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Distributional Comparative Statics

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Abstract

Distributional comparative statics is the study of how individual decisions and equilibrium outcomes vary with changes in the distribution of economic parameters (income, wealth, productivity, distortions, information, etc.). This paper develops tools to address such issues. Central to the developments is a new condition called *quasi-concave differences* which implies concavity of the policy function in optimization problems. The results are used to show how Bayesian equilibria respond to increased individual uncertainty (less precise private signals); and to derive conditions for concavity of policy functions in general stochastic dynamic programming problems. The latter generalizes Carroll and Kimball (1996) to models with borrowing constraints in the spirit of Aiyagari (1994).

Keywords: Distributional comparative statics, concave policy functions, income distribution, inequality, uncertainty, Bayesian games, dynamic stochastic general equilibrium models, arg max correspondence.

JEL Classification Codes: C61, D80, D90, E20, I30.

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1 Introduction

Imagine that the players in a Bayesian game receive less precise private signals and consequently become more uncertain about their environment. In an arms race with incomplete information, for example, countries may be uncertain about arms' effectiveness and opponents' intentions — and the degree of uncertainty is likely to change over time. How will such changes in exogenous distributions affect the Bayesian equilibria, including the mean actions and the actions' variances? In the arms race, will increased uncertainty lead to an escalation or de-escalation in the accumulation of arms?

The objective of this paper is to develop the tools needed to address such questions. Distributional comparative statics (henceforth, DCS) is the study of how changes in exogenous distributions affect endogenous distributions in models with optimizing agents. Apart from the effect on equilibrium quantities following increased uncertainty in Bayesian games; the methods developed here are able to address a number of DCS questions from different areas of economics.

- Consider an income allocation model where the population of consumers have different incomes. A macroeconomist might wish to know under what conditions on fundamentals a Lorenz dominating increase in inequality will lead to an increase in per-capita savings, or reduce the variance of savings across the population.

- In a model of international trade characterized by firms with heterogenous productivities, a trade theorist might wish to know if increased dispersion of productivities increases total output, or increases the variance in output across the firms.

- In a model of an investment decision, a financial economist might wish to know if a mean-preserving spread to the random return increases or reduces the investment.

In the income allocation setting it has been known at least since Atkinson (1970) that dominating shifts in the Lorenz curve reduce or increase aggregate savings according to whether the savings function is concave or convex. More generally, concavity or convexity of the function which maps exogenous variables into endogenous ones (the policy function) is the key to answering DCS questions about mean-preserving spreads, second-order stochastic dominance, Lorenz or generalized Lorenz shifts.[1]

To make progress, therefore, we need conditions on the primitives of a decision problem under which the policy function is concave or convex. The main technical contribution of this paper is a theorem that provides precisely that.[2] The result turns out to be intuitive: Consider for a moment the concepts of risk aversion and risk love which may be distinguished by whether more "mixed" outcomes reduce or increase an agent's total payoff (this corresponds, respectively, to concavity and convexity of the Bernoulli utility function). In a similar manner, concavity and convexity of

[1] See Section 2 for further discussion of the relationship between DCS and convexity/concavity of policy functions.

[2] Currently, the only way to do this would be by repeated use of the implicit function theorem (IFT) in an attempt to pin down the policy function's second partial derivative. Section 2 explains why that approach will not work in many situations. In addition, it is always inferior to the tools developed in this paper: even if an IFT approach is feasible, it will be substantially easier to use the theorem described here.

policy functions can be distinguished by whether more "mixed" outcomes reduce or increase the *marginal* payoff. Indeed, from the definition of concavity/convexity and the conditions describing an optimum, it follows almost immediately that these conditions on the marginal payoff lead to concavity and convexity, respectively, of the policy function (Section 3). These conditions have exact mathematical expressions captured by intuitive conditions I call *quasi-concave differences* and *quasi-convex differences*, which can also be characterized explicitly via derivatives when the functions involved are sufficiently smooth. Operationally this puts the conditions on an equal footing with, say, concavity or supermodularity/increasing differences which can be established, respectively, via the Hessian criterion and the cross-partial derivatives test of Topkis (1978). Thus we arrive at a fully tractable theory on par with existing comparative statics methods such as the implicit function theorem and monotone methods (Topkis (1978), Milgrom and Shannon (1994), Quah (2007)).[3]

The paper begins in Section 2 by further motivating and exemplifying the distributional comparative statics agenda. It then turns to quasi-concave differences, discusses the intuitive content of the definition, and shows — first in the simplest possible setting (Section 3.1), then under more general conditions (Section 3.2)— that quasi-concave differences implies concavity of the policy function in an optimization problem. An appendix treats the issue under yet more general conditions where the decision vector is allowed to live in an arbitrary topological vector lattice (Appendix III). Section 3.3 contains a practitioner's guide to the results and several fully worked through examples. Section 4 tackles the Bayesian DCS question posed at the beginning of this introduction. Section 5 derives general conditions for concavity of policy functions in stochastic dynamic programming problems; and as a concrete application extends a contribution by Carroll and Kimball (1996) to a setting with borrowing constraints (Aiyagari (1994)). Such results play an important role for various distributional comparative statics questions in macroeconomics (Huggett (2004), Acemoglu and Jensen (2015)) and also drive the analysis of inequality (Section 2.2).

2 Motivation

This section discusses the significance of convex and concave policy functions in two leading distributional comparative statics problems. In each case it is also explained why existing methods based on the implicit function theorem are not well suited to dealing with the technical issues raised.

2.1 Decisions under Uncertainty

Consider a decision maker (henceforth, DM) with objective $u(x,z)$ where $z \in Z \subseteq \mathbb{R}$ is private information and $x \in X \subseteq \mathbb{R}$ the decision variable. An outsider is affected by the DM's action and has beliefs over z represented by a probability measure μ on Z. With independent private values

[3] Note that these methods are not particularly helpful for DCS: even if exogenous distributions are parameterized so that the implicit function theorem or monotone methods apply; the conclusion will be about the direction of change in a deterministic decision variable and not in its distribution. This observation also applies to the results of Athey (2002) although there it is not necessary to parameterize the exogenous distribution. See also Section 2.1.

(or an outsider whose only source of risk is the DM's action), the DM's strategy and the outsider's beliefs about z are relevant to the outsider only through the distribution over the DM's actions that they induce. This distribution is given by

$$\mu_x(A) = \mu\{z \in Z : g(z) \in A\}, \tag{1}$$

where A is any Borel set in X and $g : Z \to X$ is the policy function

$$g(z) = \arg\max_{x \in X} u(x, z). \tag{2}$$

Consider now a shift in the outsider's beliefs μ. For example, the outsider might become more uncertain about z (a mean-preserving spread to μ), or the outsider's beliefs could be subjected to first- or second-order stochastic dominance shifts. For the reader's convenience, the formal definitions follow (see also Shaked and Shanthikumar (2007) for an in-depth treatment of stochastic orders):

Definition 1 (Stochastic Orders) Let μ and $\tilde{\mu}$ be two distributions on the same probability space $(Z, \mathscr{B}(Z))$. Then:

- $\tilde{\mu}$ *first-order stochastically dominates* μ *if* $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ *for any increasing function* $f : Z \to \mathbb{R}$ such that the integrals are well-defined.
- $\tilde{\mu}$ *is a mean-preserving spread of* μ *if* $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ *for any convex function* $f : Z \to \mathbb{R}$ *such that the integrals are well-defined.*
- $\tilde{\mu}$ *is a mean-preserving contraction of* μ *if* $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ *for any concave function* $f : Z \to \mathbb{R}$ *such that the integrals are well-defined.*[4]
- $\tilde{\mu}$ *second-order stochastically dominates* μ if $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ for any concave and increasing function $f : Z \to \mathbb{R}$ such that the integrals are well-defined.
- $\tilde{\mu}$ dominates μ in the *convex-increasing order* if $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ for any convex and increasing function $f : Z \to \mathbb{R}$ such that the integrals are well-defined.

When μ shifts (to $\tilde{\mu}$) in accordance with one of these stochastic orders, the natural question is how the distribution of the DM's actions μ_x is affected. The following observations provide the answers.[5]

1. If g is increasing, any first-order stochastic dominance increase in μ will lead to a first-order stochastic dominance increase in μ_x.

[4] Note that $\tilde{\mu}$ is a mean-preserving contraction of μ if and only if μ is a mean-preserving spread of $\tilde{\mu}$.

[5] The observations follow directly the definitions (a detailed proof is provided in Appendix I). The statement that "μ increases" is a compact way of stating that μ is replaced with a distribution $\tilde{\mu}$ that dominates μ in the given stochastic order. Similarly, "μ decreases" means that μ is replaced with a distribution $\tilde{\mu}$ that is dominated by μ in the stochastic order.

2. If g is concave, any mean-preserving spread to μ will lead to a second-order stochastic dominance decrease in μ_x.

3. If g concave and increasing, any second-order stochastic dominance increase in μ will lead to a second-order stochastic dominance increase in μ_x.

4. If g is convex, any mean-preserving spread to μ will lead to a convex-increasing order increase in μ_x.

5. If g is convex and increasing, any convex-increasing order increase in μ will lead to a convex-increasing order increase in μ_x.

To establish the condition of Observation 1 we can use the implicit function theorem or monotone methods to prove that g is increasing. Existing results therefore fully enable us to deal with first-order stochastic dominance shifts in μ. But in all of the other cases, we need to show that g is either concave or convex to determine how the distribution of the DM's actions changes. Consider the special case where,

$$u(x,z) = \int U(x,\bar{z},z)\eta(\bar{z}), \tag{3}$$

which is the payoff function of a player in a Bayesian game when η is the distribution of the opponents' actions, and z the private signal which has distribution μ. This paper's results allow us to conclude that if U is differentiable with $D_x U(x,\bar{z},z)$ concave in (x,z) for almost every $\bar{z} \in Z$ then the policy function — and more generally, the policy correspondence — will be concave whether or not the objective is concave, or even quasi-concave (see Section 3.3). Simple and intuitive conditions are also available when the objective fails to be differentiable. By Observation 2 on the previous page we can therefore conclude, for example, that if the player becomes more uncertain (a mean-preserving spread to μ), then the distribution of his actions η_x will decrease in the sense of second-order stochastic dominance. In particular, his mean action will decrease and the variance of his action will increase. In Section 4, we shall see how such results pave the way for a satisfactory treatment of distributional comparative statics in Bayesian games.

Without this paper's results, repeated "brute force" use of the implicit function theorem (IFT) provides the only existing way to address the concavity of g in this situation. It is instructive to follow this line of reasoning for a moment. If $u(x,z)$ is sufficiently smooth, concavity of u is assumed, and the solution is interior for all $z \in Z$, the following first-order condition is necessary and sufficient for an optimum

$$(D_x u(x,z) =) \int_{\bar{z}\in\bar{Z}} D_x U(x,\bar{z},z)\eta(\bar{z}) = 0. \tag{4}$$

Assume in addition that the second derivative never equals zero (strict concavity). The IFT then determines x as a function of z, $x = g(z)$ where

$$g'(z) = -\left[\int_{\bar{z}\in\bar{Z}} D^2_{xx} U(g(z),\bar{z},z)\eta(\bar{z})\right]^{-1} \int_{\bar{z}\in\bar{Z}} D^2_{xz} U(g(z),\bar{z},z)\eta(\bar{z}).$$

To determine g'' we would then need to differentiate the right-hand-side expression with respect to z and substitute in for $g'(z)$. Evidently, this is a daunting task even in the relatively simple case we are looking at here. And it may or may not lead to any useful conditions.[6] Furthermore, it imposes a host of unnecessary technical assumptions — so even when sufficient conditions for concavity of g can be found, these will not be the most general conditions. When u is *not* strictly concave, or when it is not differentiable, such brute force is inapplicable (see the examples in Section 3.3 as well as section 2.2 below). Varying constraint sets (Section 3.2) similarly confound the IFT's usefulness.

