University of Leicester

Department of Economics

Foundations and Properties of Time Discount Functions

Ali al-Nowaihi, University of Leicester, UK

Sanjit Dhami, University of Leicester, UK

Working Paper No. 14/11

September 2014

Foundations and Properties of Time Discount Functions*

Ali al-Nowaihi† **Sanjit Dhami**‡

28 August 2014

Abstract

A critical element in all discounted utility models is the specification of a discount function. We introduce three functions: the *delay*, *speedup* and *generating* functions. Each can be uniquely elicited from behaviour. The delay function determines stationarity and the common difference effect. The speedup function determines impatience. Additivity is jointly determined by the delay and speedup functions. The speedup and generating functions jointly determine a unique discount function. Conversely, a continuous discount function determines unique speedup and generating functions.

Keywords: hyperbolic discounting, nonadditivity, impatience, prospect theory, delay function, speedup function, generating function.

JEL Classification Codes: C60(General: Mathematical methods and programming); D91(Intertemporal consumer choice).

*We are grateful to the editor, Professor Marciano Siniscalchi, and to the referee for insightful comments, clear guidance and care going beyond the call of duty.

†Department of Economics, University of Leicester, University Road, Leicester. LE1 7RH, UK. Phone: +44-116-2522898. Fax: +44-116-2522908. E-mail: aa10@le.ac.uk.

‡Department of Economics, University of Leicester, University Road, Leicester. LE1 7RH, UK. Phone: +44-116-2522086. Fax: +44-116-2522908. E-mail: Sanjit.Dhami@le.ac.uk.

1 Introduction

Consider a decision maker who, today, is indifferent between one apple today and two apples tomorrow but, again today, strictly prefers two apples in 51 days to one apple in 50 days (Thaler, 1981). This is an illustration of the *common difference effect*.[1] In a seminal paper, Loewenstein and Prelec (1992) introduced their *generalized hyperbolic discount function*, henceforth LP. LP explains the common difference effect in terms of *declining impatience*, i.e., the subjective discount rate declines as the outcomes of today's choice recede into the future.[2]

A very different explanation of the common difference effect arises from the work of Read (2001) and Scholten and Read (2006a). Following the work of Read (2001) they argued that, for $r < s < t$, discounting a positive magnitude back from t to s then from s to r results in a lower present value than discounting back from t to r in one step, i.e., $D(r,s)\,D(s,t) < D(r,t)$, a property known as *subadditivity*.[3] Scholten and Read proposed their own discount function, which we call the *RS discount function*. RS can account for the common difference effect either through declining impatience, subadditivity or both. Thus, the RS discount function is the most general discount function available.

LP gave axiomatic foundations for their LP discount function.[4] No axiomatic foundations have been given for the RS discount function.

We specify our axioms in terms of three functions: The *delay function*, Ψ, the *speedup function, f*, and the *generating function*, φ. Each of these functions can be uniquely elicited from behaviour.

Ψ determines stationarity and the common difference effect (Proposition 8). f determines whether impatience is declining, constant or increasing (Proposition 9). Ψ and f jointly determine whether the preferences are subadditive, additive or superadditive (Proposition 10).

The f and φ jointly determine a unique discount function. Conversely, a continuous discount function determines unique f and φ (Proposition 7).

All proofs are in Section 10.

---

[1] By contrast, a decision maker who exhibits stationary preference over time, and who is indifferent between one apple today and two apples tomorrow, must also be indifferent between one apple in 50 days and two apples in 51 days.

[2] The common difference effect can also be explained, again through declining impatience, by the $\beta - \delta$ form of hyperbolic discounting (quasi-hyperbolic discounting) due to Phelps and Pollak (1968) and Laibson (1997).

[3] The converse property is superadditivity. Scholten and Read (2011a,b) report evidence for both subaddivity and superadditivity.

[4] See al-Nowaihi and Dhami (2006, 2008a, 2009).

2 Formulation

Consider a decision maker who, at time t_0, takes an action that results in the outcome w_i at time t_i, $i = 1, 2, ..., n$, where

$$t_0 \leq r \leq t_1 < ... < t_n. \tag{1}$$

Time r (which we call the *reference time*) is the time back to which all values are to be discounted, using a discount function, $D(r,t)$; r need not be the same as t_0. Without loss of generality, we normalize the time at which the decision is made to be $t_0 = 0$.

We assume that the decision maker has a *reference outcome level*, w_0, relative to which all outcomes are to be evaluated using the prospect theory utility function, $v(x_i)$, introduced by Kahneman and Tversky (1979), where $x_i = w_i - w_0$.

The utility function, $v : (-\infty, \infty) \to (-\infty, \infty)$, has four main properties: *reference dependence* ($v(0) = 0$), *monotonicity* (v is strictly increasing), *declining sensitivity* (v is concave for gains, $x \geq 0$, but convex for losses, $x \leq 0$), and *loss aversion* (for $x > 0$: $-v(-x) > v(x)$). There is good empirical support for these features; see, for instance, Kahneman and Tversky (2000). Furthermore, it is assumed that v is continuous.

For each *reference outcome* and *reference time* pair $(w_0, r) \in (-\infty, \infty) \times [0, \infty)$, the decision maker has a complete and transitive preference relation, $\preceq_{w_0,r}$ on $(-\infty,\infty)\times[r,\infty)$ given by

$$(w_1, t_1) \preceq_{w_0,r} (w_2, t_2) \Leftrightarrow v(w_1 - w_0) D(r, t_1) \leq v(w_2 - w_0) D(r, t_2). \tag{2}$$

Let $\mathbf{S}$ be a non-empty set of outcome-time sequences from $(-\infty, \infty) \times [0, \infty)$ of the form $(x_1, t_1), (x_2, t_2), ..., (x_i, t_i), ...$ Using (2), we extend $\preceq_{w_0,r}$ to a complete transitive preference relation on sequences in $\mathbf{S}$, as follows[5]:

$$\begin{aligned}((x_1, s_1), (x_2, s_2), ..., (x_m, s_m)) &\preceq_{w_0,r} ((y_1, t_1), (y_2, t_2), ..., (y_n, t_n)) \\ &\Leftrightarrow \Sigma_{i=1}^{m} v(x_i) D(r, s_i) \leq \Sigma_{i=1}^{n} v(y_i) D(r, t_i).\end{aligned} \tag{3}$$

Thus, the decision maker's intertemporal utility function is given by:

$$V_r((w_1, t_1), (w_2, t_2), ..., (w_n, t_n), w_0) = \Sigma_{i=1}^{n} v(x_i) D(r, t_i), \tag{4}$$

For an additive discount function (Definition 5, below) there is no loss in assuming $r = 0$. To accommodate the empirical evidence, however, we allow the discount function to be non-additive, in which case the choice of reference time, r, does matter.

[5] The following also holds for infinite sequences, provided the sums in (3) converge.

3 Discount functions and their properties

We now give a formal definition of a *discount function*.

