox reemerges if we combine the behavioral assumption of this paper with classical (non-quantum) probability theory. Hence, unlike earlier quantum models, this paper makes essential use of quantum probability.

Our derivation is parameter free, recall (1). Thus our model is more parsimonious than any of the alternatives. Our model is in accord with stylized facts 1 (*insensitivity*), 2 (*exchangeability*) and 3 (*no error*), see section 2. At the end of section 2, we suggested It may also be in accord with stylized facts 4 (*salience*) and 5 (*anonymity*); however, this could be a topic for future research.

9 Appendices

9.1 Appendix A: Experimental Instruction (translation from Chinese instruction)

General information on the experiment

You are now participating in an economic experiment. If you read the following explanations carefully, you may be able to earn some money depending on your decisions. You will receive 5 Yuan for participation. This is irrespective of your decisions in the experiment. During the experiment you are not allowed to communicate with other participants in any way. If you have questions, please raise your hand, and the experimenter will come to your desk. The experiment will be carried out only once.

This experiment is paper based. there are three tasks: Task 1, Task 2 and Task 3. In each task, there are two boxes- Box K and Box U, and each box contains 100 colored balls. The composition of the balls is known for Box K but unknown for Box U. After you complete a task, the experimenter will collect the materials for that task and you will receive the materials for the next task.

Task 1:

There are 50 *purple* balls and 50 *yellow* balls in Box K. For each of the eleven rows in Table 1, tick *exactly one* of the following boxes: "Receive x Yuan for sure", "Indifferent" or "Play Box K".

Box U contains 100 balls (purple or yellow) but in *unknown* proportions. Thus Box U can contain any number of purple balls from 0 to 100 and any number of yellow balls from 0 to 100 provided the sum of balls (purple plus yellow) is 100. The composition of Box U will be randomly decided at the end of the experiment. For each of the eleven rows in Table 2, tick *exactly one* of the following boxes: "Receive x Yuan for sure", "Indifferent" or "Play Box U".

In each table, if you believe that you are indifferent between the choice in the left column and the right column, you may tick the box under the middle column "Indifferent".

At the end of the experiment, one of the eleven rows of Table 1 or one of the eleven rows of Table 2 will be selected at random and played for real money. In Table 1, you will receive x Yuan for sure if you have ticked the box under "Receive x Yuan for sure" or, if you have ticked the box under "Play Box K", you will win 10 *Yuan* if a *purple* ball is drawn from Box K

(otherwise you win nothing). In Table 2, you will receive x Yuan for sure if you have ticked the box under "Receive x Yuan for sure" or, if you have ticked the box under "Play Box U" you will win 10 Yuan if a *purple* ball is drawn from Box U (otherwise you win nothing). In each table, if you have ticked "Indifferent" in the randomly selected row, then one of the left or right cells in this selected row will be randomly chosen to play for real.

Table 1		
Receive x Yuan for sure	Indifferent	Play Box K
$x = 10$ ☐	☐	☐
$x = 9$ ☐	☐	☐
$x = 8$ ☐	☐	☐
$x = 7$ ☐	☐	☐
$x = 6$ ☐	☐	☐
$x = 5$ ☐	☐	☐
$x = 4$ ☐	☐	☐
$x = 3$ ☐	☐	☐
$x = 2$ ☐	☐	☐
$x = 1$ ☐	☐	☐
$x = 0$ ☐	☐	☐

Table 2		
Receive x Yuan for sure	Indifferent	Play Box U
$x = 10$ ☐	☐	☐
$x = 9$ ☐	☐	☐
$x = 8$ ☐	☐	☐
$x = 7$ ☐	☐	☐
$x = 6$ ☐	☐	☐
$x = 5$ ☐	☐	☐
$x = 4$ ☐	☐	☐
$x = 3$ ☐	☐	☐
$x = 2$ ☐	☐	☐
$x = 1$ ☐	☐	☐
$x = 0$ ☐	☐	☐

After you complete Task 1, the experimenter will collect the materials for Task 1 and you will receive the materials for Task 2.