2.2 Income Allocation Models and Inequality

Standard deterministic income allocation models with HARA period utility functions satisfy the conditions of Gorman (1953) and the consumption function will consequently be linear in income (Pollak (1971)).[7] However, introducing uncertainty changes the outcome (Carroll and Kimball (1996)). Consider the stochastic income allocation model with Bellman equation

$$v(rx+wz) \quad = \max_{y\in\Gamma(x,z)} \quad u\big(rx+wz-y\big)+\beta\int v(ry+wz')\eta(dz'). \tag{5}$$

Here v is the value function and $\Gamma(x,z)=\{y\in\mathbb{R}:-b\leq y\leq rx+wz\}$ is admissible savings given past savings x and labor productivity z which follows an *i.i.d.* process with distribution η. As usual r denotes the interest factor, and w the wage rate. When $b<+\infty$, we have a model with a borrowing constraint in the spirit of workhorse incomplete market models (see *e.g.*, Aiyagari (1994) and Acemoglu and Jensen (2015)).

Let $g(rx+wz)=\arg\max_{y\in\Gamma(x,z)} u\big(rx+wz-y\big)+\beta\int v(y,z')\eta(dz')$ denote the *savings function*, and $c(rx+wz)=rx+wz-g(rx+wz)$ the *consumption function* (assuming here without further elaboration that these are well-defined). Clearly, the savings function is convex if and only if the consumption function is concave. Carroll and Kimball (1996) prove that if u belongs to the HARA class, then the consumption function is concave if there is no borrowing constraint ($b=+\infty$) and if the period utility function has a positive third derivative (precautionary savings). Technically, the proof of Carroll and Kimball (1996) relies on Euler equations and repeated application of the implicit function theorem (brute force), and so requires at least thrice differentiability of the value function v. The proof well illustrates how difficult brute force is to apply in practice, requiring ingenious guesses (*e.g.*, the class of HARA utility functions), and highly sophisticated manipulations throughout the argument (witness the proof found in their paper).

With a borrowing constraint ($b<+\infty$), the value function will not be smooth, and there simply are no existing tools capable of dealing with this situation in general.[8] Using this paper's re-

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[6] This statement is true even with one-dimensional decision variables. With multi-dimensional decision variables as explored in Appendix III, brute force becomes excessively complicated and is rarely useful.

[7] In the income allocation setting, the period utility function u exhibits Hyperbolic Absolute Risk Aversion (HARA) if $\frac{u'''u'}{(u'')^2}=k$ for a constant $k\in\mathbb{R}$ (see Carroll and Kimball (1996)).

[8] Due to the importance of borrowing/liquidity constraints for much applied work, the same authors (Carroll and Kimball (2001)) as well as Huggett (2004) address the concavity question in a framework with borrowing constraints and establish concavity of the consumption function for three special cases of the general HARA class (CRRA, CARA, and quadratic utility, respectively).

sults, a simple and direct proof of the concavity of the consumption function becomes feasible relying only upon concavity of the value function. From this follows straight-forwardly that the consumption function will be concave for the general HARA class with or without borrowing constraints (Section 5). Also, the HARA class "pops out" endogenously from our general conditions — there is no guesswork involved, and no ingenuity is required. In fact, the general results of this paper are so effective in the stochastic dynamic programming setting that little additional effort is required to prove a result on the convexity/concavity of policy functions for stochastic dynamic programming problems at the level of generality of the text book treatment of Stokey and Lucas (1989). Thus we are able to address not just the previous income allocation problem but nearly any stochastic dynamic model one can think of applying in macroeconomics. This is the topic of Section 5.

Let us finish this section with a brief look at the importance of the previous results for the study of inequality. Let $W_i = r\,x_i + w\,z_i$ denote income of agent i at a given moment in time and consider a continuum of agents $i \in [0,1]$ with identical preferences but possibly different incomes. Let η_W denote the (frequency) distribution of income. Mean consumption/per-capita consumption is then given by

$$\int c(W_i)\eta_W(d\,W_i).$$

From Atkinson (1970) we know that mean-preserving spreads to η_W are equivalent to increases in inequality in the sense of Lorenz dominance. Thus with or without borrowing constraints, any Lorenz increase in inequality will lead to *lower* mean consumption if u is in the HARA class. Notice, however, that Observations 2-4 on page 3 allow us to go considerably further. Since the consumption function c is also increasing under standard conditions, the previous statement extends to generalized Lorenz dominance by Observation 3 on page 3.[9] Of more novelty, we can go beyond considerations of the mean. For example, by Observation 2, a Lorenz increase in inequality will lead to a second-order stochastic dominance decrease in the distribution of consumption when c is concave and increasing. So we can conclude not only that the mean will decrease, but also that the variance will increase. Since the variance is a measure of the *inequality of outcomes* as opposed to the inequality of opportunities embodied in the distribution of income, such conclusions are obviously interesting. More generally, the approach developed in this paper opens up a simple and effective way of analysing the role of inequality in a variety of economic models.

3 Concavity of Policy Functions

Motivated by the previous section, we now present the paper's main results on the concavity and convexity of policy functions. The first subsection considers the simplest case with a single decision variable, a fixed constraint set, and a strictly quasi-concave objective. In the second subsection, all of these restrictions are relaxed except for the dimensionality of the decision variable

[9] The generalized Lorenz curve is constructed by scaling up the Lorenz curve by the distribution's mean and is equivalent to second-order stochastic dominance shifts, see *e.g.* Dorfman (1979).

which is treated in Appendix III. The last subsection contains a user's guide to the results as well as examples.

3.1 The Simplest Case

Let $u : X \times Z \to \mathbb{R}$ be a payoff function where $x \in X \subseteq \mathbb{R}$ is a decision variable and $z \in Z$ a vector of parameters. It is assumed throughout that X is a convex set and that Z is a convex subset of a vector space. When the associated decision problem $\max_{x \in X} u(x,z)$ has a unique solution for all $z \in Z$, we can define the *policy function*

$$g(z) = \arg\max_{x \in X} u(x,z). \tag{6}$$

We know from Topkis (1978) that $g : Z \to X$ will be *increasing* if u exhibits increasing differences, *i.e.*, if $u(x+\delta,z)-u(x,z)$ is coordinatewise increasing in z for all $x \in X$ and $\delta > 0$ such that $x+\delta \in X$. The purpose of this section is to show that *concavity* of g is ensured by a closely related condition called quasi-concave differences.

Definition 2 (Quasi-Concave Differences) A function $u : X \times Z \to \mathbb{R}$ exhibits *quasi-concave differences* if $u(x+\delta,z)-u(x,z)$ is quasi-concave on $X \times Z$ for all $\delta > 0$ in a neighborhood of 0.

If in this definition $u(x+\delta,z)-u(x,z)$ is instead required to be quasi-convex, u exhibits *quasi-convex differences*, and this condition will be shown to instead imply that g is convex. Conveniently, u exhibits quasi-convex differences if and only if $-u$ exhibits quasi-concave differences, hence there is no reason to distinguish between the two cases in the following discussion. As the following lemma shows, quasi-concavity is easy to verify for sufficiently smooth functions.

Lemma 1 (Differentiability Criterion) Assume that $u : X \times T \to \mathbb{R}$ is differentiable in $x \in X \subseteq \mathbb{R}$. Then u exhibits quasi-concave differences if and only if the partial derivative $D_x u(x,z)$ is quasi-concave in $(x,z) \in X \times Z$.

Proof. Appendix II. ■

Since quasi-concavity of $D_x u(x,z)$ is a fully tractable condition, so is quasi-concave differences. The reader is referred to the "user's guide" in Section 3.3 for analytical examples and further details on how to establish quasi-concave differences in applications. Here we turn instead to the interpretation and the relationship with concave policy functions.

Increasing differences says that the larger is z, the larger is the agent's incremental payoff from choosing a larger x. Quasi-concave differences is a complementarity condition much in the same spirit, only now the action x must not be complementary with z, but with convex combinations of (x,z). To be precise, consider two points (x_1,z_1) and (x_2,z_2) and a convex combination of the two $(x_\lambda,z_\lambda)=(\lambda x_1+(1-\lambda)x_2, \lambda z_1+(1-\lambda)z_2)$, $\lambda \in (0,1)$. Now add an increment $\delta > 0$ to the decision variable of all three bringing us to $(x_1+\delta,z_1)$, $(x_2+\delta,z_2)$, and $(x_\lambda+\delta,z_\lambda)$. Quasi-concave differences then says that the incremental payoffs at the end-points are complementary with the incremental payoff at the convex combination:

$$u(x_\lambda+\delta, z_\lambda) - u(x_\lambda, z_\lambda) \geq \min\{u(x_1+\delta, z_1) - u(x_1, z_1), u(x_2+\delta, z_2) - u(x_2, z_2)\}. \tag{7}$$

It is intuitive that this type of complementarity implies a concave policy function. To see this pick $\lambda \in (0,1)$, $z_1, z_2 \in Z$, and let $g(z_1)$ and $g(z_2)$ denote the associated optimal decisions. Consider points at a distance $\delta > 0$ below these optimal decisions, *i.e.*, consider $x_1 = g(z_1) - \delta$ and $x_2 = g(z_2) - \delta$. See Figure 1 where z_λ and x_λ are the convex combinations defined in the previous paragraph (in particular, $x_\lambda = \lambda g(z_1) + (1-\lambda)g(z_2) - \delta$ and $x_\lambda + \delta = \lambda g(z_1) + (1-\lambda)g(z_2)$).

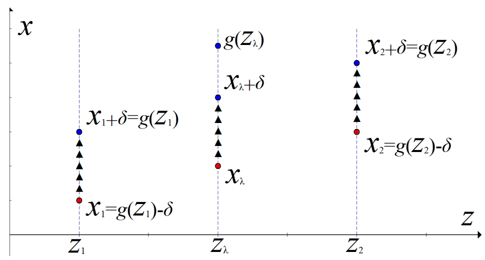

Figure 1: Quasi-concave Differences: complementarity between x and convex combinations of (x, z) implies that the arrows must point upwards at z_λ when the arrows point upwards at z_1 and z_2.

Since $g(z_1)$ is optimal given z_1, $u(\cdot, z_1)$ weakly increases between x_1 and $x_1+\delta$. Similarly $u(\cdot, z_2)$ weakly increases between x_2 and $x_2+\delta$. In the Figure this is illustrated by arrows pointing in the direction of weakly increasing payoffs. Since upward pointing arrows at the endpoints z_1 and z_2 means that the right-hand side of (7) is the minimum of non-negative numbers, quasi-concave differences immediately implies that $u(x_\lambda+\delta, z_\lambda) \geq u(x_\lambda, z_\lambda)$. So the arrows must also point upwards between between x_λ and $x_\lambda+\delta$ at the convex combination z_λ. Now, g is concave if $g(z_\lambda) \geq \lambda g(z_1) + (1-\lambda)g(z_2) = x_\lambda + \delta$. So graphically speaking, $g(z_\lambda)$ must lie above $x_\lambda+\delta$ on the vertical grid through z_λ. When u is quasi-concave in x, this follows immediately from the direction of the arrows at $x_\lambda+\delta$ (a quasi-concave function is always first non-decreasing and then non-increasing). It is clear now why complementarity between x and convex combinations of (x, z) — *i.e.*, quasi-concave differences — leads directly to a concave policy function. It does so by forcing $u(\cdot, z_\lambda)$ to be locally non-decreasing at $\lambda g(z_1) + (1-\lambda)g(z_2)$, thereby ensuring that any optimizer at z_λ is greater than or equal to $\lambda g(z_1) + (1-\lambda)g(z_2)$.

The proof below is essentially just a formalization of the previous graphical argument. There is, however, one complication related to solutions $g(z)$ touching the lower boundary of X, *i.e.*, solutions such that $g(z') = \inf X$ for some $z' \in Z$. In fact, such solutions will ruin any hope of obtaining a concave policy function for reasons that are easily seen graphically.