Definition 1 *(Discount functions): Let*

$$\Delta = \{(r, t) \in \mathbb{R} \times \mathbb{R} : 0 \leq r \leq t\}. \tag{5}$$

A discount function is a mapping, $D : \Delta \to (0, 1]$, satisfying:
(a) For each $r \in [0, \infty)$, $D(r, t)$ is a strictly decreasing function of $t \in [r, \infty)$ into $(0, 1]$ with $D(r, r) = 1$.
(b) For each $t \in [r, \infty)$, $D(r, t)$ is a strictly increasing function of $r \in [0, t]$ into $(0, 1]$.
(c) Furthermore, if D satisfies (a) with 'into' replaced with 'onto', then we call D a continuous discount function.

Outcomes that are further out into the future are less salient, hence, they are discounted more, thus, $D(r, t)$ is strictly decreasing in t. For a fixed t, if the reference point, r, becomes closer to t then an outcome at time t, when discounted back to r, is discounted less. Hence, $D(r, t)$ is strictly increasing in r.

Our terminology suggests that a continuous discount function (Definition 1(c)) is continuous. That this is partly true, is established by the following Proposition.

Proposition 1 *: A continuous discount function, $D(r, t)$, is continuous in t.*

Proposition 2 *(Time sensitivity): Let D be a continuous discount function. Suppose $r \geq 0$. If $0 < x \leq y$, or if $y \leq x < 0$, then $v(x) = v(y) D(r, t)$ for some $t \in [r, \infty)$.*

Proposition 3 *(Existence of present values): Let D be a discount function. Let $r \leq t$ and $y \geq 0$ $(y \leq 0)$. Then, for some x, $0 \leq x \leq y$ $(y \leq x \leq 0)$, $v(x) = v(y) D(r, t)$.*

Definition 2 *(Stationarity): Stationarity holds if for all outcomes $0 \leq x \leq y$ and all times $r \geq 0$, $s \geq r$, $t \geq r$, $v(x) = v(y) D(r, s)$ implies $v(x) D(r, t) = v(y) D(r, s + t)$.*

Definition 3 *(Common difference effect): The common difference effect arises if, for all outcomes $0 < x < y$ and all times $r \geq 0$, $s > r$, $t > r$, $v(x) = v(y) D(r, s)$ implies $v(x) D(r, t) < v(y) D(r, s + t)$.*

Suppose a decision maker is indifferent between one apple today and two apples tomorrow. Then, from stationarity (Definition 2), it follows that this decision maker is also indifferent between one apple in 50 days' time and two apples in 51 days' time. However, what has been observed (Thaler, 1981) is that the same decision maker, today, prefers to receive two apples in 51 days' time to receiving one apple in 50 days' time. The latter is an example of the common difference effect (Definition 3).

We now formalize the sense in which an individual may exhibit various degrees of impatience. The basic idea is to shift a time interval of a given size into the future and observe if this leads to a smaller, unchanged or larger discounting of the future.

Definition 4 *(Impatience): A discount function,* $D(r,s)$*, exhibits*[6]

$$\begin{cases} \textit{declining impatience if} & D(r,s) < D(r+t, s+t), \quad \textit{for } t>0 \textit{ and } 0 \leq r < s, \\ \textit{constant impatience if} & D(r,s) = D(r+t, s+t), \quad \textit{for } t \geq 0 \textit{ and } 0 \leq r \leq s, \\ \textit{increasing impatience if} & D(r,s) > D(r+t, s+t), \quad \textit{for } t>0 \textit{ and } 0 \leq r < s. \end{cases}$$

Definition 5 *(Additivity): A discount function,* $D(r,t)$*, is*

$$\begin{cases} \textit{Subadditive if} & D(r,s)\,D(s,t) < D(r,t), \quad \textit{for } 0 \leq r < s < t, \\ \textit{Additive if} & D(r,s)\,D(s,t) = D(r,t), \quad \textit{for } 0 \leq r \leq s \leq t, \\ \textit{Superadditive if} & D(r,s)\,D(s,t) > D(r,t), \quad \textit{for } 0 \leq r < s < t. \end{cases}$$

In Definition 5, additivity implies that discounting a quantity from time t back to time s and then further back to time r is the same as discounting that quantity from time t back to time r in one step. In other words, breaking an interval into subintervals has no effect on discounting. However, in the other two cases, it does have an effect. Under subadditive discounting, there is more discounting over the subdivided intervals (future utilities are shrunk more), while the converse is true under superadditive discounting.[7]

In theories of time discounting that do not have a reference time, the discount functions are stated under the implicit assumption that $r = 0$. Hence, we first need to restate the main discount functions for the case $r > 0$. We extend four common discount functions to the case $r > 0$. The standard versions of these functions can simply be obtained by setting $r = 0$; the reason for the choice of the acronyms corresponding to these functions will become clear in Table 1, below.

The *exponential discount function* (second row of Table 1) was introduced by Samuelson (1937). The main attraction of EDU is that it is the unique discount function that leads to time-consistent choices. The $\beta - \delta$ or *quasi-hyperbolic discount function* (3rd to 5th rows in Table 1) was proposed by Phelps and Pollak (1968) and Laibson (1997) and is popular in applied work (we use the acronym PPL for it).[8] The *generalized hyperbolic discount*

[6] Some authors use 'present bias' for what we call 'declining impatience'. But other authors use 'present bias' to mean that the discount function, $D(s,t)$ is declining in t. So we prefer 'declining impatience' to avoid confusion. It is common to use 'stationarity' for what we call 'constant impatience'. We prefer the latter, to be in conformity with 'declining impatience' and 'increasing impatience'.

[7] For empirical evidence on subadditive discounting, see Read (2001).

[8] It can be given the following psychological foundation. The decision maker essentially uses exponential discounting. But in the short run is overcome by *visceral influences* such as temptation or procrastination; see for instance Loewenstein et al. (2001).

	$D(r,t)$
Exponential	$e^{-\beta(t-r)}$, $\beta > 0$.
PPL $(r = t = 0)$	1,
PPL $(0 = r < t)$	$e^{-(\delta+\beta t)}$, $\beta > 0$, $\delta > 0$,
PPL $(0 < r \leq t)$	$e^{-\beta(t-r)}$.
LP	$\left(\frac{1+\alpha t}{1+\alpha r}\right)^{-\frac{\beta}{\alpha}}$, $t \geq 0$, $r \geq 0$, $\alpha > 0$, $\beta > 0$.
RS	$\left[1+\alpha\left(t^{\tau}-r^{\tau}\right)^{\rho}\right]^{-\frac{\beta}{\alpha}}$, $0 \leq r \leq t$, $\alpha > 0$, $\beta > 0$, $\rho > 0$, $\tau > 0$.
GRS	$e^{-Q[w(t)-w(r)]}$, $0 \leq r \leq t$, where,
	$Q : [0,\infty) \overset{\text{onto}}{\rightarrow} [0,\infty)$ is strictly increasing,
	$w : [0,\infty) \overset{\text{onto}}{\rightarrow} [0,\infty)$ is strictly increasing.