Task 2:

There are 100 balls of 10 different colors (including *purple*) in Box K. There are exactly 10 balls of each color. For each of the eleven rows in Table 1, tick *exactly one* of the following boxes: "Receive x Yuan for sure", "Indifferent" or "Play Box K".

Box U contains 100 balls of the same colors as in Box K but in *unknown* proportions. Thus, Box U could contain any number of purple balls from 0 to 100. And similarly for each of the other 9 colors (provided the sum of balls of all colors is 100). The composition of Box U will be randomly decided at the end of the experiment. For each of the eleven rows in Table 2, tick *exactly one* of the following boxes: "Receive x Yuan for sure", "Indifferent" or "Play Box U".

In each table, if you believe that you are indifferent between the choice in the left column and the right column, you may tick the box under the middle column "Indifferent".

At the end of the experiment, one of the eleven rows of Table 1 or one of the eleven rows of Table 2 will be selected at random and played for real money. In Table 1, you will receive x Yuan for sure if you have ticked the box under "Receive x Yuan for sure". However, if you have ticked "Play Box K", then you shall win 10 *Yuan* if a *purple* ball is drawn from Box K (otherwise you win nothing). In Table 2, you will receive x Yuan for sure if you have ticked the box "Receive x Yuan for sure". However, if you have ticked the box "Play Box U" then you win 10 *Yuan* if a *purple* ball is drawn from Box U (otherwise you win nothing). In each table, suppose that you ticked "Indifferent" in the randomly selected row, then one of the left or right cells in this selected row will be randomly chosen to play for real.

After you complete Task 2, the experimenter will collect the materials for Task 2 and you will receive the materials for Task 3.

Task 3:

As in task 2, there are 100 balls in Box K of 10 different colors (including *purple*). There are exactly 10 balls of each color. For each of the eleven rows in Table 1, tick *exactly one* of the following boxes: The box "Receive x Yuan for sure", "Indifferent" or "Play Box K".

As with task 2, Box U contains 100 balls of the same colors as in Box K but in *unknown* proportions. For each of the eleven rows in Table 2, tick *exactly one* of the following boxes: The box "Receive x Yuan for sure", "Indifferent" or "Play Box U".

In each table, if you believe that you are indifferent between the choice in the left column and the right column, you may tick the box under the middle column "Indifferent".

At the end of the experiment, one of the eleven rows of Table 1 or one of the eleven rows of Table 2 will be selected at random and played for real money. In Table 1, you will receive x Yuan for sure if you tick the box under "Receive x Yuan for sure". However, now if you have ticked "Play Box K", then you shall win 10 *Yuan* if a *non-purple* ball is drawn from Box K (otherwise you win nothing). In Table 2, you will receive x Yuan for sure if you tick the box "Receive x Yuan for sure". However, if you have ticked the box "Play Box U" then you win 10 *Yuan* if a *non-purple* ball is drawn from Box U (otherwise you win nothing). In each table, suppose that you tick "Indifferent" in the randomly selected row, then one of the left or right cells in this selected row will be randomly chosen to play for real.

After you have completed Task 3, the experimenter will collect the materials for Task 3 and the experiment will terminate.

9.2 Appendix B: Post-experimental Questionnaire

1. Age: ____ years old
2. Gender: (female/male)
3. Field of study: ______
4. Degree of study: ______
5. Year of study: ______
6. Have you participated in similar experiments in the past? (Yes/No)
7. Did you have statistics course(s) before? (Yes/No)
8. Does your preference of some particular color(s) affect your decisions?

A. No. B. Yes. Please specify how your preference of some particular color(s) affected your decisions below.

9.3 Appendix C

This appendix goes beyond al-Nowaihi and Dhami (2016) in two respects. First, we here give a proof of Proposition 1. Second, the role played by the *law of reciprocity* was only implicit in al-Nowaihi and Dhami (2016). Here we make it explicit.