In Figure 2, we see a policy function which at z' touches the lower boundary $\inf X = 0$ of the constraint set $X = \mathbb{R}_+$, and stays at this lower boundary point as z is further increased. It is evident that the resulting policy function will *not* be concave, even if it is concave for $z \leq$ z'. As discussed at length in the working paper version of this paper (Jensen (2012)), this observation is quite general: concave policy functions and lower boundary optimizers *cannot* coexist save for some *very*

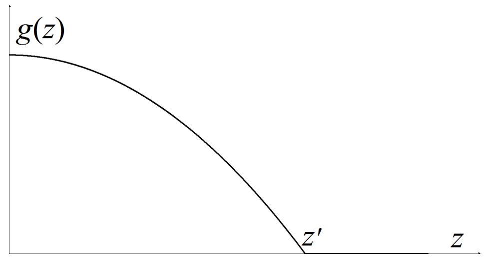

Figure 2: Concavity is destroyed when the policy function touches the lower boundary $\inf X = 0$.

pathological cases. Of course, there is no problem if the optimization problem is unconstrained below, *i.e.*, if $\inf X = -\infty$. Nor is there a problem if attention is restricted to interior optimizers (witness Figure 2 where we do have concavity when z is below z').

Theorem 1 (Concavity of the Policy Function) Let Z be a convex subset of a vector space, and $X \subseteq \mathbb{R}$ a convex subset of the reals. Assume that $u : X \times Z \to \mathbb{R}$ is strictly quasi-concave in x and that $g(z) = \arg\max_{x \in X} u(x,z) \neq \emptyset$ for all $z \in Z$. Then the policy function $g : Z \to X$ is concave if $u : X \times Z \to \mathbb{R}$ exhibits quasi-concave differences and $g(z) > \inf X$ for all $z \in Z$.

Proof. Pick $z_1, z_2 \in Z$ and let $x_1 = g(z_1)$ and $x_2 = g(z_2)$ be the optimal decisions. Since $x_1, x_2 > \inf X$, there exists $\delta > 0$ such that $u(x_1 - \delta, z_1) - u(x_1, z_1) \leq 0$ and $u(x_2 - \delta, z_2) - u(x_2, z_2) \leq 0$. Letting $\tilde{x}_q = x_q - \delta$ where $q = 1,2$, this can also be written $u(\tilde{x}_1, z_1) - u(\tilde{x}_1 + \delta, z_1) \leq 0$ and $u(\tilde{x}_2, z_2) - u(\tilde{x}_2 + \delta, z_2) \leq 0$. For $\lambda \in [0,1]$ set $\tilde{x}_\lambda = \lambda \tilde{x}_1 + (1-\lambda)\tilde{x}_2$ and $z_\lambda = \lambda z_1 + (1-\lambda) z_2$. By quasi-concave differences, $u(\tilde{x}_\lambda + \delta, z_\lambda) - u(\tilde{x}_\lambda, z_\lambda) \geq \min\{u(\tilde{x}_1 + \delta, z_1) - u(\tilde{x}_1, z_1), u(\tilde{x}_2 + \delta, z_2) - u(\tilde{x}_2, z_2)\} \geq 0$. Hence $u(\tilde{x}_\lambda + \delta, z_\lambda) - u(\tilde{x}_\lambda, z_\lambda) = u(x_\lambda, z_\lambda) - u(x_\lambda - \delta, z_\lambda) \geq 0$ where $x_\lambda = \lambda x_1 + (1-\lambda) x_2$. From strict quasi-concavity of u in x then follows that $u(\cdot, z_\lambda)$ must be non-decreasing on the interval $[\inf X, x_\lambda]$ and therefore that $g(z_\lambda) \geq x_\lambda = \lambda g(z_1) + (1-\lambda) g(z_2)$. ■

Corollary 1 (Convexity of the Policy Function) If in Theorem 1 instead of the last two conditions, it is assumed that $u : X \times Z \to \mathbb{R}$ exhibits quasi-convex differences and that $g(z) < \sup X$ for all $z \in Z$, then the policy function g is convex.

Proof. Apply Theorem 1 to the optimization problem $\max_{\tilde{x} \in -X} u(-\tilde{x}, z)$ and use that the policy function of this problem is concave if and only if g is convex. ■

3.2 Extensions

In situations such as the dynamic programming problems of Section 5, it is too restrictive to assume that the constraint set X is fixed. Further, one may face decision problems with multiple solutions possibly because the objective function is not quasi-concave in x. In such cases, we face the general decision problem

$$G(z) = \arg\max_{x \in \Gamma(z)} u(x,z). \tag{8}$$

Here $\Gamma : Z \to 2^X$ is the *constraint correspondence* and $G : Z \to 2^X$ is the *policy correspondence*. A policy function is now a selection from G, *i.e.*, a function $g : Z \to X$ with $g(z) \in G(z)$ for all $z \in Z$.

For this problem, the result of Topkis (1978) tells us that if u exhibits increasing differences and Γ is ascending then G is ascending. Conveniently, the conclusion of Theorem 1 generalizes in a very similar manner if we either maintain quasi-concavity of $u(\cdot, z)$ or else replace the definition of quasi-concave differences with the following global version:

Definition 3 (Global Quasi-Concave Differences) A function $u : X \times Z \to \mathbb{R}$ exhibits *global quasi-concave differences* if $u(x+\delta, z) - u(x, z)$ is quasi-concave on $X \times Z$ for all $\delta > 0$.

A function $u : X \times Z \to \mathbb{R}$ exhibits *global quasi-convex differences* if $-u$ exhibits global quasi-concave differences. The next lemma gives sufficient conditions for a function to exhibit global quasi-concave differences.

Lemma 2 (Global Differentiability Criterion) Assume that $u : X \times T \to \mathbb{R}$ is continuously differentiable in $x \in X \subseteq \mathbb{R}$. Then u exhibits global quasi-concave differences if the partial derivative $D_x u(x, z)$ is concave in $(x, z) \in X \times Z$.

Proof. By the fundamental theorem of calculus $u(x+\delta, z) - u(x, z) = \int_0^\delta D_x u(x+\tau, z)\, d\tau$. Since concavity is preserved under affine maps, $D_x(x+\tau, z)$ is concave for all $\tau \in \mathbb{R}$ when $D_x u(x, z)$ is concave. Since concavity is also preserved under integration, it follows that $u(x+\delta, z) - u(x, z)$ is concave in (x, z) for all $\delta > 0$. ■

Obviously, the constraint correspondence must also satisfy appropriate conditions. For this, consider the following:

Definition 4 (Concave Correspondences) A correspondence $\Gamma : Z \to 2^X$ is *concave* if for all $z_1, z_2 \in Z$, $x_1 \in \Gamma(z_1)$, $x_2 \in \Gamma(z_2)$, and $\lambda \in [0, 1]$, there exists $x \in \Gamma(\lambda z_1 + (1-\lambda) z_2)$ with $x \geq \lambda x_1 + (1-\lambda) x_2$.

By definition, $\Gamma : Z \to 2^X$ is said to be *convex* if $-\Gamma : Z \to 2^{-X}$ is concave. Convexity of correspondences was defined in Kuroiwa (1996) who also offers an extensive discussion of set-valued convexity. Concavity or convexity of Γ is not to be confused with either convexity of its values or convexity of its graph. A correspondence $\Gamma : Z \to 2^X$ is *convex-valued* if $\Gamma(z)$ is a convex subset of X for all $z \in Z$, and it has a *convex graph* if $\{(x, z) \in X \times Z : x \in \Gamma(z)\}$ is a convex subset of $X \times Z$. Neither concavity or convexity implies convex values (or vice versa). Meanwhile, convexity of a correspondence's graph is a much stronger requirement than convexity and concavity. In fact, a correspondence with a convex graph is both convex, concave, and has convex values.[10]

For most applications, the following result is sufficient to establish that a given constraint correspondence is concave or convex. Note that the result implies that in the frequently encountered case of inequality constraints where $\Gamma(z) = \{x \in X : \underline{\gamma}(z) \leq x \leq \overline{\gamma}(z)\}$, Γ is concave if and only if $\overline{\gamma}$ is concave; and Γ is convex if and only if $\underline{\gamma}$ is convex.

[10] That a correspondence with a convex graph has convex values is obvious. For the other claims, pick $x = \lambda x_1 + (1-\lambda) x_2 \in \Gamma(\lambda z_1 + (1-\lambda) z_2)$ in Definition 4.

Lemma 3 (Extremum Selection Criteria) If $\Gamma : Z \to 2^X$ admits a greatest selection, $\overline{\gamma}(z) \equiv \sup \Gamma(z) \in \Gamma(z)$ for all $z \in Z$, then Γ is concave if and only if $\overline{\gamma} : Z \to X$ is a concave function. Likewise, if Γ admits a least selection $\underline{\gamma}(z) \equiv \inf \Gamma(z) \in \Gamma(z)$ all $z \in Z$, Γ is convex if and only if $\underline{\gamma}$ is a convex function.

Proof. Only the concave case is proved. Since Γ is concave, we will for any $z_1, z_2 \in Z$, and $\lambda \in [0,1]$ have an $x \in \Gamma(\lambda z_1 + (1-\lambda)z_2)$ with $x \geq \lambda \underline{\gamma}(z_1) + (1-\lambda)\underline{\gamma}(z_2)$. Since $\overline{\gamma}(\lambda z_1 + (1-\lambda)z_2) \geq x$, $\overline{\gamma}$ is concave. To prove the converse, pick $z_1, z_2 \in Z$ and $x_1 \in \Gamma(z_1)$, $x_2 \in \Gamma(z_2)$. Since the greatest selection is concave, $x = \overline{\gamma}(\lambda z_1 + (1-\lambda)z_2) \geq \lambda \overline{\gamma}(z_1) + (1-\lambda)\overline{\gamma}(z_2) \geq \lambda x_1 + (1-\lambda)x_2$. Since $x \in \Gamma(\lambda z_1 + (1-\lambda)z_2)$, Γ is concave. ■

Theorem 2 (Concavity of the Policy Correspondence) Let Z be a convex subset of a vector space, and $X \subseteq \mathbb{R}$ a convex subset of the reals. Assume that $G(z) = \arg\max_{x \in \Gamma(z)} u(x,z) \neq \emptyset$, that $x > \inf \Gamma(z)$ whenever $x \in G(z)$, and that Γ is a concave correspondence with convex values. Then G is a concave correspondence if either (1) $u : X \times Z \to \mathbb{R}$ is quasi-concave in x and exhibits quasi-concave differences, or (2) $u : X \times Z \to \mathbb{R}$ exhibits global quasi-concave differences.

Proof. Pick any $z_1, z_2 \in Z$, $x_1 \in G(z_1)$, and $x_2 \in G(z_2)$. Setting $x_\lambda = \lambda x_1 + (1-\lambda)x_2$, and $z_\lambda = \lambda z_1 + (1-\lambda)z_2$, we must show that there exists an $x \in G(z_\lambda)$ with $x \geq x_\lambda$. Precisely as in the proof of Theorem 1, we conclude that $u(x_\lambda, z_\lambda) - u(x_\lambda - \delta, z_\lambda) \geq 0$ for some $\delta > 0$ under the first of the two alternative conditions, while under the second condition we may conclude that $u(x_\lambda, z_\lambda) - u(x_\lambda - \delta, z_\lambda) \geq 0$ and for all $\delta > 0$ with $x_\lambda + \delta \in X$. We are clearly done if there does not exist any $x \in \Gamma(z_\lambda)$ with $x < x_\lambda$. Assuming therefore that such an x exists, we can conclude from concavity and convex-values of Γ that $x_\lambda \in \Gamma(z_\lambda)$ (by convex values $[x, \tilde{x}] \subseteq \Gamma(z_\lambda)$ where $\tilde{x} \geq x_\lambda$ is in $\Gamma(z_\lambda)$ by concavity of Γ). Hence $u(x, z_\lambda) \geq u(x_\lambda, z_\lambda)$ when $x \in G(z_\lambda)$. In the case of the condition (2), it follows immediately that there must exist an $x \in G(z_\lambda)$ with $x \geq x_\lambda$. In the case of condition (1), the conclusion follows from quasi-concavity precisely as in the proof of Theorem 1. ■

Corollary 2 (Convexity of the Policy Correspondence) If in Theorem 2 it is instead assumed that Γ is a convex correspondence, that $x < \sup \Gamma(z)$ whenever $x \in G(z)$, and that u exhibits quasi-convex differences in (1) and global quasi-convex differences in (2), the conclusion becomes instead that G is a convex correspondence.

