University of Leicester

DEPARTMENT OF ECONOMICS

Hyperbolic Punishment Function

Ali al-Nowaihi, **University of Leicester, UK**
Sanjit Dhami, University of Leicester, UK

Working Paper No. 11/42
August 2011

Hyperbolic Punishment Function*

Ali al-Nowaihi† **Sanjit Dhami**‡

23 August 2011

Abstract

All models in Law and Economics use *punishment functions* (PF) that incorporates a trade-off between probability of detection, p, and punishment, F. Suppose society wishes to minimize the total costs of enforcement and damages from crime, $T(p, F)$. For a given p, an *optimal punishment function* (OPF) determines an F that minimizes $T(p, F)$. A popular and tractable PF is the *hyperbolic punishment function* (HPF). We show that the HPF is an OPF for a large class of total cost functions. Furthermore, the HPF is an upper (lower) bound for an even larger class of punishment functions. If the HPF cannot (can) deter crime then none (all) of the PF's for which the HPF is an upper (lower) bound can deter crime. Thus, if one can demonstrate that a particular policy is ineffective (effective) under the HPF, there is no need to even compute the OPF. Our results should underpin an even greater use of the HPF. We give illustrations from mainstream and behavioral economics.

Keywords: Punishment functions, Optimal punishment functions, Becker proposition, Law and economics, Behavioral models of crime and punishment.

JEL Classification: D01 (Microeconomic Behavior: Underlying Principles); D81 (Criteria for Decision Making Under Risk and Uncertainty); K42 (Illegal Behavior and the Enforcement of Law).

*We gratefully acknowledge valuable comments and guidance from the referee and editor.

†Department of Economics, University of Leicester, University Road, Leicester. LE1 7RH, UK. Phone: +44-116-2522898. Fax: +44-116-2522908. E-mail: aa10@le.ac.uk.

‡Corresponding author. Department of Economics, University of Leicester, University Road, Leicester. LE1 7RH, UK. Phone: +44-116-2522086. Fax: +44-116-2522908. E-mail: Sanjit.Dhami@le.ac.uk.

1. Introduction

The modern economic approach to crime identifies two main instruments of deterrence. The first instrument is *the probability of detection and conviction*, p. Hiring more police officers, training and equipping them better, installing CCTV cameras and formalizing accounting or legal rules in a manner that facilitates catching offenders; all these, among others, can potentially increase p. The second instrument is the *severity of the punishment.* This includes monetary fines as well as non-monetary punishments such as imprisonment or barring offenders from certain activities. We use, F, not just for fines, but also to denote all possible punishments, evaluated in monetary terms.[1]

Let $C(p,F)$ be the cost to society of law enforcement. In general, $C(p,F)$ will be increasing in each of p and F. We define an *ideal fine* to be a punishment, F, for which $C(p,F)$ is independent of F. Let $D(p,F)$ be the damage to society from crime. In general, $D(p,F)$ will be decreasing in each of p and F. However, in some cases, for example if F is severe, D could be increasing in F. Let $T(p,F) = C(p,F) + D(p,F)$ be the total cost to society from crime and from crime prevention. We assume that society aims to choose those values of p and F that minimize $T(p,F)$.

One can minimize $T(p,F)$ in two steps. First, for each p, find the F that minimizes $T(p,F)$. Let $F = \varphi(p)$ be the resulting function and call it a *punishment function.* Second, given $F = \varphi(p)$, choose the p that minimizes $T(p,\varphi(p))$.

A popular and tractable punishment function is the *hyperbolic punishment function* (HPF). This is given by

$$F = \varphi(p) = \frac{c}{p}, c > 0,$$

where c is some constant. The following quote from Polinsky and Shavell (2007) testifies to the importance of the HPF: "[The HPF] or its equivalent, was put forward by Bentham (1789, p.173), was emphasized by Becker (1968), and has been noted by many others since then."[2]

Consider an individual who (if not caught) would derive a monetary benefit, $b > 0$, from an illegal activity. Assume that this individual is an expected utility maximizer with a strictly increasing and concave utility function, u (so that this individual is risk neutral or risk averse). Becker (1968) proved that the HPF, $\varphi(p) = \frac{b}{p}$, will completely deter this individual from crime. It follows that given any probability of detection and conviction, $p > 0$, no matter how small, crime can be deterred by a sufficiently large punishment. Kolm (1973) memorably paraphrased the Becker proposition as *hang offenders with probability*

[1] Typically, p and F are taken to be substitutes. However, under certain conditions these instruments might become complements; see, for instance, Garoupa (2001). Our model is consistent with both.

[2] Polinsky and Shavell (2007, footnote 16).

zero. However, note that the HPF, $\varphi(p) = \frac{b}{p}$, need not be the *optimal punishment function* for Becker's model.[3]

In the main model of the wide ranging survey by Polinsky and Shavell (2007), decision makers are risk neutral, punishments are ideal fines and the payoff from crime is regarded as a positive contribution to total social welfare, to be balanced at the margin with the harm done to society by crime. They state, correctly, that the optimal punishment function for their model is the HPF.

We formulate a model of crime that generalizes that of Polinsky and Shavell (Section 4, below). Our model allows for the presence of risk neutrality or it's absence, presence or absence of ideal fines and for the proceeds of crime to enter or not enter the social welfare function.

The main proposition of our paper is Proposition 2, which derives the general form of the optimal punishment function for our model.

Corollary 2 gives the conditions under which our optimal punishment function is the HPF. We use this, in subsection 4.1, to prove that the HPF is the optimal punishment function for the Polinsky and Shavell model (Proposition 3). We also apply this, in subsection 4.2, to prove that the HPF is the optimal punishment function for a very different model of crime where punishments are not ideal fines and the payoff from crime is not regarded a positive contribution to total social welfare (Proposition 4).