Proposition 1 serves two purposes. First, it highlights the role played by the *law of total probability.* In general, the law of total probability, see (13) below, is not valid in quantum probability theory.[28] Instead, we use the *Feynman's rules*[29] and the *law of reciprocity*[30]. Second, Proposition 1 shows that our behavioral assumption, on its own, is not sufficient to explain the Ellsberg paradox; quantum probability is needed (recall the discussion at the end of section 3, above)

See al-Nowaihi and Dhami (2016, section 4) for the quantum foundations necessary for this appendix. See Busemeyer and Bruza (2012), chapter 2, pp. 28-98, for a more comprehensive account. In our case, working in the Hilbert space $\mathbb{C}^4$ gives the same results as working in $\mathbb{R}^4$, as can be verified by direct calculation. Hence, for simplicity, we shall work in the Hilbert space $\mathbb{R}^4$. Recall that the state of a quantum system is given by normalized vector, $\mathbf{s}$, in Hilbert space, i.e., $\mathbf{s}\dagger\mathbf{s} = (\mathbf{s}\dagger)\,\mathbf{s} = 1$, where $\mathbf{s}\dagger$ is the *conjugate transpose* of $\mathbf{s}$ (in our case, simply the transpose of $\mathbf{s}$, since we are working in $\mathbb{R}^4$).

9.3.1 States of urn U

In our experiments (as is usual in Ellsberg experiments), subjects were told that urn U contains the same number of balls of the same colors as urn K, but in unknown proportions. However, the term "unknown proportions" is not defined any further in the instructions to subjects in the experiments. Therefore, we need to conjecture how subjects may view urn U. Our conjecture is described in the construction of urn U in section 3, in which each of the two urns contains two balls labeled 1 and 2. Using this construction, we may define the following states that are based on the outcomes arising from this construction:

1. $\mathbf{s}_1$ is the state where ball 1 is drawn in each of the two rounds (each with probability p).

[28] See Busemeyer and Bruza (2012), chapter 1, p. 5.
[29] See Busemeyer and Bruza (2012), chapter 1, p13.
[30] See Busemeyer and Bruza (2012), chapter 2, p. 39.

2. $\mathbf{s}_2$ is the state where ball 1 is drawn in round one (probability p) then ball 2 is drawn in round two (probability $1-p$).

3. $\mathbf{s}_3$ is the state where ball 2 is drawn in round one (probability $1-p$) then ball 1 is drawn in round two (probability p).

4. $\mathbf{s}_4$ is the state where ball 2 is drawn in each of the two rounds (each with probability $1-p$).

urn U contains two balls labeled 1 if it is in state $\mathbf{s}_1$. It contains one ball labeled 1 and the other labeled 2 if it is either in state $\mathbf{s}_2$ or in state $\mathbf{s}_3$. In state $\mathbf{s}_4$ both balls are labeled 2. We represent these states in $\mathbb{R}^4$ by the orthonormal basis:

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $\mathbf{s}$ give the initial state of urn U (unknown composition). Then *Born's rule* leads to:

$$\mathbf{s} = p\mathbf{s}_1 + \sqrt{p\,(1-p)}\mathbf{s}_2 + \sqrt{p\,(1-p)}\mathbf{s}_3 + (1-p)\,\mathbf{s}_4, \qquad (12)$$

where there is a probability p^2 that ball 1 is drawn in each round (state $\mathbf{s}_1$), a probability $p\,(1-p)$ that ball 1 is drawn in round 1 then ball 2 is drawn in round 2 (state $\mathbf{s}_2$), a probability $p\,(1-p)$ that ball 2 is drawn in round 1 then ball 1 is drawn in round 2 (state $\mathbf{s}_3$) and, finally, a probability $(1-p)^2$ that ball 2 is drawn in each round (state $\mathbf{s}_4$).

Let the event that ball 1 is drawn from urn U be denoted by $\mathbf{t}$. We now calculate the probability of event $\mathbf{t}$ under the classical and the quantum treatments.

9.3.2 Classical (Kolmogorov) treatment (Proof of Proposition 1)

From points 1-4 of section 9.3.1, we see that starting from the initial state $\mathbf{s}$, to arrive at the state $\mathbf{t}$, where ball 1 is drawn, we must follow one of the following three paths:

1. $\mathbf{s} \to \mathbf{s}_1 \to \mathbf{t}$,

2. $\mathbf{s} \to \mathbf{s}_2 \to \mathbf{t}$,

3. $\mathbf{s} \to \mathbf{s}_3 \to \mathbf{t}$.