Proof. Apply Theorem 2 to the optimization problem $\max_{\tilde{x} \in -\Gamma(z)} u(-\tilde{x}, z)$ and use that the policy correspondence of this problem is concave if and only if G is convex. ■

If the conditions of Theorem 2 hold and the policy correspondence is single-valued $G = \{g\}$, then g must be a concave function by Lemma 3. Hence Theorem 1 is a special case of Theorem 2. From Lemma 3 also follows that when u is continuous so that G has compact values, and therefore admits a greatest selection, this greatest selection must be concave.[11] Finally, note that under the second alternative in Theorem 2, payoff functions need not be quasi-concave.

[11] Note that it is unreasonable to expect the *least* selection to be concave also. In fact, this would not characterize any reasonable concavity-type condition for a correspondence (in the case of a correspondence with a convex graph, for example, the greatest selection is concave and the least selection is convex).

3.3 A User's Guide and Some Examples

This subsection provides a practitioners' guide to Theorems 1 and 2. Focus will be on simple applications. For more involved applications, the reader is referred to the next two sections. We begin with a straight-forward consequence of Lemmas 1 and 2.

Lemma 4 (Quasi-Concave Differences for Thrice Differentiable Functions) *A thrice differentiable function $u : X \times Z \to \mathbb{R}$ where $X, Z \subseteq \mathbb{R}$ exhibits*

- *quasi-concave differences if and only if*

$$(9)\quad 2D^2_{xx}u(x,z)D^2_{xz}u(x,z)D^3_{xxz}u(x,z) \geq [D^2_{xx}u(x,z)]^2 D^3_{xzz}u(x,z) + [D^2_{xz}u(x,z)]^2 D^3_{xxx}u(x,z).$$

- *global quasi-concave differences if*

$$(10)\qquad D^3_{xxx}u(x,z) \leq 0\ \text{, and } D^3_{xxx}u(x,z)D^3_{xzz}u(x,z) - [D^3_{xxz}u(x,z)]^2 \geq 0.$$

Proof. (9) is the non-negative bordered Hessian criterion for quasi-concavity of $D_x u(x,z)$ (see *e.g.* Mas-Colell et al (1995), pp.938-939). By Lemma 1, this is equivalent to $u(x,z)$ exhibiting quasi-concave differences. (10) are the conditions for $D_x u(x,z)$ to have a negative semi-definite Hessian matrix. By Lemma 2, this implies global quasi-concave differences. ■

The previous Lemma makes it easy to apply Theorem 1 or Theorem 2 in many situations.

Heterogenous Firms in International Trade Models. Consider the model of Melitz (2003). Each firm in a continuum $[0,1]$ chooses output $x \geq 0$ in order to maximize profits. A firm with cost parameter $z > 0$, can produce x units of the output by employing $l = zx + f$ workers where $f > 0$ is a fixed overhead (Melitz (2003), p.1699).[12] The frequency distribution of the cost parameter z across the firms is η_z. With revenue function R, a firm with cost parameter z will choose $x \geq 0$ in order to maximize

$$(11)\qquad u(x,z) = R(x) - zx - f.$$

Let $G(z) = \arg\max_{x\geq 0}[R(x) - zx - f]$ denote the optimal output(s) given z. To show that G is concave or convex, we may apply Theorem 1 or the more general Theorem 2. For Theorem 1, $R(x) - zx - f$ must be strictly quasi-concave so that $G(z) = \{g(z)\}$ where g is the policy function. If (9) then holds, u exhibits quasi-concave differences and g is concave on $z \in \{z \in Z : g(z) > 0\}$, *i.e.*, when attention is restricted to the set of active firms. As the reader can easily verify, (9) holds if and only if

$$(12)\qquad R''' \leq 0.$$

So if the revenue function has a non-positive third derivative, the policy function is concave. It follows that average/aggregate output decreases when firms become more diverse (a mean preserving spread to the distribution η_z). Intuitively, this may be though of as "decreasing returns to diversity". If u exhibits quasi-convex differences (reverse the inequality (9) yielding the condition

[12] In terms of Melitz' notation, z is the inverse of the firm's productivity level φ.

$R''' \geq 0$), we instead get a convex policy function by Corollary 1 and so there is "increasing returns to diversity". By appealing to Observations 1-5 on page 3 one can further go on to predict how the distribution of the firms' outputs changes when η_z is subjected to mean-preserving spreads or other stochastic order changes. For example, by Observation 2, the distribution of the outputs of a more diverse set of firms will be second-order stochastically dominated by the distribution of outputs of a less diverse set of firms when (12) holds.

The limitations of Theorem 1 are evident in the current situation since monopolistically competitive firms' objectives will often not be strictly quasi-concave or even quasi-concave. One can then instead use Theorem 2. If $R(x) - zx - f$ is quasi-concave in x, (1) of Theorem 2 implies that G is a concave correspondence (Definition 4) under condition (12). If $R(x) - zx$ is not quasi-concave, (2) of Theorem 2 may be used and we then need to verify (10). The second condition in (10) is always satisfied since the Hessian matrix of $R'(x) - z$ is degenerate. The first condition in (10) holds if and only if (12) holds.[13] So by assuming only that solutions exist (so that G is well-defined) — and in particular with no quasi-concavity assumptions — we can conclude that G is a concave correspondence when the revenue function R has a non-positive third derivative. Hence the greatest selection from the policy correspondence will be a concave function (Lemma 3), and so the maximum aggregate output decreases with a mean-preserving spread in η_z.

If R is assumed to be strictly concave so that $R'' < 0$, we might alternatively have applied the implicit function theorem (IFT) to the first-order condition $R'(x) - z = 0$. This yields $x = g(z)$ where $D_z g = R''$. The IFT is particularly easy to use in this case, and we immediately see that (12) once again ensures concavity of g (this is because $D_z g$ decreasing precisely means that g is concave). Note, however, that Theorem 2 (and also Theorem 1) applies to many situations that the IFT approach is unable to address.

Decisions Under Uncertainty. Recall from Section 2.1 the agents' individual decision problem in a Bayesian game. We found there that the marginal payoff is $D_x u(x,z) = \int_{\bar{z}\in\bar{Z}} D_x U(x,\bar{z},z)\eta(\bar{z})$. Since integration preserves concavity, we immediately see from Lemma 2 that u exhibits global quasi-concave differences if $D_x U(x,\bar{z},z)$ is concave in (x,z) for *a.e.* $\bar{z}$. With sufficiently smooth U, this in turn holds if and only if U satisfies (10) for *a.e.* $\bar{z}$. This simple condition implies, then, that the policy correspondence is concave by Theorem 2 (if u is strictly quasi-concave in x this conclusion also follows from Theorem 1 since global quasi-concave differences implies quasi-concave differences).[14] See Section 2.1 for a discussion of the difficulties involved in applying the implicit function theorem in this example.

Investments with Random Returns and Ambiguity. As a final example, consider an agent who makes an investment $x \geq 0$ in a project whose expected return depends on a known signal z as well as a draw by nature among the possible states $\bar{z}_1, \ldots, \bar{z}_l$. The expected return is evaluated according to a Choquet/non-additive expected utility criterion with state payoff $U(\bar{z}_k, z)$ and capacities ν.

[13] Note that global quasi-concave differences is stronger than quasi-concave differences, and as shown the latter is equivalent to (12). Thus (12) is in fact both necessary and sufficient for u to exhibit global quasi-concave differences.

[14] See Lemma 5 in the next section for the non-differentiable version of the sufficient conditions for concavity of the policy function.

The cost $c(x,z)$ is assumed to be strictly convex and differentiable in x. The policy function is thus,

$$g(z) = \arg\max_x [x \min_{\bar{\mu}\in C(\nu)} \sum_{k=1}^{l} \bar{\mu}_k U(\bar{z}_k, z) - c(x,z)] \,,$$

where $C(\nu)$ denotes the core of ν.

If U is concave in z and $D_x c(x,z)$ is convex, then $\min_{\bar{\mu}\in C(\nu)} \sum_{k=1}^{l} \bar{\mu}_k U(\bar{z}_k, z) - D_x c(x,z)$ is concave in (x,z) (the first term is concave in z because the minimum of a family of concave functions is concave). By Lemma 1, it follows that the objective exhibits quasi-concave differences and by Theorem 1, $g: Z \to X$ is consequently concave. Since the first-order condition of this problem is not differentiable in z, the IFT cannot be used to reach this conclusion. In the dynamic programming set-up discussed in Section 2.2 and returned to in Section 5, the IFT cannot be used for similar reasons.

4 Distributional Comparative Statics in Bayesian Games

Consider a Bayesian game with a finite set of players $\mathscr{I} = \{1,\dots,I\}$. Player $i \in \mathscr{I}$ receives a private signal $z_i \in Z_i \subseteq \mathbb{R}$ drawn from a distribution μ_{z_i} on $(Z_i, \mathscr{B}(Z_i))$ where $\mathscr{B}(\cdot)$ denotes the Borel sets. With Bayesian equilibria defined as usual (see below), how will the set of equilibria be affected if one or more signal distributions μ_{z_i} are subjected to mean-preserving spreads or second-order stochastic dominance shifts? The purpose of this section is to use Theorem 1 to deal with this question and illustrate by means of a specific example (an arms race).

Assuming that private signals are independently distributed, an *optimal strategy* is a measurable mapping $g_i : Z_i \to X_i$ such that for almost every $z_i \in Z_i$,

$$g_i(z_i) \in \arg\max_{x_i \in X_i} \int_{z_{-i}\in Z_{-i}} u_i(x_i, g_{-i}(z_{-i}), z_i) \mu_{z_{-i}}(dz_{-i}) \,. \tag{13}$$

Here $X_i \subseteq \mathbb{R}$ is agent i's action set and $g_{-i} = (g_j)_{j\neq i}$ are the strategies of the opponents. A *Bayesian equilibrium* is a strategy profile $g^* = (g_1^*,\dots,g_I^*)$ such that for each player i, $g_i^* : Z_i \to X_i$ is an optimal strategy given the opponents' strategies $g_{-i}^* : Z_{-i} \to X_{-i}$. Obviously, $g_i : Z_i \to X_i$ is a policy function when it satisfies (13) for all $z_i \in Z_i$. The *optimal distribution* of an agent i is the measure on $(X_i, \mathscr{B}(X_i))$ given by:

$$\mu_{x_i}(A) = \mu_{z_i}\{z_i \in Z_i : g_i(z_i) \in A\} \,,\ A \in \mathscr{B}(X_i) \tag{14}$$

We begin by assuming continuity and risk aversion so that the policy functions and optimal distributions are uniquely determined:

Assumption 1 *For every i: X_i is compact and $u_i(x_i, x_{-i}, z_i)$ is strictly concave in x_i and continuous in (x_i, x_{-i}, z_i).*

Note that for given opponents' strategies this situation coincides with the decision under uncertainty example studied in Section 2.1. In particular, we know from that section how various stochastic order changes in μ_{z_i} affect the optimal distribution of the player μ_{x_i} when g_i is concave or convex (see 1-5 on page 3). Combining with Theorem 1 we immediately get:

Lemma 5 Consider a player $i \in \mathscr{I}$ and let Assumption 1 be satisfied.

