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Abstract

We show that rank dependent expected utility theory can explain the St. Petersburg paradox. This complements recent work by Blavatskyy (2005), Camerer (2005), Rieger and Wang (2006) and Pfiffelmann (2011).

Keywords: St. Petersburg paradox, Rank dependent expected utility theory.

JEL Classification: C60(General: Mathematical methods and programming); D81 (Criteria for Decision-Making under Risk and Uncertainty).

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1. Introduction

The St. Petersburg paradox runs as follows. Consider the lottery, L, that pays 2^n with probability 2^{-n}, $n = 1, 2, 3, \ldots$. The value of this lottery is infinite, yet people are prepared to pay only a (small) finite sum for it. This motivated Bernoulli (1738) to propose the logarithmic utility function, $u(x) = \ln x$, $x > 0$. Blavatskyy (2005) showed that the St. Petersburg paradox reemerges under standard parametrizations of cumulative prospect theory (Tversky and Kahneman, 1992). Camerer (2005) showed that the St. Petersburg paradox can be solved using loss aversion, with the reference point taken to be the price an individual is prepared to pay and by putting an upper bound on the amount that can be paid out. General results are given by Rieger and Wang (2006).[1] However, the example we give here falls outside their general results.[2] Since standard probability weighting functions overweight the small probabilities of large outcomes, one might expect the St. Petersburg paradox to reemerge under rank dependent expected utility theory (Quiggin, 1982, 1993). Here we show that this is not the case: Rank dependent expected utility theory resolves the St. Petersburg paradox in its original form.

2. Setup

The key to rank dependent expected utility theory is the monotonic transformation of the cumulative probability distribution of ranked outcomes. We give definitions and examples below.

Definition 1 : *By a probability weighting function we mean a strictly increasing function* $w : [0,1] \overset{onto}{\rightarrow} [0,1]$, $w(0) = 0$, $w(1) = 1$.

Example 1 : *The Prelec probability weighting function (Prelec, 1998) is given by* $w(0) = 0$, $w(p) = e^{-\beta(-\ln p)^{\alpha}}$, $p > 0$, $\alpha > 0$, $\beta > 0$.

The Prelec probability weighting function is a popular probability weighting function. It is parsimonious, fits the evidence and has an axiomatic foundation.[3] If $\alpha < 1$, then this function overweights (low) probabilities in the range $\left(0, e^{-\beta^{\frac{1}{1-\alpha}}}\right)$ and underweights (high) probabilities in the range $\left(e^{-\beta^{\frac{1}{1-\alpha}}}, 1\right)$; in fact, $\lim_{p\to 0} \frac{w(p)}{p} = \infty$ and $\lim_{p\to 1} \frac{1-w(p)}{1-p} = \infty$. For $\beta = 1$, the Prelec function overweights probabilities in the range $(0, e^{-1})$ and underweights probabilities in the range $(e^{-1}, 1)$, $e^{-1} \simeq 0.367\,88$, which is consistent with the evidence.

[1] See Pfiffelmann (2011) for a critical evaluation and a proposed new probability weighting function.

[2] For example, their first condition states that, for $\alpha > 0$, $\lim_{x\to\infty} \frac{u(x)}{x^{\alpha}} \in (0, \infty)$. However, $\lim_{x\to\infty} \frac{\ln x}{x^{\alpha}} = 0$.

[3] See al-Nowaihi and Dhami (2011) and Wakker (2010).

Definition 2 *(Rank dependent utility): Consider the lottery L that pays x_i with probability p_i, where $x_1 < x_2 < x_3 < ...$, $p_i \geq 0$, $\sum_{i=1}^{\infty} p_i = 1$. Let $u : \mathbb{R} \to \mathbb{R}$ be a strictly increasing utility function. Let the probability weighting function be w. Consider the decision weights*

$$\pi_i = w\left(\sum\nolimits_{j=i}^{\infty} p_j\right) - w\left(\sum\nolimits_{j=i+1}^{\infty} p_j\right), i = 1, 2, 3, ...$$

Then, the rank dependent expected utility of the lottery L to the decision maker is given by

$$RDU\left(L\right) = \Sigma_{i=1}^{\infty} \pi_i u\left(x_i\right).$$

Note that, since $w : [0, 1] \overset{onto}{\to} [0, 1]$ is strictly increasing, it transforms a cumulative distribution into another cumulative distribution. Hence, the decision weights, π_i, are probabilities.