Table 1: Examples of discount functions

function (6th row in Table 1) was proposed by Loewenstein and Prelec (1992) (we use the acronym LP for it). These three discount functions are *additive* (Definition 5). They can account for the common difference effect through declining impatience (Definition 4) but they cannot account for either non-additivity or intransitivity.[9] The interval discount function (7th row in Table 1) was introduced by Read (2001) and Scholten and Read (2006a) (we use the acronym RS for it). It can account for both non-additivity and intransitivity. It can account for the common difference effect through declining impatience, subadditivity or a combination of both.

Scholten and Read (2006b) present a critique of the psychological basis for discounting models (including their own). They develop an attribute model based on firmer psychological foundations. al-Nowaihi and Dhami (2008b, subsection 5.1, pp 38-40) proposed the GRS discount function (last three rows of Table 1) and argued that the Scholten and Read (2006b) tradeoff model is equivalent to a discounted utility model with the GRS discount function. Thus, their tradeoff model lends support to the GRS discount function and, in particular, their own discount function, the RS discount function.

Note that LP approaches the exponential as $\alpha \to 0$. In general, neither LP nor RS is a special case of the other. However, for $r = 0$ (and only for $r = 0$), RS reduces to LP when $\rho = \tau = 1$.[10]

It is straightforward to check that each of the exponential, LP, RS and GRS is a continuous discount function in the sense of Definition 1. It is also straightforward to check that PPL (rows 3-5 of Table 1) is a discount function. The reason that the latter is not a *continuous* discount function is that $\lim_{t\to 0+} D(0,t) = e^{-\delta} < 1 = D(0,0)$.

[9] However, note that PPL exhibits the common difference effect and declining impatience in a more restricted sense of Definitions 3, 4 where r is set equal to 0.

[10] Scholten and Read (2006a) report incorrectly that the LP-discount function is a special case of the RS-discount function.

Note that the restrictions $r \geq 0$ and $t \geq 0$ are needed in LP and RS (Table 1, rows 6 and 7). From LP we see that the further restriction $r \leq t$ is needed.[11] From PPL (Table 1, rows 3-5) we see that the 'into' in Definition 1(b) cannot be strengthened to 'onto'.

Exponential discounting exhibits constant impatience while LP exhibits declining impatience. The RS discount function (Table 1, row 7) allows for additivity, subadditivity and all the three cases in Definition 4, hence, it is of great practical importance.

4 Delay and speedup functions

Here we define two functions: The speedup function, f, and the delay function, Ψ. Each can be uniquely elicited from behaviour. Ψ determines stationarity and the common difference effect. f determines impatience. Additivity is jointly determined by f and Ψ.

4.1 Delay Function, Ψ

Let the reference time be $r \geq 0$. Suppose that a decision maker reveals the following indifference: x received at time r is equivalent to y received at time $t \geq r$, thus,

$$v(x) = v(y) D(r,t). \tag{6}$$

Now suppose that the receipt of x is *delayed* to time s. We ask, at what time, T, will y received at time T be equivalent to x received at time s, i.e., for what T does the following hold?

$$v(x) D(r,s) = v(y) D(r,T). \tag{7}$$

Let us conjecture that T depends on r, s, t through a functional relation, say, $T = \Psi(r,s,t)$. From (6), (7) we get that $\Psi(r,s,t)$ must satisfy

$$D(r,s) D(r,t) = D(r, \Psi(r,s,t)). \tag{8}$$

We shall call the function $\Psi(r,s,t)$, if it exists, a delay function (see Definition 6). For the exponential discount function (second row in Table 1), the answer is clear: $\Psi(r,s,t) = s+t$. More generally, we show that such a delay function exists, is unique and depends on r, s, t. We shall also examine its properties (see Propositions 4, 5).

Definition 6 *(Delay functions): Let D be a discount function. Suppose that the function, Ψ, has the property $D(r,s) D(r,t) = D(r, \Psi(r,s,t))$, $s \geq r$, $t \geq r$. Then we call Ψ a delay function corresponding to the discount function, D.*

[11] One alternative is to *define* $D(t,s)$ to be $1/D(s,t)$. But we do not know if people, when compounding forward, use the inverse of discount function (as they should, from a normative point of view). Fortunately, we have no need to resolve these issues in this paper.

Proposition 4 *(Properties of a delay function): Let D be a discount function and Ψ a corresponding delay function. Then Ψ has the following properties:*
(a) Ψ is unique,
(b) $\Psi(r,s,t)$ is strictly increasing in each of s and t,
(c) $\Psi(r,s,t) = \Psi(r,t,s)$,
(d) $\Psi(r,r,t) = \Psi(r,t,r) = t$,
(e) $v(x) = v(y) D(r,t)$ if, and only if, $v(x) D(r,s) = v(y) D(r, \Psi(r,s,t))$.

Proposition 5 *(Existence of a delay function): A continuous discount function has a unique delay function, $\Psi(r,s,t)$. $\Psi(r,s,t)$ is continuous in s,t.*

4.2 Speedup function, f

Suppose that x received at time r is equivalent to y received at time t, $0 \leq r \leq t$, time r being the reference time; so that (recall that $D(r,r) = 1$)

$$v(x) = v(y) D(r,t). \tag{9}$$

Suppose that the receipt of x is brought forward from time r to time 0, where time 0 is the new reference time. We ask, at what time, T, will y received at time T be equivalent to x received at time 0? Or, for what time, T, will the following hold?

$$v(x) = v(y) D(0,T). \tag{10}$$

For the exponential discount function the answer is clear: $T = t - r$. More generally, let us conjecture that T depends on r, t, so that we can write $T = f(r,t)$ where $f : \Delta \to [0,\infty)$ will be called the *speedup function*. Definition 7 formally defines a speedup function.

Definition 7 *Let $f : \Delta \to [0,\infty)$ satisfy:*
(a) For each $r \in [0,\infty)$, $f(r,t)$ is a strictly increasing function of $t \in [r,\infty)$ into $[0,\infty)$, with $f(r,r) = 0$.
(b) For each $t \in [0,\infty)$, $f(r,t)$ is a strictly decreasing function of $r \in [0,t]$ into $[0,t]$, with $f(0,t) = t$.
Then we call f a speedup function. If, in (a), 'into' is replaced with 'onto', then we call f a continuous speedup function.

A 'continuous speedup function', $f(r,t)$, is continuous in t. The proof is the same as that of Proposition 1 and, therefore, will be omitted.

5 Eliciting a discount function from observed behaviour

We define a third useful function: The generating function, φ. Like the delay function, Ψ, and the speedup function, f, the generating function, φ, can be uniquely elicited from behaviour. φ and f jointly determine a unique discount function $D(r,t) = \varphi(f(r,t))$. Conversely, a given continuous discount function determines unique generating and speedup functions φ and f (which are then continuous).