1. If $\int_{z_{-i}\in Z_{-i}} u_i(x_i, g_{-i}(z_{-i}), z_i)\mu_{z_{-i}}(dz_{-i})$ exhibits quasi-concave differences in x_i and z_i, and no element on the lower boundary of X_i ($\inf X_i$) is optimal, then a mean-preserving spread to μ_{z_i} will lead to a second-order stochastic dominance decrease in the optimal distribution μ_{x_i}.
2. If $\int_{z_{-i}\in Z_{-i}} u_i(x_i, g_{-i}(z_{-i}), z_i)\mu_{z_{-i}}(dz_{-i})$ exhibits quasi-convex differences in x_i and z_i, and no element on the upper boundary of X_i ($\sup X_i$) is optimal, then a mean-preserving spread to μ_{z_i} will lead to a convex-increasing order increase in the optimal distribution μ_{x_i}.

Lemma 5 tells us that less precise private signals (increased uncertainty) leads to higher variance of any affected player's optimal distribution. Whether the mean actions increase or decrease, however, depends on whether the payoff function exhibits quasi-convex or quasi-concave differences. The story clearly does not end there: the increase in uncertainty will transmit to other players and make *everybody's* game environments more uncertain. To deal with this, we need the following straightforward generalization of a result found in Rothschild and Stiglitz (1971).[15]

Lemma 6 Let Assumption 1 be satisfied and let $g_i(z_i, \mu_{x_{-i}}) = \arg\max_{x_i\in X_i} \int u_i(x_i, x_{-i}, z_i)\mu_{x_{-i}}(dx_{-i})$. Then for $j \neq i$:

1. *If $u_i(\tilde{x}_i, x_{-i}, z_i) - u_i(x_i, x_{-i}, z_i)$ is concave in x_j for all $\tilde{x}_i \geq x_i$, then $g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}) \leq g_i(z_i, \mu_{x_{-i}})$ whenever $\tilde{\mu}_{x_j}$ is a mean-preserving spread of μ_{x_j}.*
2. *If $u_i(\tilde{x}_i, x_{-i}, z_i) - u_i(x_i, x_{-i}, z_i)$ is concave and increasing in x_j for all $\tilde{x}_i \geq x_i$, then $g_i(z_i, \mu_{x_{-i}}) \leq g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j})$ whenever $\tilde{\mu}_{x_j}$ second-order stochastically dominates μ_{x_j}.*

If in these statements concavity in x_j is replaced with convexity, the first conclusion changes to: $g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}) \geq g_i(z_i, \mu_{x_{-i}})$ whenever $\tilde{\mu}_j$ is a mean-preserving spread of μ_{x_j}; and the second conclusion changes to $g_i(z_i, \mu_{x_{-i}}) \leq g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j})$ whenever $\tilde{\mu}_{x_j}$ dominates μ_{x_j} in the convex-increasing order.

Proof. Statement 1 is a direct application of Topkis' theorem (Topkis (1978)) which in the situation with a one-dimensional decision variable and unique optimizers says that the optimal decision will be non-decreasing [non-increasing] in parameters if the objective exhibits increasing differences [decreasing differences]. The conclusion thus follows from the fact that $\int u_i(x_i, x_{-i}, z_i)$ $\mu_{x_{-i}}(dx_{-i})$ exhibits decreasing differences in x_i (with the usual order) and μ_{x_j} (with the mean-preserving spread order $\succeq_{cx}$) if and only if the assumption of the statement holds. Also by Topkis' theorem, if $u_i(\tilde{x}_i, x_{-i}, z_i) - u_i(x_i, x_{-i}, z_i)$ is increasing in x_j for $j \neq i$ and for all $\tilde{x}_i \geq x_i$, then

[15] Rothschild and Stiglitz (1971) consider mean-preserving spreads in the differentiable case. If u is differentiable in x, the main condition of Lemma 6 is equivalent to the concavity of $D_x u(x, \cdot)$ which exactly is the assumption of Rothschild and Stiglitz (1971).

$\tilde{\mu}_{x_j} \succeq_{st} \mu_{x_j} \Rightarrow g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}) \geq g_i(z_i, \mu_{x_{-i}})$ (here $\succeq_{st}$ denotes the first-order stochastic dominance order). From this and Statement 1 follows Statement 2 because it is always possible to split a second order stochastic dominance increase $\succeq_{cvi}$ into a mean preserving contraction $\succeq_{cv}$ followed by a first order stochastic dominance increase (Formally, if $\tilde{\mu}_{x_j} \succeq_{cvi} \mu_{x_j}$, then there exists a distribution $\hat{\mu}_{x_j}$ such that $\tilde{\mu}_{x_j} \succeq_{st} \hat{\mu}_{x_j} \succeq_{cv} \mu_{x_j}$). The convex case is proved by a similar argument and is omitted. ■

For given distributions of private signals $\mu_z = (\mu_{z_1}, \ldots, \mu_{z_I})$ let $\Phi(\mu_z)$ denote the *set of equilibrium distributions, i.e.,* the set of optimal distributions (14) where $(g_1^*, \ldots, g_I^*)$ is one of the (possibly many) Bayesian equilibria. Fix a given stochastic order $\succeq$ on the probability space of optimal distributions and consider a shift in the distribution of private signals from μ_z to $\tilde{\mu}_z$. In the Theorem below, $\succeq$ will be either the second-order stochastic dominance order or the convex-increasing order; and the shift from μ_z til $\tilde{\mu}_z$ will be a mean-preserving spread. The set of equilibrium distributions then *increases* in the order $\succeq$ if

$$\forall \mu_x \in \Phi(\mu_z)\, \exists \tilde{\mu}_x \in \Phi(\tilde{\mu}_z) \text{ with } \tilde{\mu}_x \succeq \mu_x \text{ and } \forall \tilde{\mu}_x \in \Phi(\tilde{\mu}_z)\, \exists \mu_x \in \Phi(\mu_z) \text{ with } \tilde{\mu}_x \succeq \mu_x \tag{15}$$

If the order $\succeq$ is reversed in (15), the set of equilibrium distributions *decreases*. If $\Phi(\mu_z)$ and $\Phi(\tilde{\mu}_z)$ have least and greatest elements, then (15) implies that the least element of $\Phi(\mu_z)$ will be smaller than the least element of $\Phi(\tilde{\mu}_z)$ and the greatest element of $\Phi(\mu_z)$ will be smaller than the least element of $\Phi(\tilde{\mu}_z)$ (Smithson (1971), Theorem 1.7). In particular, if the equilibria are unique and we therefore have a function ϕ such that $\Phi(\mu_z) = \{\phi(\mu_x)\}$ and $\Phi(\tilde{\mu}_z) = \{\phi(\tilde{\mu}_x)\}$, we get $\phi(\mu_z) \preceq \phi(\tilde{\mu}_z)$ (so the function ϕ is increasing).

Theorem 3 (Mean Preserving Spreads in Bayesian Games) Consider a Bayesian game as described above and let $\mu_z = (\mu_{z_i})_{i \in \mathscr{I}}$ and $\tilde{\mu}_z = (\tilde{\mu}_{z_i})_{i \in \mathscr{I}}$ be two distributions of private signals.[16]

1. Suppose all assumptions of Lemma 5.1 are satisfied and $u_i(\tilde{x}_i, x_{-i}, z_i) - u_i(x_i, x_{-i}, z_i)$ is increasing and concave in x_{-i} for all $\tilde{x}_i \geq x_i$. If $\tilde{\mu}_{z_i}$ is a mean-preserving spread of μ_{z_i} for any subset of the players, then the set of equilibrium distributions decreases in the second-order stochastic dominance order (in particular the agents' mean actions will decrease, and the actions' variance will increase).

2. Suppose all assumptions of Lemma 5.2 are satisfied and $u_i(\tilde{x}_i, x_{-i}, z_i) - u_i(x_i, x_{-i}, z_i)$ is increasing and convex in x_{-i} for all $\tilde{x}_i \geq x_i$. If $\tilde{\mu}_{z_i}$ is a mean-preserving spread of μ_{z_i} for any subset of the players, then the set of equilibrium distributions increases in the convex-increasing order (in particular the agents' mean actions will increase, and again the actions' variance will increase).

If u_i is differentiable in x_i, all of these assumptions are satisfied if $D_{x_i} u_i(x_i, x_{-i}, z_i)$ is increasing in x_{-i} and either concave in (x_i, z_i) and x_{-i} [case 1] or convex in (x_i, z_i) and x_{-i} [case 2].

[16] In the following statements, it is to be understood that any distribution μ_{z_i} that is *not* replaced with a mean-preserving spread $\tilde{\mu}_{z_i}$ is kept fixed.

Proof. As in previous proofs, let $\succeq_{cvi}$ denote the concave-increasing (second-order stochastic dominance) order and $\succeq_{st}$ denote the first-order stochastic dominance order. Recast the game in terms of optimal distributions: Agent i's problem is to find a measurable function g_i which for *a.e.* $z_i \in Z_i$ maximizes $\int_{x_{-i}\in X_{-i}} u_i(x_i, x_{-i}, z_i)\mu_{x_{-i}}(dx_{-i})$. The policy function $x_i = g_i(z_i, \mu_{x_{-i}})$ determines $\mu_{x_i}(A) = \mu_{z_i}\{z_i \in Z_i : g_i(z_i, \mu_{x_{-i}}) \in A\}$ $(A \in \mathscr{B}(X_i))$. An equilibrium is a vector $\mu^*_x = (\mu^*_{x_1}, \ldots, \mu^*_{x_I})$ such that for all $i \in \mathscr{I}$: $\mu^*_{x_i}(A) = \mu_{z_i}\{z_i \in Z_i : g_i(z_i, \mu^*_{x_{-i}}) \in A\}$, all $A \in \mathscr{B}(X_i)$. Letting $f_i(\mu_{x_{-i}}, \mu_{z_i})$ denote agent i's optimal distribution given $\mu_{x_{-i}}$ and μ_{z_i}, an equilibrium is a fixed point of $f = (f_1, \ldots, f_I)$. By 2 of Lemma 6 and Observation 1 on page 3, $\tilde{\mu}_{x_j} \succeq_{cvi} \mu_{x_j} \Rightarrow f_i(\mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}, \mu_{z_i}) \succeq_{st} f_i(\mu_{x_{-i}}, \mu_{z_i})$ for all $j \neq i$. Since first-order stochastic dominance implies second-order stochastic dominance, $f_i(\mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}, \mu_{z_i}) \succeq_{st} f_i(\mu_{x_{-i}}, \mu_{z_i}) \Rightarrow f_i(\mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}, \mu_{z_i}) \succeq_{cvi} f_i(\mu_{x_{-i}}, \mu_{z_i})$. It follows that the mapping f is monotone when μ_x's underlying probability space is equipped with the product order $\succeq^I_{cvi}$. Again with the order $\succeq_{cvi}$ on optimal distributions, it follows from Lemma 5 that each f_i is decreasing in μ_{z_i} with the convex (mean-preserving spread) order on μ_{z_i}'s underlying probability space. f will also be continuous (it is a composition of continuous functions) and so the theorem's conclusions follow directly from Theorem 3 in Acemoglu and Jensen (2015) (the conditions of that Theorem are immediately satisfied when f is viewed as a correspondence). For the second statement of the theorem the argument is precisely the same except that one now equips the set of optimal distributions with the convex-increasing order and notes that f_i is monotone when the private distributions μ_{z_i}'s underlying probability spaces are equipped with the mean preserving spread order. The differentiability conditions presented at the end of the theorem follow from Lemma 1. ■

Note that under the assumptions of Theorem 3, the game is supermodular. Under the additional conditions of the following corollary, the game is monotone (Van Zandt and Vives (2007)). The proof follows along the same lines as the proof of Theorem 3 and is omitted.

Corollary 3 (Second-Order Stochastic Dominance Changes) If in addition to the assumptions of Theorem 3, it is assumed that $u_i(\tilde{x}_i, x_{-i}, z_i) - u_i(x_i, x_{-i}, z_i)$ is increasing in z_i, then if $\tilde{\mu}_{z_i}$ second-order stochastically dominates μ_{z_i} for any subset of the players, the set of equilibria decreases in the second-order stochastic dominance order in case 1. In case 2., the set of equilibria increases in the convex-increasing order when $\tilde{\mu}_{z_i}$ dominates μ_{z_i} in the convex-increasing order for any subset of the players.