In many cases of interest, however, the optimal punishment function may be quite intractable and the researcher might not be interested in deriving an *optimal punishment function* but rather in using a sensible and tractable punishment function. The tractability of the HPF is not in doubt.[4] Suppose we have a model for which the HPF is an upper bound for the optimal punishment function. In other words, for any feasible level of p, the punishment recommended by the HPF is at least as high as the optimal punishment function. Then, if the HPF cannot deter crime for that model, then neither can the optimal punishment function. And, conversely, if the HPF is a lower bound to the optimal punishment function, and if the HPF can deter crime, then so can the optimal punishment function. These considerations reduce the need to compute an optimal punishment function (a non-trivial exercise) in many cases.

Corollary 3 gives conditions under which our optimal punishment function (Proposition 2) is bounded above by the HPF. Subsection 4.3 gives a model similar to that of Polinsky and Shavell, except that individuals are risk averse. There we use Corollary 3 show that the HPF is an upper bound for the optimal punishment function of that model.

[3] The formal definition of an *optimal punishment function* is given in Definition 3 below. We use this term in the usual sense as a punishment function that for any feasible values of p, F minimizes the total cost of crime, $T(p, F)$.

[4] Indeed it would seem to be the simplest punishment function other than $\varphi(p) = constant$.

Subsections 5.2 and 5.4 illustrate two different points. First, these two subsections illustrate the applicability of Corollary 3. Second, they serve as introductions to two models of crime based on the popular alternative non-expected utility (non-EU) theories, namely, *rank dependent expected utility theory* (RDU) and the more radical *cumulative prospect theory* (CP). We introduce the elements of RDU and CP in the paper below. RDU and CP explain the evidence in decision theoretic contexts far better than EU.[5]. We are, therefore, of the opinion that RDU and, even more so, CP, provide better approaches to modelling in law and economics.

However, whatever position one takes on the issue of an appropriate decision theory (EU or non-EU), our results illustrate the utility of the HPF.

2. The model

Let $p \in [0, 1]$ be the probability of detection (and conviction) and let $F \geq 0$ be the monetary equivalent of all punishments. These are the only two instruments of law enforcement in our model.

Consider an individual who can engage in either a legal activity that generates the income, y_0, or an illegal activity that generates the income, y_1. Hence, the benefit, b, to the individual from the illegal activity is

$$b = y_1 - y_0. \tag{2.1}$$

If engaged in the illegal activity, the individual is caught with probability p, $0 \leq p \leq 1$. If caught, a punishment is imposed on the individual whose monetary equivalent to that individual is $F \in [0, \infty]$. Given the enforcement parameters p and F, the individual makes only one choice: To commit the crime or not.

2.1. Cost and damages functions

Let $C(p, F)$ be the *cost to society of law enforcement*. Also, let $D(p, F)$ be the *damage to society caused by crime*. We assume that C and D are continuous functions of p and F with continuous first and second partial derivatives[6], i.e., $C, D \in C^2$. We denote partial derivatives with subscripts, e.g., $C_p = \frac{\partial C}{\partial p}$ and $C_{pF} = \frac{\partial^2 C}{\partial p \partial F}$. We also assume

$$C_p > 0,\ C_F \geq 0. \tag{2.2}$$

Thus, the cost of law enforcement can be reduced by reducing the probability of detection and conviction, p. In general, an increase in the punishment, F, will increase the cost

[5] See, for instance, Kahneman and Tversky (2000), Starmer (2000) and Wakker (2010).

[6] C^n is the class of continuous functions with continuous partial derivatives up to order n.

of law enforcement (for example, increasing the length of prison sentences). We note, for future reference, the special case below.

Definition 1 *(Ideal fine): The case $C_F = 0$ can be thought of as that of an ideal fine, which has a fixed administrative cost and involves a transfer from the offender to the victim or society (so there is no aggregate loss to society other than the fixed administrative cost).*[7]

2.2. Society's objective

Let

$$T(p,F) = C(p,F) + D(p,F), \tag{2.3}$$

be the total cost to society of crime. Society aims to choose p and F so as to minimize $T(p,F)$. We assume that

$$[T_F]_{F=0} < 0. \tag{2.4}$$

Lemma 1 *: If $F = F_{opt}$ minimizes $T(p,F)$ with respect to F, given p, then $F_{opt} > 0$.*

Proof: (2.4) ensures that total costs can be reduced by raising the punishment to just above zero, hence $F = 0$ is not optimal. Hence, $F_{opt} > 0$. ■.

3. Punishment functions

Society aims to choose p and F so as to minimize the total cost, $T(p,F)$, of crime and of law enforcement. We carry out this optimization problem in two steps. First, we ask whether, for each given p, there is a level of punishment, $F = \varphi(p)$, that minimizes $T(p, \varphi(p))$. If the existence of such an 'optimal punishment function' is assured, then we could ask whether there exists a probability, p, that minimizes $T(p, \varphi(p))$. Below, we give formal definitions. First, we define a punishment function (optimal or otherwise), then we define an optimal punishment function.

Definition 2 *(Punishment function): By a punishment function we mean a function $\varphi(p) : [0,1] \to [0,\infty]$ that assigns to each probability of detection and conviction, $p \in [0,1]$, a punishment $\varphi(p) \in [0,\infty]$.*

Note that we allow for the possibility of infinite punishments.

Definition 3 *(Optimal punishment function): Let $\varphi : [0,1] \to [0,\infty]$ be a punishment function. We call φ an optimal punishment function if, for all $p \in [0,1]$ such that $\varphi(p) < \infty$, and for all $F \in [0,\infty)$, $T(p, \varphi(p)) \leq T(p,F)$.*

[7] An example is the payment in some societies of blood money, where money is paid by the guilty party directly to the victim's family, and no further punishment is carried out.

Thus, an optimal punishment function ensures that the total costs to society (of enforcement and damages) are the lowest possible.

Proposition 1 *(Existence): (a) An optimal punishment function,* $\varphi(p) : [0,1] \to [0,\infty]$, *exists.*
(b) If $\varphi(p) < \infty$ *then* $[T_F(p,F)]_{F=\varphi(p)} = 0$ *and* $[T_{FF}(p,F)]_{F=\varphi(p)} \geq 0$.