By the *law of total probability*, we then have:

$$P(\mathbf{t}) = P(\mathbf{t}|\mathbf{s}_1)\, P(\mathbf{s}_1) + P(\mathbf{t}|\mathbf{s}_2)\, P(\mathbf{s}_2) + P(\mathbf{t}|\mathbf{s}_3)\, P(\mathbf{s}_3)\,. \tag{13}$$

From points 1-4 of section 9.3.1, we see that $P(\mathbf{t}|\mathbf{s}_1) = 1$, $P(\mathbf{s}_1) = p^2$, $P(\mathbf{t}|\mathbf{s}_2) = \frac{1}{2}$, $P(\mathbf{s}_2) = p(1-p)$, $P(\mathbf{t}|\mathbf{s}_3) = \frac{1}{2}$, $P(\mathbf{s}_3) = (1-p)\,p$. Hence, from (13), we get:

$$P(\mathbf{t}) = p. \tag{14}$$

Hence, if the probability of drawing ball 1 from the known urn K is p, then the classical probability of drawing ball 1 from the unknown urn U is also p. This establishes Proposition 1.

9.3.3 Quantum treatment

In general, the law of total probability (13) is not valid in quantum probability theory. Instead, we use the *Feynman's rules* and the *law of reciprocity.*

Reciprocity Let $\mathbf{t}$ be the state where ball 1 is drawn from U. We wish to calculate the probability, $P(\mathbf{s} \to \mathbf{t})$, of the transition $\mathbf{s} \to \mathbf{t}$. By the quantum law of reciprocity, $P(\mathbf{s} \to \mathbf{t}) = P(\mathbf{t} \to \mathbf{s})$, both being equal to $(\mathbf{s}\dagger\mathbf{t})^2$.[31] But $P(\mathbf{t} \to \mathbf{s})$ is the probability of the state of U conditional on drawing ball 1 from U. Let $\mathbf{w}$ be this state. To find $\mathbf{w}$, we first project $\mathbf{s}$ onto the subspace spanned by $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$, then normalize. This gives

$$\mathbf{w} = \sqrt{\frac{p}{2-p}}\mathbf{s}_1 + \sqrt{\frac{1-p}{2-p}}\mathbf{s}_2 + \sqrt{\frac{1-p}{2-p}}\mathbf{s}_3. \tag{15}$$

Feynman's rules To arrive at the state, $\mathbf{w}$, the state of urn U conditional on ball 1 being drawn, we must follow one of the three paths:

1. $\mathbf{s} \to \mathbf{s}_1 \to \mathbf{w}$,

2. $\mathbf{s} \to \mathbf{s}_2 \to \mathbf{w}$.

[31] Recall we are working in a real Hilbert space. For a complex Hilbert space, we would have $P(\mathbf{s} \to \mathbf{t}) = P(\mathbf{t} \to \mathbf{s}) = (\mathbf{s}\dagger\mathbf{t})\,(\mathbf{s}\dagger\mathbf{t})^*$, where $(\mathbf{s}\dagger\mathbf{t})^*$ is the complex conjugate of $\mathbf{s}\dagger\mathbf{t}$.

3. $\mathbf{s} \to \mathbf{s}_3 \to \mathbf{w}$.