Example 2 *(The St. Petersburg paradox): Consider the identity probability weighting function $w\left(p\right) = p$. Then $\pi_i = p_i$. Consider the identity utility function $u\left(x\right) = x$. Set $x_i = 2^i$ and $p_i = 2^{-i}$, $i = 1, 2, 3, ...$. Then Definition 2 gives the infinite expected value $E\left(L\right) = \Sigma_{i=1}^{\infty} \pi_i u\left(x_i\right) = \Sigma_{i=1}^{\infty} 2^{-i} 2^i = \Sigma_{i=1}^{\infty} 1 = \infty$. Although this lottery has an infinite expected value, people are prepared to pay only a (small) finite amount of money for it.*

Example 3 *(Bernoulli's resolution, 1738, of the St. Petersburg paradox): As in Example 2 but now take the utility function to be $u\left(x\right) = \ln x$. We then get the finite expected utility $EU\left(L\right) = \sum_{i=1}^{\infty} \frac{1}{2^i} \ln 2^i = \sum_{i=1}^{\infty} \frac{i}{2^i} \ln 2 = 2 \ln 2 \simeq 1.3863$.[4] The certainty equivalent of this is $e^{2 \ln 2} = 4$.[5]*

3. The St. Petersburg paradox under rank dependent expected utility theory

Take the probability weighting function to that of Prelec (Example 1) with $\alpha = \frac{1}{2}$ and $\beta = 1$, i.e., $w(p) = e^{-(-\ln p)^{\frac{1}{2}}}$. These values are consistent with the empirical evidence. For instance, Bleichrodt and Pinto (2000) find that $\alpha = 0.53$ and $\beta = 1.08$. Taking the probabilities to be $p_i = 2^{-i}$, $i = 1, 2, 3, ...$, as in Example 2, we get $\pi_i = w\left(\sum_{j=i}^{\infty} p_j\right) - w\left(\sum_{j=i+1}^{\infty} p_j\right) = w\left(\sum_{j=i}^{\infty} \frac{1}{2^j}\right) - w\left(\sum_{j=i+1}^{\infty} \frac{1}{2^j}\right)$

[4] Let $u_i = \frac{i}{2^i}$ then $\frac{u_{i+1}}{u_i} = \frac{1}{2}\left(1 + \frac{1}{i}\right) \to \frac{1}{2} < 1$ as $i \to \infty$. Hence, by D'Alembert's ratio test, $\sum_{i=1}^{n} \frac{i}{2^i}$ converges as $n \to \infty$. Thus, we can set $S = \sum_{i=1}^{\infty} \frac{i}{2^i}$. Hence, $2S = \sum_{i=1}^{\infty} \frac{i}{2^{i-1}} = \sum_{i=0}^{\infty} \frac{i+1}{2^i} = \sum_{i=0}^{\infty} \frac{i}{2^i} + \sum_{i=0}^{\infty} \frac{1}{2^i} = S + 2$. Hence, $S = 2$.

[5] Note that the monies allocated to evaluating the St. Petersburg lottery are separated from total wealth. Thus some form of mental accounting is used (Thaler, 1999). This is implicit in much of the literature.

$= w\left(\frac{1}{2^{i-1}}\right) - w\left(\frac{1}{2^i}\right) = e^{-\left[-\ln\left(\frac{1}{2^{i-1}}\right)\right]^{\frac{1}{2}}} - e^{-\left[-\ln\left(\frac{1}{2^i}\right)\right]^{\frac{1}{2}}} = e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} - e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}}$, $i = 1, 2, 3, \ldots$.