Proof: (a) Let $p \in [0,1]$. The following two cases, (ai) and (aii), are mutually exclusive and exhaustive.

(ai) Suppose that there exists an $F_{\max} \in [0,\infty)$ such that,

$$\text{for all } F \geq F_{\max},\ T(p,F) \geq T(p,F_{\max}). \tag{3.1}$$

In this case we can restrict our search for an optimal F to the non-empty compact set $[0,F_{\max}]$. Since $T(p,F)$ is a continuous function of F on the non-empty compact set, $[0,F_{\max}]$, it attains a minimum at some $F_{opt} \in [0,F_{\max}]$. Set $\varphi(p) = F_{opt}$.[8]

(aii) Suppose that for each $F \in [0,\infty)$, there exists a $G > F$ such that $T(p,G) < T(p,F)$. In this case set $\varphi(p) = \infty$.

(b) From Lemma 1 we get that $\varphi(p) > 0$. Hence, $\varphi(p) \in (0,\infty)$. It follows that, necessarily,$[T_F(p,F)]_{F=\varphi(p)} = 0$ and $[T_{FF}(p,F)]_{F=\varphi(p)} \geq 0$. $\blacksquare$.

3.1. The hyperbolic punishment function (HPF)

A useful punishment function, with a long history in the law and economics literature, and is extremely tractable is the *hyperbolic punishment function* (HPF).

Definition 4 *(Hyperbolic punishment function, HPF): The hyperbolic punishment function,* $H(p)$, *is defined by*

$$H(p) = \frac{c}{p},\ c > 0. \tag{3.2}$$

The name derives from the fact that in p,F space, the HPF plots as a rectangular hyperbola. Note that, for (3.2), $H(0) = \infty$.

4. The hyperbolic punishment function as a bound for the optimal punishment function.

We start this section with the main proposition of our paper (Proposition 2, below). All the following results of this paper are consequences of this proposition. Proposition 2 establishes the general form of the optimal punishment function for a broad class of

[8] We have here made an implicit use of the axiom of choice here.

models. Corollaries 2, 3 and 4, that follow, establish conditions under which the HPF is, respectively, the optimal punishment function, an upper bound for the optimal punishment function or a lower bound for the optimal punishment function. Subsections 4.1, 4.2 and 4.3 then give applications. Section 5 gives two further applications; the latter to non-expected utility models.

Proposition 2 : *Consider the following class of total cost functions:*

$$T\left(p,F\right)=\Pi\left(p\right)+\pi\left(p\right)\Phi\left(\alpha\left(p\right)+\beta\left(p\right)\Psi\left(F\right)\right), \tag{4.1}$$

where

$$\Pi,\pi,\Phi,\alpha,\beta,\Psi \text{ are } C^{2}, \tag{4.2}$$

$$\pi\left(p\right)>0,\beta\left(p\right)\geq 0,\Psi'\left(F\right)>0, \tag{4.3}$$

$$\Phi'\left(\alpha\left(p\right)+\beta\left(p\right)\Psi\left(0\right)\right)<0, \tag{4.4}$$

$$\Phi'\left(c\right)=0,\Phi''\left(c\right)\geq 0 \text{ for, at most, one } c\in\mathbb{R}. \tag{4.5}$$

Let φ be the optimal punishment function (Definition 3 and Proposition 1). Then, either

$$\varphi\left(p\right)=\infty, \tag{4.6}$$

or

$$\varphi\left(p\right)=\Psi^{-1}\left(\frac{c-\alpha\left(p\right)}{\beta\left(p\right)}\right),\ c\in\mathbb{R},\ \Phi'\left(c\right)=0,\Phi''\left(c\right)\geq 0. \tag{4.7}$$

Proof: From (4.1) and (4.2) we get

$$T_{F}\left(p,F\right)=\pi\left(p\right)\beta\left(p\right)\Psi'\left(F\right)\Phi'\left(\alpha\left(p\right)+\beta\left(p\right)\Psi\left(F\right)\right), \tag{4.8}$$

$$\begin{aligned} T_{FF}\left(p,F\right)=&\ \pi\left(p\right)\beta\left(p\right)\Psi''\left(F\right)\Phi'\left(\alpha\left(p\right)+\beta\left(p\right)\Psi\left(F\right)\right)\\ &+\pi\left(p\right)\left[\beta\left(p\right)\Psi'\left(F\right)\right]^{2}\Phi''\left(\alpha\left(p\right)+\beta\left(p\right)\Psi\left(F\right)\right). \end{aligned} \tag{4.9}$$

From (4.3), (4.4) and (4.8) we get that (2.4) is satisfied. Hence, by Proposition 1, an optimal punishment function, $\varphi:[0,1]\rightarrow[0,\infty]$, exists. If $\varphi\left(p\right)=\infty$, then this establishes (4.6). Suppose now that $\varphi\left(p\right)<\infty$. Then from Proposition 1, $\left[T_{F}\left(p,F\right)\right]_{F=\varphi(p)}=0$ and $\left[T_{FF}\left(p,F\right)\right]_{F=\varphi(p)}\geq 0$. Hence, from (4.3) and (4.8), $\Phi'\left(\alpha\left(p\right)+\beta\left(p\right)\Psi\left(\varphi\left(p\right)\right)\right)=0$ and $\Phi''\left(\alpha\left(p\right)+\beta\left(p\right)\Psi\left(\varphi\left(p\right)\right)\right)\geq 0$. But, by (4.5), $\Phi'\left(c\right)=0,\Phi''\left(c\right)\geq 0$ hold for at most, hence exactly one, $c\in\mathbb{R}$. Hence $\alpha\left(p\right)+\beta\left(p\right)\Psi\left(\varphi\left(p\right)\right)=c$. Hence, $\varphi\left(p\right)=\Psi^{-1}\left(\frac{c-\alpha(p)}{\beta(p)}\right)$. ■.

The following four corollaries are immediate consequences of Proposition 2.