Using Feynman's first rule (single path), $A(\mathbf{s} \to \mathbf{s}_i \to \mathbf{w})$ $= A(\mathbf{s} \to \mathbf{s}_i)\, A(\mathbf{s}_i \to \mathbf{w})$, the relevant transition amplitudes are:
$A(\mathbf{s} \to \mathbf{s}_1) = \mathbf{s}\dagger\mathbf{s}_1 = p$, $A(\mathbf{s}_1 \to \mathbf{w}) = \mathbf{s}_1\dagger\mathbf{w} = \sqrt{\frac{p}{2-p}}$, $A(\mathbf{s} \to \mathbf{s}_1 \to \mathbf{w}) = \sqrt{\frac{p^3}{2-p}}$.
$A(\mathbf{s} \to \mathbf{s}_2) = \mathbf{s}\dagger\mathbf{s}_2 = \sqrt{p(1-p)}$, $A(\mathbf{s}_2 \to \mathbf{w}) = \mathbf{s}_2\dagger\mathbf{w} = \sqrt{\frac{1-p}{2-p}}$, $A(\mathbf{s} \to \mathbf{s}_2 \to \mathbf{w})$ $= (1-p)\sqrt{\frac{p}{2-p}}$.
$A(\mathbf{s} \to \mathbf{s}_3) = \mathbf{s}\dagger\mathbf{s}_3 = \sqrt{p(1-p)}$, $A(\mathbf{s}_3 \to \mathbf{w}) = \mathbf{s}_3\dagger\mathbf{w} = \sqrt{\frac{1-p}{2-p}}$, $A(\mathbf{s} \to \mathbf{s}_3 \to \mathbf{w})$ $= (1-p)\sqrt{\frac{p}{2-p}}$.

We shall treat the paths $\mathbf{s} \to \mathbf{s}_2 \to \mathbf{w}$ and $\mathbf{s} \to \mathbf{s}_3 \to \mathbf{w}$ as indistinguishable from each other but both distinguishable from path $\mathbf{s} \to \mathbf{s}_1 \to \mathbf{w}$. Our argument for this is as follows. The path $\mathbf{s} \to \mathbf{s}_1 \to \mathbf{w}$ results in urn U containing two balls labeled 1. This is clearly distinguishable from paths $\mathbf{s} \to \mathbf{s}_2 \to \mathbf{w}$ and $\mathbf{s} \to \mathbf{s}_3 \to \mathbf{w}$, each of which result in urn U containing one ball labeled 1 and one ball labeled 2. From examining urn U, it is impossible to determine whether this arose by selecting ball 1 first (path $\mathbf{s} \to \mathbf{s}_2 \to \mathbf{w}$), then ball 2 (path $\mathbf{s}\to\mathbf{s}_3 \to \mathbf{w}$), or the other way round.

We apply Feynman's second rule (multiple indistinguishable paths) to find the amplitude of the transition $\mathbf{s} \to \mathbf{w}$, via $\mathbf{s}_2$ or via $\mathbf{s}_3$. We add the amplitudes of these two paths. Thus, $A(\mathbf{s} \to \mathbf{w})$, via $\mathbf{s}_2$ or $\mathbf{s}_3$ is $A(\mathbf{s} \to \mathbf{s}_2 \to \mathbf{w}) + A(\mathbf{s} \to \mathbf{s}_3 \to \mathbf{w}) = 2(1-p)\sqrt{\frac{p}{2-p}}$. The probability of this transition is $\left(2(1-p)\sqrt{\frac{p}{2-p}}\right)^2 = \frac{4p(1-p)^2}{2-p}$. The probability of the transition $\mathbf{s} \to \mathbf{s}_1 \to \mathbf{w}$ is $\left(\sqrt{\frac{p^3}{2-p}}\right)^2 = \frac{p^3}{2-p}$. We apply Feynman's third rule (multiple distinguishable paths) to get the total probability of the transition $\mathbf{s} \to \mathbf{w}$, via all paths. We add the two probabilities. This gives $P(\mathbf{s} \to \mathbf{w}) = \frac{p^3}{2-p} + \frac{4p(1-p)^2}{2-p} = \frac{5p^3-8p^2+4p}{2-p}$.

Quantum probability Recall that $\mathbf{s}$ is the initial state of urn U, $\mathbf{t}$ is the state in which ball 1 is drawn and $\mathbf{w}$ is the state of urn U conditional on ball 1 having been drawn. We wish to calculate the probability, $P(\mathbf{s} \to \mathbf{t})$, of the transition $\mathbf{s} \to \mathbf{t}$. By the quantum law of reciprocity, $P(\mathbf{s} \to \mathbf{t}) = P(\mathbf{t} \to \mathbf{s})$. But $P(\mathbf{t} \to \mathbf{s})$ is the probability of the state of U conditional on drawing ball

1 from U. We have already calculated this to be $\frac{5p^3-8p^2+4p}{2-p}$.