There are many interesting applications of Theorem 3, ranging from auction theory to the Diamond search model. Here we will study a Bayesian version of the classical arms race game from the field of conflict resolution (see *e.g.* Milgrom and Roberts (1990), p.1272), and ask whether increased uncertainty about arms' effectiveness and opponents' intentions leads to an intensification of the arms race or not.

Consider two countries, $i = 1, 2$, with identical state payoff functions $u_i(x_i, x_{-i}, z_i) = B(x_i - x_{-i} - z_i) - c x_i$. B is a strictly concave function and $c > 0$ a constant cost parameter. z_i is a random variable that reflects the relative effectiveness of the arms — real or imagined (for example a domestic media frenzy might correspond to a mean-preserving spread to z_i).[17] Assuming that B is sufficiently smooth, we can use the conditions at the end of Theorem 3. By strict concavity, $D_{x_i} u(x_i, x_{-i}, z_i) = B'(x_i - x_{-i} + z_i) - c$ is increasing in x_{-i}, and the question is therefore whether it is also either convex or concave in (x_i, z_i) and in x_{-i}. Obviously, this depends entirely on whether B' is convex or concave, *i.e.*, on whether the third derivative of B is positive or negative. In the convex case (positive third derivative), the countries' policy functions are convex. Hence greater uncertainty will increase the affected country's (or countries') expected stock of arms as well as the variance (Lemma 5, which specifically says that given the other country's strategy, greater uncertainty will lead to a convex-increasing shift in the arms strategy). This will transmit to a more uncertain environment for the other country and make it accumulate more arms (Lemma 6). This escalation continues until an equilibrium is reached with higher mean stocks of arms and greater uncertainty about the exact size of the arsenals (Theorem 3).[18] Note that a positive third derivative means that the countries are "prudent" (Kimball (1990)) — a well-understood behavioral trait that also plays a key role in other settings such as in income allocation problems (Carroll and Kimball (1996)). Of course, prudence, which in the words of Kimball (1990) (p.54) is "the propensity to prepare and forearm oneself in the face of uncertainty", has rather more beneficial consequences in income allocation models than it does in arms races. It is therefore not uniformly good news that experimental evidence seems to suggest that most people are prudent (Nussair et al (2011)). But of course, prudence may be situation-dependent or imprudent politicians may be elected. In this case B will have a negative third derivative, and the countries' policy functions will be concave so that greater uncertainty lowers the mean stock of arms in equilibrium. Note however, that according to Theorem 3, the variance will still increase, so whether decision makers are prudent or not, the risk of exceptionally high stocks of arms and the negative consequences in case of war still increases when the environment becomes more uncertain.

5 Stochastic Dynamic Programming: Convexity and Concavity of the Policy Function

This section uses Theorem 2 to study the concavity/convexity of policy functions in an infinite horizon stochastic setting. The issue is important for a variety of distributional comparative statics questions. Thus the relationship between earning risk and wealth accumulation is guided by whether the consumption function is concave or convex (Huggett (2004)). In dynamic stochastic general equilibrium models, it more generally determines the effect of increased individual uncertainty on aggregate market outcomes (Acemoglu and Jensen (2015)).

The treatment and notation follows Chapter 9 of Stokey and Lucas (1989). The dynamic programming problem is,

[17] A myriad of other specifications would of course be possible, for example costs could instead be random. This section's results may be applied for any such specification.

[18] Note that since the conditions of Corollary 3 are satisfied, these conclusions are valid not just for mean-preserving spreads but for second-order stochastic dominance decreases more generally.

$$\text{(16)} \qquad \begin{array}{ll} \max & \mathbb{E}_0[\sum_{t=0}^{\infty}\beta^t u(x_t, x_{t+1}, z_t)] \\ & \text{s.t.} \begin{cases} x_{t+1} \in \Gamma(x_t, z_t)\ ,\ t = 0, 1, 2, \dots \\ \text{where } (x_0, z_0) \gg 0 \text{ are given.} \end{cases} \end{array}$$

In comparison with Stokey and Lucas (1989), two structural restrictions are made to simplify the exposition (but the results easily generalize, see Remarks 1-2 at the end of this section): first, the z_t's are assumed to be *i.i.d.* with distribution μ_z; second, only the one-dimensional case is considered, *i.e.*, it is assumed that $x_t \in X \subseteq \mathbb{R}$ and $z_t \in Z \subseteq \mathbb{R}$. Aside from these restrictions, everything is completely parallel to Stokey and Lucas (1989). Both X and Z are assumed to be convex sets equipped with their Borel σ-algebras.[19] The *value function* $v : X \times Z \to \mathbb{R}$ associated with the above problem is determined by the functional equation,

$$v(x,z) = \sup_{y \in \Gamma(x,z)} \left[u\big(x,y,z\big) + \beta \int v(y,z')\mu_z(dz') \right]. \tag{17}$$

The following standard assumptions are imposed (see Stokey and Lucas (1989), Chapter 9).

Assumption 2 $\Gamma : X \times Z \to 2^X$ *is non-empty, compact-valued, continuous, and has a convex graph, i.e., for all* $x, \tilde{x} \in X$, $z \in Z$, *and all* $\lambda \in [0,1]$: $\lambda y + (1-\lambda)\tilde{y} \in \Gamma(\lambda x + (1-\lambda)\tilde{x}, z)$ *whenever* $y \in \Gamma(x,z)$ *and* $\tilde{y} \in \Gamma(\tilde{x}, z)$.

Assumption 3 $u : X \times X \times Z \to \mathbb{R}$ *is bounded and continuous, and* $\beta \in (0,1)$. *Furthermore,* $u(x,y,z)$ *is concave in* (x,y) *and strictly concave in* y.

Note that Assumption 2 in particular requires Γ to have a *convex graph*. As discussed in the first paragraph after Definition 4, this implies that Γ is a concave as well as a convex correspondence (*cf.* the conditions of Theorem 2). Under Assumptions 2-3, the value function $v = v(x,z)$ is uniquely determined, continuous, and concave in x. Furthermore, the *policy function* $g : X \times Z \to X$ is a well-defined and continuous function:

$$g(x,z) = \arg \sup_{y \in \Gamma(x,z)} \left[u\big(x,y,z\big) + \beta \int v(y,z')\mu_z(dz') \right]. \tag{18}$$

Theorem 4 (Convex Policy Functions in Dynamic Stochastic Programming Problems) Consider the stochastic dynamic programming problem (16) under Assumptions 2-3 and let $g : X \times Z \to X$ denote the policy function (18). Assume that $u(x,y,z)$ is differentiable and satisfies the following upper boundary condition: $\lim_{y^n \uparrow \sup \Gamma(x,z)} D_y u(x, y^n, z) = -\infty$ (or in some other way ensure that $\sup \Gamma(x,z)$ will never be optimal given (x,z)). Then the policy function g is convex in x if $D_x u(x,y,z)$ is non-decreasing in y and there exists a $k \geq 0$ such that $\frac{1}{1-k}[-D_y u(x,y,z)]^{1-k}$ is concave in (x,y) and $\frac{1}{1-k}[D_x u(x,y,z)]^{1-k}$ is convex in (x,y).[20] If in addition $\Gamma(x,\cdot)$ is a convex correspondence and $\frac{1}{1-k}[-D_y u(x,y,z)]^{1-k}$ is concave in (y,z), then the policy function g will also be convex in z.

[19]For our result on the policy function $g(x,z)$'s convexity in x, it may alternatively be assumed that Z is a countable set equipped with the σ-algebra consisting of all subsets of Z (see Stokey and Lucas (1989), Assumption 9.5.a.).

[20]In the limit case $k = 1$, $\frac{1}{1-k}[f(x)]^{1-k}$ is by convention equal to $\log(f(x))$.

Proof. See Section 5.1. ■

Theorem 4 has a host of applications in macroeconomics. For example, it is applied in Acemoglu and Jensen (2015) to study how increased uncertainty affects the equilibria in large dynamic economies. As a concrete example, consider the income allocation problem discussed in Section 2.2. Let $r > 0$ and $w > 0$ denote, respectively, the factor of interest and wage rate. Let $\Gamma(x,z) = \{y \in [-\underline{b}, \overline{b}] : y \leq rx + wz\}$, and let $\tilde{u}$ be a strictly concave and strictly increasing period utility function. The income allocation problem can then be written as a dynamic programming problem:

$$\begin{array}{ll} \max & \mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t \tilde{u}(rx_t + wz_t - x_{t+1})] \\ \text{s.t.} & \begin{cases} x_{t+1} \in \Gamma(x_t, z_t),\ t = 0,1,2,\ldots \\ \text{where } (x_0, z_0) \gg 0 \text{ are given.} \end{cases} \end{array} \tag{19}$$

Note that this formulation explicitly incorporates borrowing limits in the spirit of Aiyagari (1994). Note also that x_t is savings at date t, so the policy function $g(x,z)$ is the *savings function* and the *consumption function* is $c(x,z) = rx + wz - g(x,z)$.[21] As seen, c is concave in x ("concavity of the consumption function") if and only if g is convex in x ("convexity of the savings function"). In terms of this section's general notation we have $u(x,y,z) = \tilde{u}\big(rx + wz - y\big)$. It is easy to verify (and well known) that Assumptions 2-3 are satisfied. Under a standard boundary condition on $\tilde{u}$, we will never have $g(x,z) = \sup\Gamma(x,z)$, *i.e.*, the consumer will not choose zero consumption at any date.

We have $D_y u(x,y,z) = -\tilde{u}'(rx+wz-y)$ and $D_x u(x,y,z) = r\tilde{u}'(rx+wz-y)$. $D_x u$ is strictly increasing in y since $\tilde{u}$ is strictly concave. By Theorem 4, the consumption function is consequently concave if $\frac{1}{1-k}[\tilde{u}'(rx+wz-y)]^{1-k}$ is concave and $\frac{1}{1-k}[r\tilde{u}'(rx+wz-y)]^{1-k}$ is convex in (x,y). Assuming that $\tilde{u}$ is thrice differentiable, the relationship with the result of Carroll and Kimball (1996) is explicit. Since the Hessian determinants of $D_x u$ and $D_y u$ equal zero, it is straightforward to verify that this will hold if and only if

$$\frac{\tilde{u}'\tilde{u}'''}{(\tilde{u}'')^2} = k \geq 0. \tag{20}$$

Thus we precisely find the condition that $\tilde{u}$ must be of the HARA-form, as is assumed by Carroll and Kimball (1996) in their proof of concavity of the consumption function. One easily verifies that the condition for convexity in z is also satisfied when $\tilde{u}$ is of the HARA-form. Thus we have generalized the result of Carroll and Kimball (1996) to the incomplete markets setting with borrowing constraints.

5.1 Proof of Theorem 4

The value and policy functions will equal the pointwise limits of the sequences $(v^n)_{n=0}^{\infty}$ and $(g^n)_{n=0}^{\infty}$ determined by:

$$v^{n+1}(x,z) = \sup_{y\in\Gamma(x,z)} \left[u\big(x,y,z\big) + \beta \int v^n(y,z')\mu_z(dz') \right], \text{ and} \tag{21}$$

[21] Note that if we let $W = rx + wz$ (income), we will always have that g and c are functions of W. See Section 2.2 where this was made explicit.

$$g^n(x,z) = \arg\sup_{y\in\Gamma(x,z)}\left[u\big(x,y,z\big)+\beta\int v^n(y,z')\mu_z(dz')\right]. \tag{22}$$

Under the Theorem's conditions, v^n is concave in x for all n. Since a concave function is absolutely continuous, v^n will be absolutely continuous in x for all n. The following result presents conditions under which an absolutely continuous function is quasi-concave. It generalizes the sufficiency part of Lemma 1.