Corollary 1 : *Suppose $\Phi'(x) \neq 0$ for any $x \in \mathbb{R}$. Then, for all $p \in [0,1]$, $\varphi(p) = \infty$. Thus, in this case and for any given probability of detection, $p \in [0,1]$, the optimal punishment function specifies an infinite punishment.*

Corollary 2 : *Suppose that, in Proposition 2, $\alpha(p) = 0$, $\beta(p) = p$ and $\Psi(F) = F$. Then $\varphi(p) = \frac{c}{p}$, i.e., the optimal punishment function is the hyperbolic punishment function.*

Corollary 3 : *Suppose that, in Proposition 2, $\frac{c-\alpha(p)}{\beta(p)} \leq \Psi\left(\frac{c}{p}\right)$. Then $\varphi(p) \leq \frac{c}{p}$. Thus, in this case, the hyperbolic punishment function is an upper bound for the optimal punishment function.*

Corollary 4 : *Suppose that, in Proposition 2, $\frac{c-\alpha(p)}{\beta(p)} \geq \Psi\left(\frac{c}{p}\right)$. Then $\varphi(p) \geq \frac{c}{p}$. Thus, in this case, the hyperbolic punishment function is a lower bound for the optimal punishment function.*

4.1. The Polinsky and Shavell model of crime.

In their wide ranging survey, Polinsky and Shavell (2007, p.413) considered the model of crime described immediately below. We will see that it is a special case of our framework. Applying Corollary 2, we will see that the optimal punishment function for this model is the hyperbolic punishment function.

We now outline the Polinsky and Shavell model. An individual earns income, y_0, from a legal activity or income, y_1, from an illegal activity. Hence, the benefit to the individual from the illegal activity, if not caught, is $b = y_1 - y_0$. If caught, a fine, F, is levied on the individual. Individuals are expected value maximizers. With these assumptions an individual engages in crime if, and only if,

$$(1-p)\,y_1 + p\,(y_1 - F) \geq y_0, \tag{4.10}$$

which simplifies to

$$b \geq pF. \tag{4.11}$$

Thus, an individual will commit a crime if, and only if, the benefit exceeds the expected cost of crime, pF, to the individual.

Punishments in the Polinsky-Shavell model are ideal fines. Hence, the cost of enforcement, $C(p)$, is a function of the probability of detection, p, only, so $C_F = 0$. It is also assumed that $C_p > 0$. There is a distribution of the benefits from crime, b, given by the density function $z(b) > 0$.[9] The harm to society from an act of crime by any individual is $h > 0$, the same for all individuals. Damage to society from crime is given by

$$D(p,F) = \int_{pF}^{\infty} (h-b)\,z(b)\,db. \tag{4.12}$$

[9] In Polinsky and Shavell, z has finite support. Changing to an infinite support actually simplifies the exposition without altering any of the conclusions.

We can view $D(p,F)$, in (4.12), as the sum of three terms:

$$D(p,F) = h\int_{pF}^{\infty} z(b)\,db - \int_{pF}^{\infty}(b-pF)\,z(b)\,db - pF\int_{pF}^{\infty} z(b)\,db. \tag{4.13}$$

The first term on the right hand side of (4.13) is the total harm to society from crime. The second term is the total benefit to individuals who commit crime net of the expected punishment. The third term is the total tax revenue from punishments (which are ideal fines).

Society desires to minimize the following total cost function

$$T(p,F) = C(p) + \int_{pF}^{\infty}(h-b)\,z(b)\,db. \tag{4.14}$$

The Polinsky and Shavell model can be recast in the form of Proposition 2 with the following choices:

$$\Pi(p) = C(p),\, \pi(p) = 1,\, \alpha(p) = 0,\, \beta(p) = p,\ \Psi(F) = F, \tag{4.15}$$

$$\Phi(pF) = \int_{pF}^{\infty}(h-b)\,z(b)\,db. \tag{4.16}$$

Proposition 3 *(Polinsky and Shavell, 2007): The optimal punishment function for the Polinsky and Shavell model of crime is the hyperbolic punishment function.*

Proof: From (4.16) we get

$$\Phi'(pF) = -(h-pF)\,z(pF), \tag{4.17}$$

$$\Phi'(0) = -hz(0) < 0, \tag{4.18}$$

$$\Phi''(pF) = z(pF) - (h-pF)\,z'(pF). \tag{4.19}$$

From (4.15) and (4.18) we see that $\Phi'(\alpha(p)) = \Phi'(0) < 0$. Hence (4.4) holds. From (4.17), we see that $\Phi'(pF) = 0 \Leftrightarrow pF = h$. Moreover, from (4.19), we see that $\Phi''(h) = z(h) > 0$. So, (4.5) holds. Thus, all the conditions of Corollary 2 hold. Hence, the optimal punishment function is the hyperbolic punishment function, $\varphi(p) = \frac{h}{p}$. ■.

4.2. A model where punishments are not ideal fines and where the proceeds of crime do not enter the social welfare function.

In this subsection, we model the behaviour of individuals exactly as in Polinsky and Shavell (2007); see subsection 4.1, above. In particular, an individual will engage in crime if, and only, the benefit from crime, b, exceeds the expected monetary equivalent of the cost of punishment, i.e., if, and only if, (4.11) holds. However, we modify the Polinsky and Shavell (2007) in the following respects.

1. In the Polinsky-Shavell model, punishment, F, takes the form of monetary fines. On the other hand, here we assume that no monetary fines can be collected and punishment takes the form of imprisonment.

2. We generalize the enforcement cost, $C(p, F)$, beyond ideal punishments. Specifically,
$$C(p, F) = C_0(p) + kpF,\ k \geq 0,\ C_0'(p) > 0. \tag{4.20}$$
In (4.20), $C_0(p)$ is the component of law enforcement that depends only on the probability of detection and conviction, p. Increasing p by, say, increasing the number of police officers, increases $C_0(p)$. If $k = 0$, then we have the case of ideal fines, as in subsection 4.1 in the Polinsky and Shavell model. We allow for the possibility that $k > 0$ and so the cost of crime prevention increases proportionally with an increase in expected punishment.