Definition 8 *(The generating function): Let $\varphi : [0,\infty) \to (0,1]$ be a strictly decreasing function with $\varphi(0) = 1$. Then we call φ a generating function. If, in addition, φ is onto, we call φ a continuous generating function.*

A 'continuous generating function' is continuous. The proof is the same as that of Proposition 1 and, therefore, will be omitted.

Several axiom systems have been proposed for behaviour under risk that generate a continuous prospect theory utility function representation; and methods have been proposed and used to elicit this prospect theory function from observed behaviour. See, for example, Wakker (2010). The important point for us is that, because of reference dependence ($v(0) = 0$), such a function is measurable on the ratio scale, i.e., u and v represent the same prospect theory preferences if, and only if, $u = \alpha v$ for some positive constant, α. The following proposition is an immediate consequence of this.

Proposition 6 *(Uniqueness of the generating function): The generating function, φ, is uniquely determined by behaviour.*

Proposition 7 *(representation theorem):*
(a) Suppose φ is a generating function (Definition 8) and f is a speedup function (Definition 7). Then $D = \varphi \circ f$ is a discount function (Definition 1). In particular, $D(0,t) = \varphi(t)$. If φ and f are continuous, then so is D.
(b) If D is a continuous discount function, then there are unique generation and speedup function, φ and f, such that $D = \varphi \circ f$. Moreover, φ and f are also continuous.
(c) D is additive if, and only if, $D(r,t) = \frac{\varphi(t)}{\varphi(r)}$ for all $0 \leq r \leq t$.

6 Stationarity, impatience and additivity

Here we show how the properties of stationarity, common difference effect, impatience and subadditivity/additivity/superadditivity determine, and are determined by, the properties of the delay and speedup functions. In what follows, Ψ and f are the delay and speedup functions corresponding to the discount function, D.

Proposition 8 *: A continuous discount function with the delay function, Ψ, exhibits:*
(a) stationarity if, and only if, $\Psi(r,s,t) = s + t$, *for all* $r \geq 0$, $s \geq r$, $t \geq r$,
(b) the common difference effect if, and only if, $\Psi(r,s,t) > s + t$, *for all* $r \geq 0$, $s > r$, $t > r$.

Proposition 9 *: A continuous discount function with the speedup function, f, exhibits:*
(a) declining impatience if, and only if, $f(r,s) > f(r+t, s+t)$, *for all* $t > 0$, $s > r$,
(b) constant impatience if, and only if, $f(r,s) = f(r+t, s+t)$, *for all* $t \geq 0$, $s \geq r$,
(c) increasing impatience if, and only if, $f(r,s) < f(r+t, s+t)$, *for all* $t > 0$, $s > r$.

Proposition 10 *: A continuous discount function with delay function, Ψ, and speedup function, f, exhibits:*
(a) subadditivity if, and only if, $f(r,t) < \Psi(0, f(r,s), f(s,t))$, *for all* $0 \leq r < s < t$,
(b) additivity if, and only if, $f(r,t) = \Psi(0, f(r,s), f(s,t))$, *for all* $0 \leq r \leq s \leq t$,
(c) superadditivity, and only if, $f(r,t) > \Psi(0, f(r,s), f(s,t))$, *for all* $0 \leq r < s < t$.

Proposition 11 *: The Table 2, below, gives the generating function, φ, and the speedup function, f, of each of the discount functions $D(r,t)$, given in Table 1.*

	$\varphi(t)$	$f(r,t)$	$\Psi(r,s,t)$
Exponential	$e^{-\beta t}$	$t - r$	$s + t$
PPL $(r = t = 0)$	1	0	s
PPL $(0 = r < t)$	$e^{-(\delta + \beta t)}$	t	$\frac{\delta}{\beta} + s + t$
PPL $(0 < r \leq t)$	$e^{-\beta t}$	$t - r$	$s + t$
LP	$(1 + \alpha t)^{-\frac{\beta}{\alpha}}$	$\frac{t-r}{1+\alpha r}$	$s + t + \alpha s t$
RS	$\left[1 + \alpha t^{\tau\rho}\right]^{-\frac{\beta}{\alpha}}$	$(t^{\tau} - r^{\tau})^{\frac{1}{\tau}}$	$\left[s^{\tau\rho} + t^{\tau\rho} + \alpha (st)^{\tau\rho}\right]^{\frac{1}{\tau\rho}}$
Generalized RS	$e^{-Q[w(t)]}$	$w^{-1}[w(t) - w(r)]$	$w^{-1}Q^{-1}[Q(w(s)) + Q(w(t))]$

Table 2: Generating, speedup and delay functions for the discount functions in Table 1

Proposition 12 *: Let $D(r,t)$ be the RS-discount function (Table 1, row 7), then:*
(a) If $0 < \rho \leq 1$, *then D is subadditive.*
(b) If $\rho > 1$, *then D is neither subadditive, additive nor superadditive.*
(ci) If $0 < \tau < 1$, *then D exhibits declining impatience.*
(cii) If $\tau = 1$, *then D exhibits constant impatience.*
(ciii) If $\tau > 1$, *then D exhibits increasing impatience.*

In the light of Proposition 12, we can now see the interpretation of the parameters τ and ρ in the RS-discount function (Table 1, row 7).[12] τ controls impatience, independently

[12] Scholten and Read (2006), bottom of p1425, state: $\alpha > 0$ implies subadditivity (incorrect), $\rho > 1$ implies superadditivity (incorrect) and $0 < \tau < 1$ implies declining impatience (correct but incomplete).

of the values of the other parameters α, β and ρ. $0 < \tau < 1$, gives declining impatience, $\tau = 1$ gives constant impatience and $\tau > 1$ gives increasing impatience. If $0 < \rho \leq 1$, then we get subadditivity, irrespective of the values of the other parameters α, β and τ. However, if $\rho > 1$, then the RS-discount function (Table 1, row 7) can be neither subadditive, additive nor superadditive[13].

7 Are intransitivities due to shifts in reference time?

We now investigate if some observed intransitive choices can be explained by shifts in the reference time. Consider the following hypothetical situation. A decision maker prefers a payoff of 1 now to a payoff of 2 next period. The decision maker also prefers a payoff of 2 next period to a payoff of 3 two periods from now. Finally, the same decision maker prefers a payoff of 3 two periods from now to a payoff of 1 now. Schematically:

$$(1, \text{now}) \succ (2, \text{next period}) \succ (3, 2 \text{ two periods from now}) \succ (1, \text{now}) . \tag{11}$$

Ok and Masatlioglu (2007, p215) use a similar example to motivate their intransitive theory of relative discounting.