Thus, if the probability of drawing ball 1 from the known urn K is p, then the quantum probability of drawing ball 1 from the unknown urn U is

$$Q(p) = \frac{5p^3 - 8p^2 + 4p}{2-p}.$$

This completes the proof of Proposition 2.

References

[1] Abdellaoui M. A., Baillon A., Placido L. and Wakker P. P. (2011). The rich domain of uncertainty: source functions and their experimental implementation. *American Economic Review* **101**(**2**). 695–723.

[2] al-Nowaihi A. and Dhami, S. (2016). The Ellsberg paradox: A challenge to quantum decision theory? *Journal of Mathematical Psychology.* DOI: 10.1016/j.jmp.2016.08003.1.

[3] al-Nowaihi A., Ian Bradley I. and Dhami S. (2008) The Utility Function Under Prospect Theory. *Economics Letters* 99, p.337–339.

[4] Basieva I. and Khrennikov A. (2015). On the possibility to combine the order effect with sequential reproducibility for quantum measurements. *Foundations of Physics* **45**(**10**). 1379-1393.

[5] Busemeyer J. R. and Bruza P. D. (2012). *Quantum Models of Cognition and Decision.* Cambridge University Press.

[6] Chow C. C. and Sarin R. K. (2001). Comparative Ignorance and the Ellsberg Paradox. *Journal of Risk and Uncertainty* **22**. 129–139.

[7] Chow C. C. and Sarin R. K. (2002). Known, Unknown and Unknowable Uncertainties. *Theory and Decision* **52**. 127–138.

[8] Curley S.P., Yates J.F. and Abrams R.A. (1986). Psychological sources of ambiguity avoidance. *Organizational Behavior and Human Decision Processes* **38**. 230–256.

[9] Dhami, S. (2016) Foundations of behavioral economic analysis. Oxford University Press: Oxford.

[10] Dimmock S. G., Kouwenberg R. and Wakker, P. P. (2015). Ambiguity attitudes in a large representative sample. *Management Science*. Article in Advance. Published Online: November 2, 2015. 10.1287/mnsc.2015.2198

[11] Ellsberg D. (1961). Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics* **75**. 643-669.

[12] Ellsberg D. (2001). *Risk, Ambiguity and Decision*. Garland Publishers, New York. Original Ph.D. dissertation: Ellsberg D. 1962. Risk, Ambiguity and Decision. Harvard, Cambridge, MA.

[13] Fox C. R. and Tversky A. (1995). Ambiguity aversion and comparative ignorance. *Quarterly Journal of Economics* **110**(**3**). 585–603.

[14] Harrison, G. W. (1986). An experimental test for risk aversion. *Economics Letters* **21**. 7–11.

[15] Kahneman D. and Tversky A. (2000). *Choices, Values and Frames*. Cambridge University Press.

[16] Keynes J. M. (1921). *A treatise on probability*. London: Macmillan Co.

[17] Khrennikov A., Basieva I., Dzhafarov E. N. and Busemeyer J. R. (2014). Quantum models for psychological measurement: An unsolved problem. PLoS ONE **9**(**10**) e110909.

[18] Kothiyal A., Spinu, V. and Wakker P. P. (2014). An experimental test of prospect theory for predicting choice under ambiguity. *Journal of Risk and Uncertainty* **48**(**1**). 1–17.

[19] La Mura P. (2009). Projective expected utility. *Journal of Mathematical Psychology* **53**(**5**), 408-414.

[20] Prelec D. (1998) The probability weighting function. *Econometrica* **60**. 497-524.

[21] Pulford B. D. and Colman A. M. (2008). Ambiguity aversion in Ellsberg urns with few balls. *Experimental Psychology* **55**(**1**). 31-37.

[22] Trautmann S. T., Vieider F.M. and Wakker P. P. (2008). Causes of ambiguity aversion: known versus unknown preferences. *Journal of Risk and Uncertainty* **36**(**3**). 225–243.

[23] Wooldridge J. M. (2015). *Introductory econometrics: A modern approach*. Nelson Education.