As in Examples 2 and 3, take the outcomes to be $x_i = 2^i$, $i = 1, 2, 3, \ldots$. Consider the lottery L that pays 2^i with probability 2^{-i}, $i = 1, 2, 3, \ldots$, as in the original St. Petersburg paradox. As in Example 3, take the utility function to be $u(x) = \ln x$, as in the original St. Petersburg paradox. From Definition 2, we get that the rank dependent expected utility of L is $RDU(L) = \Sigma_{i=1}^{\infty} \pi_i u(x_i) = \Sigma_{i=1}^{\infty} \left[e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} - e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}}\right] \ln 2^i$
$= \Sigma_{i=1}^{\infty} \left[e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} - e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}}\right] i \ln 2 = (\ln 2)\left[\Sigma_{i=1}^{\infty} i e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} - \Sigma_{i=1}^{\infty} i e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}}\right]$
$= (\ln 2)\left[\Sigma_{i=1}^{\infty}(i-1+1) e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} - \Sigma_{i=1}^{\infty} i e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}}\right]$
$= (\ln 2)\left[\Sigma_{i=1}^{\infty}(i-1) e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} + \Sigma_{i=1}^{\infty} e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} - \Sigma_{i=1}^{\infty} i e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}}\right]$
$= (\ln 2)\left[\Sigma_{i=0}^{\infty} i e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}} + \Sigma_{i=1}^{\infty} e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} - \Sigma_{i=1}^{\infty} i e^{-(\ln 2)^{\frac{1}{2}}(i)^{\frac{1}{2}}}\right] = (\ln 2)\Sigma_{i=1}^{\infty} e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}}$.

We want to show that the partial sums $(\ln 2)\Sigma_{i=1}^{n} e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}}$ converge, as $n \to \infty$, and find lower and upper bounds for $(\ln 2)\Sigma_{i=1}^{\infty} e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}}$. Let $f(x) = (\ln 2) e^{-(\ln 2)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}$, then $(\ln 2)\Sigma_{i=1}^{\infty} e^{-(\ln 2)^{\frac{1}{2}}(i-1)^{\frac{1}{2}}} = \Sigma_{i=1}^{\infty} f(i)$. By a version of Maclaurin's integral test we have $\Sigma_{i=1}^{n-1} f(i) + \int_{x=n}^{\infty} f(x)\,dx < \Sigma_{i=1}^{\infty} f(i) < \Sigma_{i=1}^{n} f(i) + \int_{x=n}^{\infty} f(x)\,dx$.[6] In particular, for $n = 2$, we get $f(1) + \int_{x=2}^{\infty} f(x)\,dx < \Sigma_{i=1}^{\infty} f(i) < f(1) + f(2) + \int_{x=2}^{\infty} f(x)\,dx$. Hence, $0.693\,15 + \int_{x=2}^{\infty} f(x)\,dx < \Sigma_{i=1}^{\infty} f(i) < 0.301\,48 + 0.693\,15 + \int_{x=2}^{\infty} f(x)\,dx$. Elementary calculus gives[7] $\int f(x)\,dx = -2\left[1 + (\ln 2)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}\right] e^{-(\ln 2)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}$ and, hence, $\int_{x=n}^{\infty} f(x)\,dx = 2\left[1 + (\ln 2)^{\frac{1}{2}}(n-1)^{\frac{1}{2}}\right] e^{-(\ln 2)^{\frac{1}{2}}(n-1)^{\frac{1}{2}}}$. In particular, $\int_{x=2}^{\infty} f(x)\,dx = 2\left(1 + (\ln 2)^{\frac{1}{2}}\right) e^{-(\ln 2)^{\frac{1}{2}}}$ $= 1.594\,1$. Hence, $2.287\,3 < \Sigma_{i=1}^{\infty} f(i) < 2.588\,7$, i.e., $2.287\,3 < RDU(L) < 2.588\,7$. Let $CE(L)$ be the certainty equivalent of the lottery L under rank dependent expected utility. Then, since the utility function is logarithmic, we get $\ln(CE) = RDU(L)$. Hence, $CE(L) = e^{RDU(L)}$. The bounds on $RDU(L)$ derived above then give $e^{2.287\,3} < CE(L) < e^{2.588\,7}$, i.e., $9.848\,3 < CE(L) < 13.312$. Hence, a crude estimate of $CE(L)$ is $CE(L) \simeq \frac{9.848\,3 + 13.312}{2} \simeq 11.58$. We can get an arbitrarily high degree of accuracy by taking n to be sufficiently large. For example, $n = 18$ gives $11.438 < CE(L) < 11.697$ and $CE(L) \simeq 11.568$.