Alternatively, we may view (11) as due to a framing effect resulting in a shift in the reference point for time. Assume that the choice of reference time in each pairwise comparison is the sooner of the two dates, in conformity with Assumption A0. Then (11) can be formalized as follows:

$$V_0 (1, 0) > V_0 (2, 1) \, , \; V_1 (2, 1) > V_1 (3, 2) \, , \; V_0 (3, 2) > V_0 (1, 0) \, . \tag{12}$$

Thus, the decision maker prefers a payoff of 1 now to a payoff of 2 next period, both discounted back to the present. The decision maker also prefers a payoff of 2 next period to a payoff of 3 the following period, both discounted back to next period. Finally, the decision maker prefers a payoff of 3 in two periods from now to a payoff of 1 now, both discounted back to the present. If this view is accepted, then the apparent intransitivity in (11) arises from conflating $V_0 (3, 2)$ with $V_1 (3, 2)$ and $V_1 (2, 1)$ with $V_0 (2, 1)$. The following example shows that (12) is consistent with a reference-time theory of intertemporal choice.

Example 1 *: Take the reference point for wealth to be the current level of wealth, so each payoff is regarded as a gain to current wealth. Take the utility function to be*[14]

$$v (x) = x^{\frac{1}{2}} (1 + x)^{\frac{1}{2}} \, , \; x \geq 0. \tag{13}$$

[13] In this case, depending on the particular values of r, s and t, we may have $D (r, s) D (s, t) < D (r, t)$, $D (r, s) D (s, t) = D (r, t)$ or $D (r, s) D (s, t) > D (r, t)$).

[14] This is the SIE utility function of al-Nowaihi and Dhami (2009), with $\mu = 1$, $\theta_+ = 0.5$, $\lambda = 2$, $0 < \sigma = \gamma = 0.5$.

Thus (working to five significant figures),

$$v(1) = 1.4142,\ v(2) = 2.4495 \text{ and } v(3) = 3.4641. \tag{14}$$

As our discount function we take the Read-Scholten discount function, RS (7th row in Table 1) with $\alpha = \beta = 1$ and $\rho = \tau = \frac{1}{2}$:

$$D(r,t) = \left(1 + \left(t^{\frac{1}{2}} - r^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-1}. \tag{15}$$

Thus,

$$D(0,1) = \frac{1}{2},\ D(1,2) = 0.60842 \text{ and } D(0,2) = 0.45679. \tag{16}$$

From (14) and (16) we get

$$V_0(1,0) = v(1)\,D(0,0) = 1.4142, \tag{17}$$

$$V_0(3,2) = v(3)\,D(0,2) = 1.5824, \tag{18}$$

$$V_1(3,2) = v(3)\,D(1,2) = 2.1076, \tag{19}$$

$$V_1(2,1) = v(2)\,D(1,1) = 2.4495, \tag{20}$$

$$V_0(2,1) = v(2)\,D(0,1) = 1.2248. \tag{21}$$

From (17) to (21), we get

$$V_0(1,0) > V_0(2,1),\ V_1(2,1) > V_1(3,2),\ V_0(3,2) > V_0(1,0), \tag{22}$$

confirming (12).

No additive discount function (e.g., Exponential, PPL or LP in Table 1) can explain (apparently) intransitive choices as exhibited in (11). The reason is that, under additivity, all utilities can be discounted back to time zero and, hence, can be compared and ordered.

8 The tradeoff model of intertemporal choice

Read and Scholten's critique of discounting models, including their own, led them to develop their *tradeoff model* of intertemporal choice (Scholten and Read, 2006b). We argue that the tradeoff model of Scholten and Read (2006b) can be incorporated within RT-theory. If this is accepted, then their tradeoff model lends further support to the RT-theory and, in particular, their own discount function, RS, and its generalization, GRS (Table 1).

We proceed by first recasting their model in a more general form (and indicate how their model is to be obtained as a special case). However, there should be no presumption that they would agree with our reformulation. They develop their model through three successive versions. We concentrate on their fourth and final version, page 15.

Let $r \geq 0$ be the reference point for time.[15] The tradeoff model establishes preference relationships, $\prec_r$ and $\sim_r$ between outcome-time pairs (x, s) and (y, t) when both outcomes are discounted back to the reference time, r. Thus $(x, s) \prec_r (y, t)$ if, and only if, y received at time t is strictly preferred to x received at time s. Similarly, $(x, s) \sim_r (y, t)$ if, and only if, y received at time t is equivalent to x received at time s. These relationship are established using three functions, a *value function*, u, a *tradeoff function* Q and a *delay-perception function*, ϕ.

We assume that $Q : [0, \infty) \overset{\text{onto}}{\rightarrow} [0, \infty)$ is strictly increasing, $\phi : [0, \infty) \overset{\text{onto}}{\rightarrow} [0, \infty)$ is strictly increasing.[16]

Let $x > 0$, $y > 0$ and $s \geq r \geq 0$, $t \geq r$, then the decision criteria in this model is given by:

$$(x, s) \sim_r (y, t) \Leftrightarrow Q[\phi(t) - \phi(r)] - Q[\phi(s) - \phi(r)] = u(y) - u(x), \tag{23}$$
$$(x, s) \prec_r (y, t) \Leftrightarrow Q[\phi(t) - \phi(r)] - Q[\phi(s) - \phi(r)] < u(y) - u(x). \tag{24}$$

To understand these inequalities, suppose that $x < y$ and $s < t$. The decision maker then has a choice between a *smaller-sooner* (SS) reward, (x, s), and a *larger later* (LL) reward, (y, t). The two *attributes* are outcome and time. The *advantage of the SS reward* is along the time dimension because it is available at an earlier date. This is indicated by the LHS of the inequalities in (23), (24). Alternatively this may be termed as the *disadvantage of the LL reward*. The *advantage of the LL reward* (alternatively the *disadvantage of the SS reward*) is that it offers a higher outcome; this is indicated by the RHS of the inequalities in (23), (24). Thus, in (24), the LL reward is strictly preferred to the SS reward if the advantage of the SS reward is smaller than the advantage of the LL reward.

We now state the analogue of the decision criteria in (23), (24) when the outcomes are losses: $x < 0$, $y < 0$ and (as before) $s \geq r \geq 0$, $t \geq r$. In this case:

$$(x, s) \sim_r (y, t) \Leftrightarrow Q(\phi(t) - \phi(r)) - Q(\phi(s) - \phi(r)) = u(x) - u(y), \tag{25}$$
$$(x, s) \prec_r (y, t) \Leftrightarrow Q(\phi(t) - \phi(r)) - Q(\phi(s) - \phi(r)) > u(x) - u(y). \tag{26}$$

For losses, let $y < x < 0$. Then, the RHS of the inequality in (26), $u(x) - u(y)$, is the advantage of the SS reward, (x, s). Since both rewards are losses, the LHS of (26) becomes the advantage of the LL reward, (y, t).

[15] In Read and Scholten (2006b), $r = 0$. To ease the burden of notation, we shall suppress reference to the reference point for wealth, w_0. Thus, in what follows, we write $\prec_r$ and $\sim_r$ when we should have written $\prec_{r,w_0}$ and $\sim_{r,w_0}$, respectively.