3. The damages function takes the form
$$D(p, F) = h\int_{pF}^{\infty} z(b)\, db,\ h > 0. \tag{4.21}$$
Comparing (4.21) with (4.13) note the absence from (4.21) of the second and third terms on the right hand side of (4.13). The last term on the right hand side of (4.13), $pF\int_{pF}^{\infty} z(b)\, db$, is the total tax revenue collected from punishments. Its absence from the right hand side of (4.21) signifies that we are assuming that prisoners do not generate any revenue for the government (see assumption 1 above). However, imprisonment does deter crime, which is reflected in the lower limit of integration in (4.21). The second term on the right hand side of (4.13), $\int_{pF}^{\infty} (b - pF)\, z(b)\, db$, is the total of all benefits to individuals committing crime net of the expected monetary equivalent of the punishment. Its absence from the right hand side of (4.21) signifies that we are assuming that the personal benefit that individuals derive from crime should not be counted as part of total social welfare. To take an extreme case, the pleasure that sadists derive from torturing their victims should not be counted as part of social welfare.

Thus total social costs from crime is now
$$T(p, F) = C_0(p) + kpF + h\int_{pF}^{\infty} z(b)\, db,\ k \geq 0. \tag{4.22}$$
We need two further assumptions, absent from the Polinsky and Shavell model. The first of them is
$$k < hz(0). \tag{4.23}$$
If the marginal cost of punishment, k, is too high, then it might be optimal not to punish crime at all. Condition (4.23) guarantees that this is not the case. The second of our two extra assumptions (Axiom 1) immediately follows the next definition.

Definition 5 *(Single peakedness): A probability density function, $z(b)$, is single peaked if any horizontal line cuts the graph of $z(b)$ in, at most, two points: $(b_1, z(b_1))$, $(b_2, z(b_2))$, $b_1 \leq b_2$. If $b_1 = b_2 = b$, then $z'(b) = 0$. If $b_1 < b_2$, then $z'(b_1) > 0$ and $z'(b_2) < 0$.*

Axiom 1 *(Single peakedness): The probability density function, $z(b)$, in (4.22) is single peaked (Definition 5).*

Most probability density functions in common use do satisfy Axiom 1; the leading example being the density function of the normal distribution. Axiom 1, along with the other assumptions, will guarantee that condition (4.5), of Proposition 2, holds. In particular, they will guarantee that the conditions of Corollary 2 hold. Consequently, the optimal punishment function for the model of this subsection will be, again, the hyperbolic punishment function.

The model of this subsection can be recast in the form of Proposition 2 with the following choices:

$$\Pi(p) = C_0(p), \pi(p) = 1, \alpha(p) = 0, \beta(p) = p, \Psi(F) = F, \tag{4.24}$$

$$\Phi(pF) = kpF + h\int_{pF}^{\infty} z(b)\, db. \tag{4.25}$$

Proposition 4 : *For the alternative model of crime formulated in this subsection, the optimal punishment function is the hyperbolic punishment function.*

Proof: From (4.25) we get

$$\Phi'(pF) = k - hz(pF), \tag{4.26}$$

$$\Phi'(0) = k - hz(0) < 0, \text{ from (4.23)}, \tag{4.27}$$

$$\Phi''(pF) = -hz'(pF). \tag{4.28}$$

From (4.24) and (4.27) we see that $\Phi'(\alpha(p)) = \Phi'(0) < 0$. Hence (4.4) holds. From (4.26), we see that $\Phi'(pF) = 0 \Leftrightarrow z(pF) = \frac{k}{h}$. From Axiom 1, it follows that $z^{-1}\left(\frac{k}{h}\right)$ contains no more than two points: $\left(b_1, \frac{k}{h}\right)$, $\left(b_2, \frac{k}{h}\right)$, $b_1 \leq b_2$. If $b_1 = b_2 = b$, then $z'(b) = 0$. In this case (4.5) holds. If $b_1 < b_2$, then $z'(b_1) > 0$ and $z'(b_2) < 0$. In this case (4.5) again holds, with $c = b_2$. Thus, all the conditions of Corollary 2 hold. Hence, the optimal punishment function is the hyperbolic punishment function, $\varphi(p) = \frac{c}{p}$, where $c = b_2$. ■.

4.3. A model with risk averse individuals

Here we formulate a model that is similar to the Polinsky and Shavell model except that individuals are expected utility maximizers rather than expected revenue maximizers. Assume that all individuals have the same utility function, u. Then an individual will commit a crime if, and only if,

$$pu\left(y_0+b-F\right)+\left(1-p\right)u\left(y_0+b\right)\geq u\left(y_0\right), \tag{4.29}$$

where b is the benefit an individual derives from crime if not caught.

In particular, let each individual have the constant absolute risk aversion utility function

$$u\left(x\right)=-e^{-x}, \tag{4.30}$$

then (4.29) becomes

$$-pe^{-\left(y_0+b-F\right)}-\left(1-p\right)e^{-\left(y_0+b\right)}\geq -e^{-y_0}, \tag{4.31}$$

which simplifies to

$$b\geq \ln\left(1-p+pe^F\right). \tag{4.32}$$

As in the Polinsky and Shavell model, let the benefits from crime (if not caught) be distributed with density $z\left(b\right)>0$. And as in Polinsky and Shavell, let an individual act of crime inflict the harm, h, on society. The total damage to society from crime is given by

$$D\left(p,F\right)=\int_{b=\ln\left(1-p+pe^F\right)}^{\infty}\left(h-b\right)z\left(b\right)db. \tag{4.33}$$

As in Polinsky and Shavell, assume punishments are ideal fines, with cost to society of law enforcement given by $C\left(p\right)$, which is independent of the punishment, F. Hence, the total cost to society of crime and law enforcement is

$$T\left(p,F\right)=C\left(p\right)+\int_{b=\ln\left(1-p+pe^F\right)}^{\infty}\left(h-b\right)z\left(b\right)db. \tag{4.34}$$