Lemma 7 Assume that $u : X\times Z \to \mathbb{R}$ is absolutely continuous in $x \in X$, *i.e.*, assume that $u(x,z) = \alpha(a,z)+\int_a^x p(\tau,z)\,d\tau$ for a Lebesgue integrable function $p : X\times Z \to \mathbb{R}$ (here $a = \inf X$ and α is a function that does not depend on x). Then u exhibits quasi-convex differences [quasi-concave differences] if $p(x,z)$ is quasi-convex [quasi-concave].

Proof. The statement can be verified by going through the proof of Lemma 1 and everywhere replace u's derivative with p. ■

Next we need a result on how to ensure that the sum of two functions exhibits quasi-concave or quasi-convex differences.

Lemma 8 Let u be of the form $u(x,z) = f(x,z)+h(-x,z)$ where $f,h : X\times Z \to \mathbb{R}$ are differentiable with $D_x f(x,z) = f'_x(x,z) \geq 0$ and $D_x h(x,z) = -h'_x(-x,z) \leq 0$. Then u exhibits quasi-convex differences on the subset $\{(x,z)\in X\times Z : D_x u(x,z) \leq 0\}$ if there exists a $k \geq 0$ such that $\frac{1}{1-k}[f'_x]^{1-k}$ is convex and $\frac{1}{1-k}[h'_x]^{1-k}$ is concave.[22] If instead $\frac{1}{1-k}[f'_x]^{1-k}$ is concave and $\frac{1}{1-k}[h'_x]^{1-k}$ is convex, then u exhibits quasi-concave differences on the subset $\{(x,z)\in X\times Z : D_x u(x,z)\geq 0\}$.

Proof. In the online appendix (Jensen (2015), Lemma 3). ■

From now on, call a function $f : X \to \mathbb{R}$ *k-convex* [*k-concave*] if $\frac{1}{1-k}[f(x)]^{1-k}$ is convex [concave] where, as previously mentioned, the case $k = 1$ is taken to mean log-convex [log-concave] by convention.[23]

Lemma 9 Given z, let a denote the least point at which $v^n(\cdot,z)$ is defined. Assume that $-D_y u(x,y,z)$ is k-concave in (x,y) [k-concave in (y,z)], and that $v^n(y,z) = \int_a^y p^n(\tau,z)\,d\tau$ where $p^n(\cdot,z)$ is k-convex. Then $g^n(x,z)$ is convex in x [convex in z].

Proof. This is a direct application Theorem 2's corollary (Corollary 2) to the optimization problem in (22). We consider here only the convexity of g^n in x (the exact same argument implies convexity in z under the Lemma's square-bracketed assumption). Except for quasi-convex differences in (x,y), all the assumptions of Corollary 2 are clearly satisfied (in particular, Γ is a convex correspondence as mentioned immediately after Assumption 3). To see that quasi-convex differences

[22] In the limit case $k = 1$, $\frac{1}{1-k}[f'_x]^{1-k}$ is by convention equal to $\log(f'_x)$ (similarly for h'_x).

[23] Quite a bit can be said about such functions, but since this is mainly a mathematical distraction from the point of view of this paper, further investigation has been relegated to the online appendix.

holds, we use Lemma 7 and must thus verify that,

$$D_y u(x,y,z) + \beta \int p^n(y,z')\mu_z(dz'), \tag{23}$$

is quasi-convex in (x,y) on the relevant set which, allowing for solutions at lower boundary points is $A \equiv \{(x,y) \in X^2 : D_y u(x,y,z) + \beta \int D_y v^n(y,z')\mu_z(dz') \leq 0\}$. To see that this holds, first use that k-convexity is preserved under integration (Jensen (2015), Lemma 2) to conclude that when $p^n(y,z')$ is k-convex in y, $\beta \int p^n(y,z')\mu_z(dz')$ is k-convex in y. Then use Lemma 8. ■

To finish the proof we need just one last technical result.

Lemma 10 Assume that $D_x u(x,y,z)$ is k-convex in (x,y) and non-decreasing in y and that $g^n(x,z)$ is convex in x. Then $v^{n+1}(x,z) = \int_a^x p^{n+1}(\tau,z)\, d\tau$ where $p^{n+1}(\cdot,z)$ is k-convex.

Proof. Since v^{n+1} is absolutely continuous, we can (abusing notation slightly) write it as: $v^{n+1}(x,z) = \int_a^x D_x v^{n+1}(\tau,z)\, d\tau$. In particular, $D_x v^{n+1}(x,z)$ exists almost everywhere and when it exists $D_x v^{n+1}(x,z) = D_x u(x, g^n(x,z), z)$ by the envelope theorem. k-convexity of $p^{n+1}(x,z) \equiv D_x u(x, g^n(x,z), z)$ in x now follows immediately from the fact that k-convexity is preserved under convex, increasing transformations (Jensen (2015), Lemma 1). ■

To prove that g is convex, consider the value and policy function iterations (21)-(22). Start with any value function v^0 such that $v^0(y,z) = \int_a^y p^0(\tau,z)\, d\tau$ where $p^0(\cdot,z)$ is k-convex. Then by Lemma 9, g^0 is convex. Hence by Lemma 10, $v^1(y,z) = \int_a^y p^1(\tau,z)\, d\tau$ where $p^1(\cdot,z)$ is k-convex. Repeating the argument, g^1 is convex and $v^2(y,z) = \int_a^y p^2(\tau,z)\, d\tau$ where $p^2(\cdot,z)$ is k-convex. And so on ad infinitum. The pointwise limit of a sequence of convex function is convex, hence $g(\cdot,z) = \lim_{n\to\infty} g^n(\cdot,z)$ is convex. The same argument applies for convexity in z, concluding the proof of Theorem 4.

Remark 1 (General Markov Processes) The previous proof goes through without any modifications if z_t is allowed to be a general Markov process, *i.e.*, if the functional equation (17) is replaced with:

$$v(x,z) = \sup_{y \in \Gamma(x,z)} \left[u\big(x,y,z\big) + \beta \int v(y,z')Q(z,dz') \right], \tag{24}$$

where Q is z_t's transition function. Indeed, the previous proof goes through line-by-line if we instead begin with the functional equation (24).

Remark 2 (Multidimensional Strategy Sets) The proof also easily extends to the case where X and Z are multidimensional (a case treated in the Appendix). The only modification needed is in the proof of Lemma 9 where now Theorem 5 in the appendix is needed to conclude that g^n is convex, in place of Theorem 2. Thus Theorem 4 extends to the multidimensional case if we in addition assume that u is supermodular in y, that Γ's values are lower semi-lattices, and that optimizers stay away from the upper boundary.[24]

[24] In particular, the objective function in (22) is supermodular in y when u is supermodular in y because supermodularity/increasing differences is preserved under integration (Topkis (1998), Theorem 2.7.6.) and $v^n(y,z')$ is therefore supermodular in y for all n.

6 Conclusion

This paper contributes to distributional comparative statics (DCS), *i.e.*, to the study of how changes in exogenous distributions affect endogenous distributions in economic models. Most DCS questions can be answered if suitable policy functions are either concave or convex. In the main theoretical contribution of the paper (Theorem 1), it is shown that concavity of the policy function hinges on an intuitive as well as easily verifiable condition on the primitives of a model, namely *quasi-concave differences*. That observation parallels Topkis' theorem (Topkis (1978)) which ensures that the policy function is increasing (strategic complementarity) when the objective function exhibits increasing differences. Theorem 2, as well as Theorem 5 in the Appendix, extends the result to policy correspondences (multiple optimizers), objectives that may not be quasi-concave, and multi-dimensional action sets.

Several areas of application were discussed including uncertainty comparative statics, international trade models of heterogenous firms (Melitz (2003)), the macroeconomic modeling of inequality, and stochastic dynamic programming. In all of these, the concavity of suitably defined policy functions turns out to drive the conclusions, ultimately owing to Observations 1-5 in Section 2.1 (page 3) and the fact that Lorenz dominance is equivalent to mean-preserving spreads and generalized Lorenz dominance is equivalent to decreases in second-order stochastic dominance (see the discussion at the end of Section 2.2). As a concrete illustration of uncertainty comparative statics, a Bayesian arms race is studied and it is found that "prudence" (Kimball (1990)) determines whether mean stocks of arms increase or decrease when uncertainty goes up — but in all cases, a more uncertain environment also leads to higher equilibrium variance and thus greater uncertainty about the scale of destruction in the event of a war. The stochastic dynamic programming results are illustrated by generalizing a result due to Carroll and Kimball (1996) to allow for borrowing constraints. These results play a key role for distributional comparative statics in dynamic stochastic general equilibrium (DSGE) models — a theme taken up in Acemoglu and Jensen (2015) who study, for example, how increased uncertainty about future earnings prospects affects output per worker in the Aiyagari (1994) model.

7 Appendices

Appendix I: Proof of Observations 1-5 in Section 2.1

The distribution μ_x is the image measure of μ under g. Hence $\int f(x)\mu_x(dx) = \int f(g(z))\mu(dz)$ for any function $f : X \to \mathbb{R}$ such that the integrals are well-defined. Each claim thus amounts to saying that for classes of functions $\mathscr{F}$ and $\mathscr{F}_x$, if $\int h(z)\tilde{\mu}(dz) \geq \int h(z)\mu(dz)$ for all $h \in \mathscr{F}$, then $\int f(g(z))\tilde{\mu}(dz) \geq \int f(g(z))\mu(dz)$ for all $f \in \mathscr{F}_x$. In the case of Observation 1, $\mathscr{F}$ and $\mathscr{F}_x$ both equal the class of increasing functions and the claim follows from the fact that $f \circ g$ is increasing when both f and g are increasing. For Observation 2, $\mathscr{F}$ is the class of convex functions and $\mathscr{F}_x$ the class of decreasing, convex functions, and the claim follows because $f \circ g$ is convex when g is concave and f is convex and decreasing. For Observation 3, both $\mathscr{F}$ and $\mathscr{F}_x$ equal the class of increasing, concave functions and the conclusion follows because $f \circ g$ is increasing and concave when both f and g are increasing and concave. Observations 4-5 are proved by the same arguments as Observations 2-3 and may be omitted.

Appendix II: Proof of Lemma 1

To facilitate the results on multi-dimensional strategy sets in Appendix III, the general case where $X \subseteq \mathbb{R}^n$ is considered. The statement to be provided is that if u is differentiable in x, it will exhibit quasi-concave differences if and only if $D_{x_j} u(x,z)$ is quasi-concave in (x,z) for all $j = 1,\ldots,n$ (evidently, Lemma 1 is a special case of this statement).