[16] They explicitly state two assumptions: $Q' > 0$, $Q'' < 0$. However, in the next paragraph, they say that $Q'' > 0$ for sufficiently small intervals. So, we make no assumptions on Q''. They explicitly state no further assumptions on Q and w. However, we believe our other assumptions on Q and w are in line with what they intend (see their equations (2) and (5) for the earlier, and simpler, versions of their model).

For completeness, we also need (again, $s \geq r \geq 0$, $t \geq r$) to specify the following properties:

$$(0, s) \sim_r \ (0, t), \tag{27}$$
$$x < 0 \Rightarrow (x, s) \prec_r (0, t), \tag{28}$$
$$y > 0 \Rightarrow (0, s) \prec_r (y, t), \tag{29}$$
$$x < 0, y > 0 \Rightarrow (x, s) \prec_r (y, t). \tag{30}$$

From (27), the time at which a zero outcome is received is irrelevant. From (28), a reward of zero is always preferred to a negative reward, irrespective of the time. From (29), a positive reward is always preferred to a zero reward, irrespective of the time. From (30), a positive outcome is always preferred to a negative one, irrespective of the time.

To get the tradeoff model of Read and Scholten, set $r = s$ in the above equations.[17] However, RT theory allows for the reference time $r \geq s$.

To define a discount function, D, that expresses these preferences, let

$$v(x) = e^{u(x)}, \text{ for } x > 0, \tag{31}$$
$$v(x) = -e^{-u(x)}, \text{ for } x < 0. \tag{32}$$

Then all the above relations, (23) to (30), can be summarized by the following. For all x, y and all r, s, t such that $s \geq r \geq 0$, $t \geq r$:

$$(x, s) \sim_r (y, t) \Leftrightarrow v(x) e^{-Q[\phi(s)-\phi(r)]} = v(y) e^{-Q[\phi(t)-\phi(r)]}, \tag{33}$$
$$(x, s) \prec_r (y, t) \Leftrightarrow v(x) e^{-Q(\phi(s)-\phi(r))} < v(y) e^{-Q[\phi(t)-\phi(r)]}. \tag{34}$$

(33) and (34) suggest we take our discount function to be the generalized RS function, GRS, which is a generalization of the discount function RS of Scholten and Read (2006a). Thus, RT-theory can incorporate the tradeoff model.

9 Summary

A continuous discount function, D, determines a unique (and continuous) generating function, $\varphi(t) = D(0, t)$, and a unique (and continuous) speedup function, f, so that $D(r, t) = \varphi(f(r, t))$. In particular, D is additive if, and only if, $D(r, t) = \frac{\varphi(t)}{\varphi(r)}$.

A continuous discount function, D, also determine a unique (and continuous) delay function, Ψ. Whether preferences exhibit stationarity or the common difference effect is determined by Ψ. Whether impatience is declining, constant or increasing is determined by f. Ψ and f jointly determine whether preferences are subadditive, additive or super-additive.

[17] They explicitly state only (23) and (25) (with $r = s$). However, we believe that our other equations are in line with their framework.

10 Proofs

Proposition 1: Let $r \in [0, \infty)$ and $t \in [r, \infty)$. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[r, \infty)$ converging to t. We want to show that $\{D(r, t_n)\}_{n=1}^{\infty}$ converges to $D(r, t)$. It is sufficient to show that any monotone subsequence of $\{D(r, t_n)\}_{n=1}^{\infty}$ converges to $D(r, t)$. In particular, let $\{D(r, t_{n_i})\}_{i=1}^{\infty}$ be a decreasing subsequence of $\{D(r, t_n)\}_{n=1}^{\infty}$. Since $\{D(r, t_{n_i})\}_{i=1}^{\infty}$ is bounded below by $D(r, t)$, it must converge to, say, q, where $D(r, t) \leq q \leq D(r, t_{n_i})$, for all i. Since D is onto, there is a $p \in [r, \infty)$ such that $D(r, p) = q$. Moreover, $t_{n_i} \leq p \leq t$, for each i. Suppose $D(r, t) < q$. Then $t_{n_i} < p$, for each i. Hence also $t_{n_i} < t$, for each i. But this cannot be, since $\{t_{n_i}\}_{i=1}^{\infty}$, being a subsequence of the convergent sequence $\{t_n\}_{n=1}^{\infty}$, must also converge to the same limit, t. Hence, $D(r, t) = q$. Hence, $\{D(r, t_{n_i})\}_{i=1}^{\infty}$ converges to $D(r, t)$. Similarly, we can show that any increasing subsequence of $\{D(r, t_n)\}_{n=1}^{\infty}$ converges to $D(r, t)$. Hence, $\{D(r, t_n)\}_{n=1}^{\infty}$ converges to $D(r, t)$. Hence, $D(r, t)$ is continuous in t. $\blacksquare$.

Proposition 2: Let $D(r, t)$ be a continuous discount function and $r \geq 0$. Suppose $0 < x \leq y$. Since v is strictly increasing with $v(0) = 0$, it follows that $0 < v(x) \leq v(y)$ and, hence, $0 < \frac{v(x)}{v(y)} \leq 1$. Since, by Definition 1(c), $D(r, t) : [r, \infty) \overset{\text{onto}}{\rightarrow} (0, 1]$, it follows that $\frac{v(x)}{v(y)} = D(r, t)$ for some $t \in [r, \infty)$. A similar argument applies if $y < x < 0$. $\blacksquare$.

Proposition 3: Let $r \leq t$ and $y \geq 0$. Then, $0 < D(r, t) \leq 1$. Hence, $0 = v(0) \leq v(y) D(r, t) \leq v(y)$. Since v is continuous and strictly increasing, it follows that $v(y) D(r, t) = v(x)$ for some $x \in [0, y]$. Similarly, if $y \leq 0$, then $v(y) D(r, t) = v(x)$ for some $x \in [y, 0]$. $\blacksquare$.

Proposition 4: Let D be a discount function and Ψ and Φ two corresponding delay functions. Let $s, t \in [r, \infty)$. Then $D(r, \Phi(r, s, t)) = D(r, s) D(r, t) = D(r, \Psi(r, s, t))$. Since $D(r, \tau)$ is strictly decreasing in τ, we must have $\Phi(r, s, t) = \Psi(r, s, t)$. This establishes (a). Using Definition 1, it is straightforward to check that properties (b) to (d) follow from Definition 6. Now, suppose $v(x) = v(y) D(r, t)$. Multiply both sides by $D(r, s)$ to get $v(x) D(r, s) = v(y) D(r, s) D(r, t) = v(y) D(r, \Psi(r, s, t))$. Conversely, suppose $v(x) D(r, s) = v(y) D(r, \Psi(r, s, t))$. Then $v(x) D(r, s) = v(y) D(r, s) D(r, t)$. Since $D(r, s) > 0$, we can cancel it to get $v(x) = v(y) D(r, t)$. This establishes (e). $\blacksquare$.