This model can be recast in the form of Proposition 2 with the following choices

$$\Pi\left(p\right)=C\left(p\right),\ \pi\left(p\right)=1,\ \alpha\left(p\right)=1-p,\ \beta\left(p\right)=p,\ \Psi\left(F\right)=e^F. \tag{4.35}$$

$$\Phi\left(1-p+pe^F\right)=\int_{b=\ln\left(1-p+pe^F\right)}^{\infty}\left(h-b\right)z\left(b\right)db. \tag{4.36}$$

Proposition 5 *: The hyperbolic punishment function is an upper bound for the optimal punishment function of this subsection.*

Proof: From (4.36) we get

$$\Phi'\left(1-p+pe^{F}\right)=-\left[h-\ln\left(1-p+pe^{F}\right)\right]\frac{z\left(\ln\left(1-p+pe^{F}\right)\right)}{1-p+pe^{F}}, \tag{4.37}$$

$$\begin{aligned}\Phi''\left(1-p+pe^{F}\right)&=\frac{z\left(\ln\left(1-p+pe^{F}\right)\right)}{\left(1-p+pe^{F}\right)^{2}}\\&-\left[h-\ln\left(1-p+pe^{F}\right)\right]\left[\frac{z\left(\ln\left(1-p+pe^{F}\right)\right)}{1-p+pe^{F}}\right]'.\end{aligned} \tag{4.38}$$

From (4.35) we get

$$\alpha(p)+\beta(p)\Psi(0)=1 \tag{4.39}$$

From (4.37) and (4.39) we get

$$\Phi'(1)=-hz(0)<0. \tag{4.40}$$

From (4.40) we see that (2.4) is satisfied. From (4.37) we get

$$\Phi'\left(1-p+pe^{F}\right)=0\Leftrightarrow h-\ln\left(1-p+pe^{F}\right)=0, \tag{4.41}$$

from which we get

$$\Phi'\left(1-p+pe^{F}\right)=0\Leftrightarrow 1-p+pe^{F}=e^{h}. \tag{4.42}$$

From (4.38) and (4.42) we get

$$\Phi''\left(e^{h}\right)=\frac{z(h)}{e^{2h}}>0. \tag{4.43}$$

From (4.42) and (4.43) we see that (4.5) is satisfied with $c=e^{h}$. Hence, Proposition 2 applies.

Since

$$\frac{e^{h}+p-1}{p}\leq\frac{e^{h}}{p}\leq e^{\frac{e^{h}}{p}}, \tag{4.44}$$

we get, from Corollary 3, that the HPF is an upper bound for the optimal punishment function for this model. ■.

Remark 1 : *The optimal punishment function for the model of this subsection is* $\varphi(p)=\ln\frac{e^{h}+p-1}{p}$, *which is clearly less tractable than the HPF,* $H(p)=\frac{e^{h}}{p}$.

5. Non-expected utility models of crime.

Both the Polinsky and Shavell (2007) model of crime (subsection 4.1, above) and the alternative model of crime (subsection 4.2, above) assume that decision makers are expected value maximizers. This is equivalent to assuming that decision makers are risk neutral. Since, at least, Bernoulli (1738) the assumption of expected revenue maximization has been known to be very restrictive. For example, under this assumption, no person will gamble or insure.

Suppose we have n states of the world, state i occurring with probability $p_i > 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^{n} p_i = 1$. Consider the lottery, L, that pays the monetary value, y_i, if state i occurs. We shall call, y_i, an outcome. It could be, for example, income, wealth or the value of an asset. The expected value of this lottery is

$$E(L) = \sum_{i=1}^{n} p_i y_i. \tag{5.1}$$

An expected value maximizer will choose that action that maximizes (5.1). Note that (5.1) is a bilinear form, i.e., it is linear in p_i given y_i and it is also linear in y_i given p_i. It is this bilinearity that makes an expected value maximizer risk neutral. To incorporate more general attitudes to risk, such as risk aversion or risk seeking, we have to allow non-linearity in p_i or non-linearity in y_i (or both).

Expected utility theory (EU) allows non-linearity in y_i but retains linearity in p_i (recall the example of subsection 4.3, above). Specifically, it is assumed that a decision maker has a utility function, u. The *expected utility* of the lottery, L, is then

$$EU(L) = \sum_{i=1}^{n} p_i u(y_i). \tag{5.2}$$

Because of the linearity of (5.2) in p_i, the attitude to risk of an expected utility maximizer is entirely captured by the shape of his utility function, u. For example, he is risk averse if, and only if, u is strictly concave (as in the example of subsection 4.3, above). He is risk seeking if, and only if, u is strictly convex.

However, it has been well known, at least since Allais (1953), that EU is not a satisfactory theory of decision making under risk. The utility function, u, can either be strictly concave or strictly convex but not both. However, risk aversion and risk seeking behaviour are often seen together in the same individual. Kunreuther et al. (1978) show that EU is unable to explain the poor take up of insurance against earthquakes, floods and hurricanes. Rabin (2000) shows that a reasonable degree of risk aversion for low stake gambles implies an absurdly high degree of risk aversion for high stake gambles. For example, it follows from his "calibration theorem" that an individual who (quite reasonably) would reject the gamble lose 9c with probability half or win 10c with probability half, at all levels of wealth, would also, necessarily, reject the following gamble: Lose \$1 with probability half or win an infinite amount of wealth with probability half; which is absurd.

In the context of criminal activity, specifically tax evasion, Dhami and al-Nowaihi (2007) show that the predictions of EU are both quantitatively incorrect (by factors of up to 100) and qualitatively incorrect.[10] They also show that, by contrast, the evidence on tax evasion is easily explained by prospect theory (PT).