4. Summary

We have addressed the St. Petersburg paradox using rank dependent expected utility theory with the Prelec probability weighting function. For the latter, we used the parameter values $\alpha = \frac{1}{2}$ and $\beta = 1$. These are consistent with the parameter estimates $\alpha = 0.53$ and

[6] It is easy to give a rigorous proof. The intuition behind the proof can be seen by sketching a diagram.

[7] Differentiate $-2\left[1 + (\ln 2)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}\right] e^{-(\ln 2)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}$ to get $f(x)$.

$\beta = 1.08$ of Bleichrodt and Pinto (2000). We have used the logarithmic utility function as in Bernoulli (1738) and as is common in much of the literature. These assumptions give a certainty equivalent close to 11.58 for the St. Petersburg lottery. According to Camerer (2005) "... when asked how much they would pay to buy such a gamble, people routinely report sums around $20". However, Pfiffelmann (2011) states "According to a number of experiments, the maximum an individual is willing to pay for this gamble is around $3". Thus our figure of 11.58 is well within the reported range for the sums people are willing to pay. This suggests that rank dependent expected utility theory has the potential to easily explain the St. Petersburg paradox.

5. References

al-Nowaihi, A., Dhami, S, 2011. Probability weighting functions, in: Cochran, J.J., Cox, Jr., L.A., Keskinocak, P., Kharoufeh, J.P., Smith, J.C. (Eds.), Wiley Encyclopaedia of Operations Research and Management Science. John Wiley and Sons, New Jersey, pp. 4201-4219.

Bernoulli, D., 1738. Exposition of a New Theory of the Measurement of Risk. English translation in Econometrica. 1954. 22, 23-36.

Blavatskyy, P.R., 2005. Back to the St. Petersburg paradox? Management Science. 51, 677-678.

Bleichrodt, H., Pinto, J.L., 2000. A Parameter–Free Elicitation of the Probability Weighting Function in Medical Decision Analysis. Management Science. 46, 1485–1496.

Camerer, C., 2005. Three Cheers - Psychological, Theoretical, Empirical - for Loss Aversion. Journal of Marketing Research. 42, 129-133.

Pfiffelmann, M., 2011. Solving the St. Petersburg Paradox in cumulative prospect theory: the right amount of probability weighting. Theory and Decision, 71: 325-341.

Prelec, D., 1998. The probability weighting function. Econometrica. 60, 497-528.

Quiggin, J., 1982. A Theory of Anticipated Utility. Journal of Economic Behaviour and Organization. 3, 323–343.

Quiggin, J., 1993. Generalized Expected Utility Theory. Kluwer Academic Publishers, Massachusetts.

Rieger, M.O., Wang, M., 2006. Cumulative prospect theory and the St. Petersburg paradox. Economic Theory. 28, 665-679.

Samuelson, P., 1977. St. Petersburg paradoxes: defanged, dissected, and historically described. Journal of Economic Literature. 15, 24-55.

Thaler, R.H., 1999. Mental accounting matters. Journal of Behavioral Decision Making. 12, 183-206.

Tversky, A., Kahneman, D., 1992. Advances in prospect theory: Cumulative representation of uncertainty. Journal of Risk and Uncertainty. 5, 297-323.

Wakker, P.P., 2010. Prospect Theory for Risk and Ambiguity. Cambridge University Press, Cambridge.