Proposition 5: Let D be a continuous discount function. Let $s, t \in [r, \infty)$. Then $D(r, s), D(r, t) \in (0, 1]$. Since $\tau \mapsto D(r, \tau)$ is onto $(0, 1]$, there is some $T \in [0, \infty)$ such that $D(r, s) D(r, t) = D(r, T)$. Since $D(r, \tau)$ is strictly decreasing in τ, this T is unique. Set $T = \Psi(r, s, t)$. The function, $\Psi(r, s, t)$, thus defined, is a delay function corresponding to D. Let $\psi(t) = D(r, t)$. Then $\psi : [r, \infty) \overset{onto}{\rightarrow} (0, 1]$ is strictly decreasing and continuous. Hence it has an inverse, ψ^{-1}, which is also strictly decreasing and continuous (proof as in Proposition 1). From Definition 6, we get $\psi(s) \psi(t) = \psi(\Psi(r, s, t))$. Hence, $\Psi(r, s, t) = \psi^{-1}(\psi(s) \psi(t))$ is continuous. $\blacksquare$.

Proposition 6: Suppose a decision maker, with prospect theory preferences, is indifferent between receiving $x > 0$ now and receiving $y > x$ at time $t > 0$. Let u and v be a prospect theory utility functions that represent this decision maker's preferences. Then, necessarily, $u(x) = u(y) D_1(0,t) = u(y) \varphi_1(t)$ and $v(x) = v(y) \varphi_2(t)$ for some discount functions D_1 and D_2. Hence, $\varphi_1(t) = \frac{u(x)}{u(y)} = \frac{\alpha v(x)}{\alpha v(y)} = \frac{v(x)}{v(y)} = \varphi_2(t)$. $\blacksquare$.

Proposition 7: (a) Since f maps Δ into $[0,\infty)$ and φ maps $[0,\infty)$ into $(0,1]$, it follows that $D = \varphi \circ f$ maps Δ into $(0,1]$. Since $f(r,r) = 0$ and $\varphi(0) = 1$, it follows that $D(r,r) = \varphi(f(r,r)) = 1$. Since $f(r,t)$ is strictly decreasing in r and strictly increasing in t, and since φ is strictly decreasing, it follows that $D(r,t) = \varphi(f(r,t))$ is strictly increasing in r and strictly decreasing in t. Hence, $D = \varphi \circ f$ is a discount function. In particular, $D(0,t) = \varphi(f(0,t)) = \varphi(t)$, by Definition 7b. Let $r \in [0,\infty)$. Suppose $f(r,.)$ maps $[r,\infty)$ onto $[0,\infty)$ and φ maps $[0,\infty)$ onto $(0,1]$, it follows that $D = \varphi \circ f$ maps Δ onto $(0,1]$. Hence, if φ and f are continuous, then so is D.

(b) Let D be a continuous discount function. Define $\varphi(t) = D(0,t)$. Then φ maps $[0,\infty)$ onto $(0,1]$. Since $D(0,t)$ is strictly decreasing in t, so is φ. Finally, $\varphi(0) = D(0,0) = 1$. Hence, φ is a continuous generating function. Since $\varphi : [0,\infty) \overset{onto}{\to} (0,1]$ is strictly decreasing, it follows that it has a unique inverse, $\varphi^{-1} : (0,1] \overset{onto}{\to} [0,\infty)$, which is also strictly decreasing. Define $f = \varphi^{-1} \circ D$. Since D is strictly increasing in r and strictly decreasing in t, and since φ^{-1} is strictly decreasing, it follows that f is strictly decreasing in r and strictly increasing in t. Furthermore, $f(r,r) = \varphi^{-1}(D(r,r)) = \varphi^{-1}(1) = 0$ and $f(0,t) = \varphi^{-1}(D(0,t)) = \varphi^{-1}(\varphi(t)) = t$. Since $D : \Delta \overset{onto}{\to} (0,1]$ and since $\varphi^{-1} : (0,1] \overset{onto}{\to} [0,\infty)$, it follows that $f = \varphi^{-1} \circ D$ maps Δ onto $[0,\infty)$. Hence, f is a continuous generating function. Suppose, $D = \varphi' \circ f'$ for generating and speedup functions $\varphi'$and f'. Then $\varphi(t) = D(0,t) = \varphi'(f'(0,t)) = \varphi'(t)$. Hence, $\varphi = \varphi'$. Hence, $D = \varphi \circ f'$. Hence, $f' = \varphi^{-1} \circ D = f$. Hence, φ and f are unique.

(c) If $D(r,t) = \frac{\varphi(t)}{\varphi(r)}$ for all $0 \leq r \leq t$, then $D(r,s) D(s,t) = \frac{\varphi(s)}{\varphi(r)} \frac{\varphi(t)}{\varphi(s)} = \frac{\varphi(t)}{\varphi(r)} = D(r,t)$ for all $0 \leq r \leq s \leq t$. Conversely, suppose $D(r,s) D(s,t) = D(r,t)$ for all $0 \leq r \leq s \leq t$, then, in particular, $D(0,r) D(r,t) = D(0,t)$ for all $0 \leq r \leq t$, i.e., $\varphi(r) D(r,t) = \varphi(t)$ for all $0 \leq r \leq t$. Hence, $D(r,t) = \frac{\varphi(t)}{\varphi(r)}$ for all $0 \leq r \leq t$. $\blacksquare$.

Proposition 8: Consider a continuous discount function, D. We first prove part (a).

(i) Assume D is stationary. Let $r \geq 0$, $s \geq r$, $t \geq r$. Choose $y > 0$. Hence, $v(y) > 0$. Since $v(0) = 0$, v is strictly increasing and $0 < D(r,s) \leq 1$, $v(y) D(r,s) = v(x)$ for some $x \in (0,1]$. Hence, by stationarity (Definition 2), $v(x) D(r,t) = v(y) D(r,s+t)$. Hence, $v(y) D(r,s) D(r,t) = v(y) D(r,s+t)$. Hence, $D(r,s) D(r,t) = D(r,s+t)$. By Definition 6 and Proposition 5, the continuous discount function, D, has a unique delay function, Ψ, and $D(r,s) D(r,t) = D(r,\Psi(r,s,t))$. Hence, $D(r,\Psi(r,s,t)) = D(r,s+t)$. Since $D(r,t)$ is strictly decreasing in t (Definition 1), we get $\Psi(0,s,t) = s+t$.

(ii) Assume that $\Psi(r,s,t) = s+t$, for all $r \geq 0$, $s \geq r$, $t \geq r$. Let $v(x) = v(y) D(r,s)$.

Then $v(x)D(r,t) = v(y)D(r,s)D(r,t) = v(y)D(r,\Psi(r,s,t)) = v(y)D(r,s+t)$. Hence, stationarity holds.

The proof of part (b) is similar except, where appropriate, equality is replaced with strict inequality. ■.

Proposition 9: Follows from Definition 4 (impatience) and the facts that $D(r,t) = \varphi(f(r,t))$ (Proposition 7) and φ is strictly decreasing (Definition 8). ■.