These are just some examples from the large literature that has documented the systematic and robust failure of EU. As long ago as 1957, Luce and Raiffa (two of the founders) described the then available evidence against EU as "bolstered by a staggering amount of empirical data".[11] More recently, Camerer and Loewenstein (2004) wrote "... the statistical evidence against EU is so overwhelming that it is pointless to run more studies testing EU against alternative theories ... ".[12] Comprehensive reviews of the violation of EU can be found in Kahneman and Tversky (2000), Starmer (2000) and Wakker (2010). It is surprising, therefore, that the bulk of the research in economics in general and law and economics in particular, is still conducted in an EU framework.

The most successful of the alternatives to EU are rank dependent expected utility (RDU) and cumulative prospect theory (CPT). Both employ non-linear transformation of probability by the device of a probability weighting function, which we turn to next.

5.1. Probability weighting functions.

Definition 6 *(probability weighting functions): A probability weighting function is a strictly increasing map from* $[0,1]$ *onto* $[0,1]$.

Remark 2 *: It follows from Definition 6 that a probability weighting function,* w*, is continuous with a continuous inverse,* w^{-1}*, and* $w(0)=0$ *and* $w(1)=1$.

Definition 7 *(Inverse-S shaped probability weighting functions): A probability weighting function,* $w:[0,1]\rightarrow[0,1]$ *is inverse-S shaped if there exists* $\widetilde{p}\in(0,1)$ *such that* $w(p)>p$ *for* $p<\widetilde{p}$ *and* $w(p)<p$ *for* $p>\widetilde{p}$.

An inverse-S shaped probability weighting function overweights low probabilities but underweights high probabilities, in agreement with the evidence; see, for instance, Starmer (2000) and Wakker (2010).

[10] The qualitative incorrectness in this case refers to the Yitzhaki puzzle. Under plausible attitudes to risk, EU predicts that when the tax rate increases, taxpayers will evade less. In the limit when the tax rate is a 100%, all income is declared. This contradicts the bulk of evidence; see Dhami and al-Nowaihi (2007) for the details.

[11] See Luce and Raiffa (1957), p. 37, lines 14,15.

[12] See Camerer and Loewenstein, 2004, p20, second paragraph.

Example 1 : *A popular example of a probability weighting function is that of Prelec (1998):*

$$w(p) = e^{-\beta(-\ln p)^{\alpha}},\ \alpha > 0,\ \beta > 0. \tag{5.3}$$

The Prelec probability weighting function is inverse-S shaped if, and only if, $\alpha < 1$. A sketch of the Prelec function for $\alpha = 0.65$ and $\beta = 1$, as estimated by Prelec (1998), is given below.

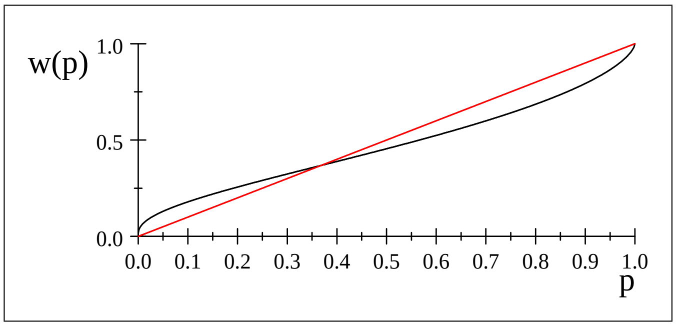

Figure 5.1: The Prelec (1998) function, $w(p) = e^{-(-\ln p)^{0.65}}$.

Remark 3 : *The Prelec probability weighting function, $w(p) = e^{-(-\ln p)^{\alpha}}$, $\alpha > 0$, has the fixed point, $\widetilde{p} = e^{-1} \simeq 0.37$, in broad agreement with the evidence. For most illegal activities, probabilities of detection and conviction are well below this value. Other probability weighting functions, when estimated, give similar results.*

Lemma 2 : *Let $w : [0,1] \rightarrow [0,1]$ be an inverse-S shaped probability weighting function. Let $w(\widetilde{p}) = \widetilde{p} \in (0,1)$, $w(p) \geq p$ for $p \in [0, \widetilde{p}]$ and $w(p) \leq p$ for $p \in [\widetilde{p}, 1]$. Let $\lambda \geq 1$. Then $\frac{\lambda w(p)}{\lambda w(p) + w(1-p)} \geq p$ for all $p \in [0, \widetilde{p}]$.*

Proof: Suppose that for some $p \in [0, \widetilde{p}]$, $\frac{\lambda w(p)}{\lambda w(p)+w(1-p)} < p$ (hence, $p \neq 0$). It follows that $\frac{\lambda p}{\lambda w(p)+w(1-p)} < p$ and, hence, $\frac{\lambda w(p)+w(1-p)}{\lambda} > 1$. Hence $\frac{\lambda(1-p)}{\lambda w(p)+w(1-p)} < 1-p$. But $1-p \in [\widetilde{p}, 1]$ and, hence, $w(1-p) \leq 1-p$. Hence, $\frac{\lambda w(1-p)}{\lambda w(p)+w(1-p)} < 1-p$. Hence, $\frac{w(1-p)}{\lambda w(p)+w(1-p)} < 1-p$, since $\lambda \geq 1$. Hence, $\frac{\lambda w(p)}{\lambda w(p)+w(1-p)} + \frac{w(1-p)}{\lambda w(p)+w(1-p)} < p + 1 - p$, i.e., $1 < 1$, which cannot be. Hence, for all $p \in [0, \widetilde{p}]$, $\frac{\lambda w(p)}{\lambda w(p)+w(1-p)} \geq p$. $\blacksquare$.

5.2. Rank dependent expected utility (RDU).

Here we give a brief introduction to rank dependent expected utility (RDU) as a prelude to reformulating the Polinsky and Shavell model (subsection 4.1) under RDU, below. RDU was first developed by Quiggin (1982, 1993). Mark Machina (2008) described RDU as the most notable[13] non-expected utility theory of decision making. It is a generalization of EU. So all phenomena that can be explained by EU can also be explained by RDU. However, the converse is not true. A large number of phenomena that can be easily explained by RDU cannot be explained by EU. The Allais paradoxes are, maybe, the most famous examples.