"$\Rightarrow$": Since $D_{x_j} u(x,z) = \lim_{\delta\to 0} \frac{u(x+\delta\epsilon_j,z)-u(x,z)}{\delta}$ where ϵ_j denotes the j'th unit vector, and quasi-convexity is preserved under pointwise limits (Johansen (1972)), each partial derivative $D_{x_j} u(x,z)$ is quasi-convex at (x,z) when u exhibits quasi-convex differences at (x,z). "$\Leftarrow$": This direction is not easy. The idea is to prove the contrapositive by contradiction (note that since quasi-convexity is not preserved under integration, we cannot use the fundamental theorem of calculus). So we assume that u does not exhibit quasi-convex differences, that each partial derivative $D_{x_j} u(x,z)$ is quasi-convex, and then derive a contradiction. For $\alpha \in [0,1]$ set $x_\alpha \equiv \alpha x_0 + (1-\alpha)x_1$ and $z_\alpha = \alpha z_0 + (1-\alpha)z_1$. Say that u exhibits quasi-convex differences in the direction $\eta > 0$ at (x_0, z_0, α) if for all $\delta_n > 0$ in some neighborhood of 0:

$$(25) \quad u(x_\alpha + \delta_n \eta, z_\alpha) - u(x_\alpha, z_\alpha) \leq \max\{u(x_0 + \delta_n \eta, z_0) - u(x_0, z_0), u(x_1 + \delta_n \eta, z_1) - u(x_1, z_1)\}$$

It is easy to see that if u exhibits quasi-convex differences (on all of $X \times Z$), then it exhibits quasi-convex differences in all directions $\eta > 0$ at all $(x,z,\alpha) \in X \times Y \times [0,1]$. Let ϵ_j denote the j'th unit vector (a vector with 1 in the j'th coordinate and zeroes everywhere else). Since a function is quasi-convex in all directions if and only if it is quasi-convex in all unit/coordinate directions ϵ_j, we may (as always) restrict attention to the directions of the coordinates in the previous statement. Hence if u *does not* exhibit quasi-convex differences, there will exist a coordinate direction ϵ_j, $(x_0, z_0), (x_1, z_1) \in X \times Y$, $\hat{\alpha} \in [0,1]$ and a sequence $\delta_n \downarrow 0$ such that for all n:

$$(26)\ u(x_{\hat{\alpha}} + \delta_n \epsilon_j, z_{\hat{\alpha}}) - u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) > \max\{u(x_0 + \delta_n \epsilon_j, z_0) - u(x_0, z_0), u(x_1 + \delta_n \epsilon_j, z_1) - u(x_1, z_1)\}$$

Note that we necessarily have $\hat{\alpha} \in (0,1)$ when the previous inequality holds. Intuitively, the inequality says that there exists a point (x_α, z_α) on the line segment between (x_0, z_0) and (x_1, z_1) at which $u(\cdot + \delta_n \epsilon_j, \cdot) - u(\cdot, \cdot)$ takes a strictly higher value than at any of the endpoints. Now, divide through (26) with δ_n and take limits:

$$D_{x_j} u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) \geq \max\{D_{x_j} u(x_0, z_0), D_{x_j} u(x_1, z_1)\}$$

Since $D_{x_j} u(\cdot,\cdot)$ is quasi-convex, it follows that: $D_{x_j} u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) = \max\{D_{x_j} u(x_0, z_0), D_{x_j} u(x_1, z_1)\}$. Assume without loss of generality that $D_{x_j} u(x_0, z_0) \geq D_{x_j} u(x_1, z_1)$. Since $D_{x_j} u(x_0, z_0)$ is quasi-convex and $0 < \hat{\alpha} < 1$, it follows that either (i) $D_{x_j} u(x_0, z_0) = D_{x_j} u(x_\alpha, z_\alpha)$ for all $\alpha \in [0, \hat{\alpha}]$ or (ii)

$D_{x_j} u(x_1, z_1) = D_{x_j} u(x_\alpha, z_\alpha)$ for all $\alpha \in [\hat{\alpha}, 1]$ (or both).[25] Consider case (i) (the proof in case (ii) is similar). When (i) holds, u's restriction to the line segment between (x_0, z_0) and $(x_{\hat{\alpha}}, z_{\hat{\alpha}})$ must necessarily be of the form: $u(x, z) = c\, x_j + g(x_{-j}, z)$ where $c_n = D_{x_n} u(x_0, z_0)$ (a constant) and x_{-j} denotes all coordinates of x except for the j'th one (remember that j is the positive coordinate of e_j). But then $u(x_{\hat{\alpha}} + \delta_n \epsilon_j, z_{\hat{\alpha}}) - u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) = u(x_0 + \delta_n \epsilon_j, z_0) - u(x_0, z_0) = c\, \delta_n \epsilon_j$ which implies that:

(27) $u(x_{\hat{\alpha}} + \delta_n \epsilon_j, z_{\hat{\alpha}}) - u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) \leq \max\{u(x_0 + \delta_n \epsilon_j, z_0) - u(x_0, z_0), u(x_1 + \delta_n \epsilon_j, z_1) - u(x_1, z_1)\}$

Comparing (27) with (26) we have a contradiction, and the proof is complete.

Appendix III: Multi-dimensional Decision Variables

This appendix treats concavity of the policy correspondence in the case where the decision vector is allowed to live in an arbitrary ordered topological vector lattice V, $x \in X \subseteq V$. Note that in this setting, the order mentioned in Definition 4 is the order inherited from V. So if $V = \mathbb{R}^N$ with the usual Euclidean order, the theorem below implies that each coordinate correspondence $G_n : Z \to 2^{X^n} \subseteq 2^{\mathbb{R}}$ is concave in the sense discussed in detail in Section 3.2.

The multi-dimensional setting forces us to make some additional assumptions. In comparison with Theorem 2, X must be a lattice, u must be supermodular in the decision vector, and Γ must be upper semi-lattice valued. Finally, Γ's values must be order convex rather than merely convex.[26] Finally, the boundary conditions must be suitably generalized as we turn to first. It should be noted that all of these assumptions automatically are satisfied when X is one-dimensional. Hence the result to follow encompasses Theorem 2 (focus is below on the quasi-concave case but one easily establishes a parallel to (2) of Theorem 2).

First, the basic definitions. Say that a point $x \in \Gamma(z)$ lies on the *upper* [*lower*] *boundary* of $\Gamma(z)$ if there does not exist an $x' \in \Gamma(z)$ with $x' \gg x$ ($x' \ll x$). The upper boundary is denoted by $\overline{\mathrm{B}}(\Gamma(z))$ and the lower boundary is denoted by $\underline{\mathrm{B}}(\Gamma(z))$. What we are going to require in the theorem below is precisely as in Theorem 2 except that the infimum is replaced with the lower boundary.

Next, X must be a *lattice*, *i.e.*, if x and x' lie in X so do their infimum $x \wedge x'$ and supremum $x \vee x'$. If $X \subseteq \mathbb{R}^n$ with the usual Euclidean/coordinatewise order, the infimum (supremum) is simply the coordinatewise minimum (maximum). Assuming that X is a lattice is actually a very weak additional requirement in the present framework because it is the constraint correspondence Γ that determines the feasible set. It is the next assumption that really has "bite". A lower semi-lattice [upper semi-lattice] is a subset $A \subset X$ with the property that if $x, x' \in A$ then the infimum $x \wedge x'$

[25] A quasi-convex function's restriction to a convex segment as the one considered here can always be split into two segments, one which is non-increasing and one which is non-decreasing (and in the present situation, there *must* first be non-increasing segment since the function's value weakly decreases between the endpoints). On the convex line segment between (x_0, z_0) and (x_1, z_1) we have in the present situation that the function begins at $D_{x_j} u(x_0, z_0)$, again takes the value $D_{x_j} u(x_0, z_0)$ at (x_α, z_α) and then moves to a weakly lower value $D_{x_j} u(x_1, z_1)$ at the end-point (x_1, z_1). It follows that if $D_{x_j} u(\cdot, \cdot)$ is not constant on the first interval (corresponding to $\alpha \in [0, \alpha]$) it must strictly decrease and then strictly increase on this interval, which implies that $D_{x_j} u(\cdot, \cdot)$ is constant on the second of the two intervals.

[26] Note that this once again precisely parallels monotone comparative statics. In that setting supermodularity and lattice-type assumptions are also unnecessary/trivially satisfied in the one-dimensional case but must be imposed in multiple dimensions.

[supremum $x \vee x'$] also lies in A. Either is of course weaker than being a lattice. The lower or upper semi-lattice is *order-convex* if $a, b \in A$, and $a \leq a' \leq b$ imply $a' \in A$ (in words, if the set contains an ordered pair of elements, it contains the entire order interval marked by these elements). Note that order-convexity is stronger than convexity in general, although the two coincide in the one-dimensional case. A budget set is an order-convex lower semi-lattice (it is not a lattice), and a firm's input requirement set is an order-convex upper semi-lattice (but again not a lattice). As these examples indicate, the fact that we avoid assuming that Γ's values are lattices greatly expands the scope of the theorem below.

Finally, u must be supermodular in the choice variables. The well-known definition is as follows.

Definition 5 (Topkis (1978)) *The objective function $u : X \times Z \to \mathbb{R}$ is supermodular in x if $u(x \vee x', z) + u(x \wedge x', z) \geq u(x, z) + u(x', z)$ for all $x, x' \in X$ and for all $z \in Z$. If u is twice differentiable in x and $X \subseteq \mathbb{R}^n$, it is supermodular in x if and only if the Hessian matrix $D^2_{xx} u(x, z) \in \mathbb{R}^{n \times n}$ has non-negative off-diagonal elements (for all x and z).*

We are now ready to state and prove the main result with multi-dimensional action sets. Note that as in theorems 1-2, the boundary condition is trivially satisfied if the optimization problem is unrestricted or attention is restricted to interior solutions.

Theorem 5 (Concavity of the Policy Correspondence, Multidimensional Case) Let Z be a convex subset of a vector space and X a convex lattice. Define the policy correspondence $G(z) = \arg\sup_{x \in \Gamma(z)} u(x, z)$ where $u : X \times Z \to \mathbb{R}$ is quasi-concave and supermodular in x and $\Gamma : Z \to 2^X$ has order-convex values. Assume that $G(z)$ is non-empty and compact for all $z \in Z$. Then:

1. The policy correspondence G is concave if $u : X \times Z \to \mathbb{R}$ exhibits quasi-concave differences, Γ is concave and upper semi-lattice valued, and $x \in G(z) \Rightarrow x \notin \underline{\mathrm{B}}(\Gamma(z))$ for all $z \in Z$.

2. The policy correspondence G is convex if $u : X \times Z \to \mathbb{R}$ exhibits quasi-convex differences, Γ is convex and lower semi-lattice valued, and $x \in G(z) \Rightarrow x \notin \overline{\mathrm{B}}(\Gamma(z))$ for all $z \in Z$.

Proof of Theorem 5. The convex case 2. is proved (the proof of the concave case is similar). Pick $z_1, z_2 \in Z$, $x_1 \in G(z_1)$, and $x_2 \in G(z_2)$. Exactly as in the proof of Theorem 1, we can use quasi-convex differences to conclude that for some $\delta \gg 0$, $u(x_\alpha, z_\alpha) \geq u(x_\alpha + \delta, z_\alpha)$ for all $\alpha \in [0, 1]$. Hence by quasi-concavity of u in x, $u(x, z_\alpha)$ is non-increasing for $x \geq x_\alpha$. We wish to show that for all α there exists $\hat{x} \in G(z_\alpha)$ with $\hat{x} \leq x_\alpha$. Pick any $x \in G(z_\alpha)$. I am first going to prove that,

$$x \wedge x_\alpha \in \Gamma(z_\alpha) \tag{28}$$

Since Γ has convex values, there exists some $\tilde{x} \in \Gamma(z_\alpha)$ with $\tilde{x} \leq x_\alpha$. We have $x \in \Gamma(z_\alpha)$ (since $x \in G(z_\alpha)$) and so since Γ's values are lower semi-lattices, $x \wedge \tilde{x} \in \Gamma(z_\alpha)$. But $x \wedge \tilde{x} \leq x \wedge x_\alpha \leq x$, hence $x \wedge x_\alpha \in \Gamma(z_\alpha)$ because Γ has order-convex values. That was what we wanted to show. Next use supermodularity of $u(\cdot, t)$ and the fact that $u(\cdot, z_\alpha)$ is non-increasing for $x \geq x_\alpha$ (implies that $u(x_\alpha, z_\alpha) \geq u(x \vee x_\alpha, z_\alpha)$) to conclude that:

$$u(x, z_\alpha) - u(x \wedge x_\alpha, z_\alpha) \leq u(x \vee x_\alpha, z_\alpha) - u(x_\alpha, z_\alpha) \leq 0 \tag{29}$$

(28)-(29) imply that $x \wedge x_\alpha \in G(z_\alpha)$. But since clearly $x \wedge x_\alpha \leq x_\alpha$ this completes the proof. ■

As mentioned above, one easily establishes a version similar to (2) of Theorem 2 that dispenses with quasi-concavity in the decision vector. Since the argument is identical to the one presented in Section 3.2, it is omitted.

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