Proposition 10: By Definition 6, $D(0,s)D(0,t) = D(0,\Psi(0,s,t))$, $s \geq 0$, $t \geq 0$. Hence, from Proposition 7b, we get $\varphi(s)\varphi(t) = \varphi(\Psi(0,s,t))$, $s \geq 0$, $t \geq 0$. Since this holds for all $s \geq 0$, $t \geq 0$, we get $\varphi(f(r,s))\varphi(f(s,t)) = \varphi(\Psi(0,f(r,s),f(s,t)))$, $0 \leq r < s < t$.

By Definition 5, a discount function, $D(r,t)$, is Subadditive if, and only if, $D(r,s)D(s,t) < D(r,t)$ for all $0 \leq r < s < t$, i.e., if, and only if, $\varphi(f(r,s))\varphi(f(s,t)) < \varphi(f(r,t))$, for all $0 \leq r < s < t$ (using Proposition 7b), i.e., if, and only if, $\varphi(\Psi(0,f(r,s),f(s,t))) < \varphi(f(r,t))$, for all $0 \leq r < s < t$ (using what has been established above), i.e., if, and only if, $\Psi(0,f(r,s),f(s,t)) > f(r,t)$, for all $0 \leq r < s < t$ (since φ is strictly decreasing, Definition 8). This establishes part (a). Parts (b) and (c) are similar. ■.

Proposition 11: All the claims can be verified by straightforward calculations. ■.

Proposition 12: We shall use the following two simple mathematical results (for proofs, see al-Nowaihi and Dhami, 2008b, p49):

Result 1 *: Let $x \geq 0$ and $y \geq 0$. Then $0 < \rho \leq 1 \Rightarrow x^\rho + y^\rho \geq (x+y)^\rho$.*

Result 2 *: Let $\tau > 0$, $0 \leq s < t$ and $r > 0$. Let $F(r) = (t+r)^\tau - (s+r)^\tau - (t^\tau - s^\tau)$. Then:*
(i) $0 < \tau < 1 \Rightarrow F(r) < 0$,
(ii) $\tau = 1 \Rightarrow F(r) = 0$,
(iii) $\tau > 1 \Rightarrow F(r) > 0$.

(a) Suppose $0 < \rho \leq 1$. Let $0 \leq r < s < t$. For RS (Table 2), we get: $f(r,s) = (s^\tau - r^\tau)^{\frac{1}{\tau}}$ and $f(s,t) = (t^\tau - s^\tau)^{\frac{1}{\tau}}$. Hence, $\Psi(0,f(r,s),f(s,t)) =$
$[(s^\tau - r^\tau)^\rho + (t^\tau - s^\tau)^\rho + \alpha(s^\tau - r^\tau)^\rho(t^\tau - s^\tau)^\rho]^{\frac{1}{\tau\rho}} >$
$[(s^\tau - r^\tau)^\rho + (t^\tau - s^\tau)^\rho]^{\frac{1}{\tau\rho}} \geq (t^\tau - r^\tau)^{\frac{1}{\tau}} = f(r,t)$, where we have used Result 1. Hence, by Proposition 10a, preferences exhibit subadditivity.

(b) It is sufficient to give an example. Let $\alpha = \tau = 1$ and $\rho = 2$. Hence, $D(0,1)D(1,2) = 4^{-\beta} > 5^{-\beta} = D(0,2)$. Hence, for $\alpha = \tau = 1$ and $\rho = 2$, D cannot be additive or subadditive. However, for the same parameter values, we have $D(0,10)D(10,20) = 10201^{-\beta} < 401^{-\beta} = D(0,20)$. Hence, D cannot be superadditive either.[18]

(c) Similar to part a, except that we use Result 2 and Proposition 9. ■.

[18]Other examples can be given to show that there is nothing special about $r = 0$, $\alpha = 1$, $\tau = 1$, or $\rho = 2$, as long as $\rho > 1$.

References

al-Nowaihi, A., Dhami, S. 2006. A Note On The Loewenstein-Prelec Theory Of Intertemporal Choice. *Mathematical Social Sciences*, Volume 52, Issue 1, Pages 99-108.

al-Nowaihi, A., Dhami, S. 2008a. A note on the Loewenstein-Prelec theory of intertemporal choice: Corrigendum. *Mathematical Social Sciences.* 52, 99-108.

al-Nowaihi, A., Dhami, S. 2008b. A general theory of time discounting: The reference-time theory of intertemporal choice. *University of Leicester Working Paper* No. 08/22.

al-Nowaihi, A., Dhami, S. 2009. A value function that explains the magnitude and sign effects. *Economics Letters* 105, 224-229.

Kahneman, D., Tversky, A. 1979. Prospect theory : an analysis of decision under risk. *Econometrica* 47, 263-291.

Kahneman D., Tversky A. 2000. *Choices, Values and Frames, Cambridge University Press*, Cambridge.

Laibson, D. 1997. Golden eggs and hyperbolic discounting. *Quarterly Journal of Economics* 112, 443-477.

Loewenstein G. 1988. Frames of mind in intertemporal choice. *Management Science* 34, 200-214.

Loewenstein G., Prelec D. 1992. Anomalies in intertemporal choice : evidence and an interpretation. *The Quarterly Journal of Economics* 107, 573-597.

Loewenstein, G. F., Weber, E. U., Hsee, C. K., and Welch, E. S. (2001) Risk as feelings, *Psychological Bulletin*, 127: 267–286.

Ok, E. A., and Y. Masatlioglu, 2007. A theory of (relative) discounting. Journal of Economic Theory 137, 214-245.

Phelps, E. S., Pollak, R. 1968. On second-best national saving and game-equilibrium growth. *Review of Economic Studies* 35, 185-199.

Read, D. 2001. Is time-discounting hyperbolic or subadditive? *Journal of Risk and Uncertainty*, 23, 5-32.

Samuelson, P. 1937. A note on measurement of utility. *Review of Economic Studies*, 4, 155-161.

Scholten, M., and Read, D. 2006a. Discounting by intervals: A generalized model of intertemporal choice, *Management Science*, 52, 1426-1438.

Scholten, M. and Read, D., 2006b. Beyond Discounting: The Tradeoff Model of Intertemporal. Choice. Working Paper LSE-OR 06.88. London: London School of Economics and Political Science.

Scholten, M., and Read, D. 2011a. Descriptive Models of Intertemporal Choice Part 1: Anomalies in Choices Between Less Sooner and More Later. mimeo.

Scholten, M., and Read, D. 2011b. Descriptive Models of Intertemporal Choice Part 2: The Delay-Speedup Asymmetry and Other Anomalies. mimeo.

Thaler, R.H. 1981. Some empirical evidence on dynamic consistency. *Economic Letters* 8, 201-207.

Wakker, P.P. 2010. *Prospect Theory for Risk and Ambiguity*. Cambridge University Press.