Suppose that the possible outcomes of the lottery, L, are ordered from lowest to highest:

$$y_1 \leq y_2 \leq ... \leq y_n. \tag{5.4}$$

As before, let the probability of y_i be $p_i \geq 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^n p_i = 1$. Let w be a probability weighting function. We define decision weights, $\pi_1, \pi_2, ..., \pi_n$, as follows.

Definition 8 *(Decision weights under RDU): Consider ranked outcomes, $y_1 \leq y_2 \leq ... \leq y_n$. Let the probability of outcome y_i be $p_i \geq 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^n p_i = 1$. Let w be a probability weighting function (Definition 6). The decision weights, $\pi_1, \pi_2, ..., \pi_n$, are defined by*[14] $\pi_i = w\left(\sum_{j=i}^n p_j\right) - w\left(\sum_{j=i+1}^n p_j\right)$, $i = n, n-1, ..., 2, 1$.

Remark 4 *: The decision weights in Definition 8 might look computationally complex but they have an intuitive explanation based on the shape of the underlying probability weighting function. For instance, if the weighting function is convex (respectively concave) throughout then the decision maker is pessimistic (respectively optimistic) in the sense that he/she places relatively higher (respectively lower) decision weight on smaller outcomes. These results hold even if the utility function is linear (which in the case of EU would have implied that the decision maker is risk neutral.*[15]

Example 2 *(Decision weights under RDU, $n = 2$): Consider ranked outcomes, $y_1 \leq y_2$. Let the probability of outcome y_i be $p_i \geq 0$, $i = 1, 2$, $p_1 + p_2 = 1$. Let w be a probability weighting function (Definition 6). The decision weights, π_1, π_2, are then $\pi_2 = w(p_2)$, $\pi_1 = w(p_1 + p_2) - w(p_2) = 1 - w(p_2)$.*

Definition 9 *(Rank dependent expected utility): Consider a decision maker with utility function, u, over outcomes (such as income or wealth levels). Consider the ranked outcomes, $y_1 \leq y_2 \leq ... \leq y_n$. Let the probability of outcome y_i be $p_i \geq 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^n p_i = 1$. Let w be a probability weighting function (Definition 6). Let $\pi_1, \pi_2, ..., \pi_n$, be the decision weights as in Definition 8. Then his/her rank dependent expected utility function is*

$$\textstyle\sum_{i=1}^n \pi_i u(y_i). \tag{5.5}$$

[13] At the end of the fourth paragraph, p519.

[14] In what follows, it might be useful to recall the mathematical convention that if $N > n$ then $\sum_{j=N}^n p_j = 0$ and, hence, $w\left(\sum_{j=N}^n p_j\right) = 0$.

[15] For a textbook treatment of these topics, the reader can consult Wakker (2010).

Axiom 2 : *The objective of a decision maker under RDU is to maximize his/her rank dependent expected utility (Definition 9), given the constraints he/she faces.*

Remark 5 : *EU is a special case of RDU. To see this, take the probability weighting function in Definition 8 to be $w(p) = p$, then $\pi_i = p_i$ and (5.5) reduces to (5.2).*

5.3. Crime under rank dependent expected utility (RDU).

We reformulate the Polinsky and Shavell model of crime under RDU, then use Corollary 3 to show that the hyperbolic punishment function is an upper bound for the optimal punishment function for the reformulated Polinsky and Shavell model. A similar construction and a similar result can be derived for the alternative model of subsection 4.2.

As before, we consider an individual who can earn either y_0 from a legal activity or y_1 from an illegal activity. Hence, the benefit from crime is $b = y_1 - y_0$. If the individual engages in the illegal activity he is caught with probability, p, and bears punishment whose monetary value is $F \geq 0$. Clearly, the ranking of the two outcomes of the illegal activity is $y_1 - F \leq y_1$. Let w be the probability weighting function of the decision maker. From Example 2, it follows that the decision weights are $\pi_2 = w(1-p)$ and $\pi_1 = 1 - w(1-p)$. As in the Polinsky and Shavell model, let the utility of the individual be $u(y) = y$. Hence, the decision maker's rank dependent expected utility from the illegal activity is $[1 - w(1-p)](y_1 - F) + [w(1-p)]\, y_1$. It follows that the decision maker will engage in the illegal activity if, and only if, $[1 - w(1-p)](y_1 - F) + [w(1-p)]\, y_1 \geq y_0$, which simplifies to

$$b \geq [1 - w(1-p)]\, F. \tag{5.6}$$

Remark 6 : *1. From the assumption that $u(y) = y$, it does not follow that the decision maker is risk neutral (unlike the case of EU). The reason is that non-linear weighting of probabilities can introduce risk aversion (or risk seeking) despite a linear utility function (see Remark 4 above).*
2. For the special case, $w(1-p) = 1 - p$, (5.6) reduces to (4.11), as is to be expected. Comparing (5.6) and (4.11) the reader may conjecture, correctly as it will turn out, that the following development will exactly mirror that of the Polinsky and Shavell model but with p replaced by $1 - w(1-p)$.

As in the Polinsky and Shavell model, punishments are ideal fines, so the cost of detection and punishment, $C(p)$, is independent of F. Damage to society from crime is slightly more general and is given by[16]

$$D(p, F) = \int_{[1-w(1-p)]F}^{\infty} (h - b)\, z(b)\, db. \tag{5.7}$$

[16]For the special case, $w(1-p) = 1 - p$, (5.7) reduces to (4.12), as is to be expected.

This model can be recast in the form of Proposition 2 with the following choices:

$$\Pi(p) = C(p), \pi(p) = 1, \alpha(p) = 0, \beta(p) = 1 - w(1-p), \Psi(F) = F, \tag{5.8}$$

$$\Phi([1 - w(1-p)]F) = \int_{[1-w(1-p)]F}^{\infty} (h - b) z(b) \, db. \tag{5.9}$$

Proposition 6 : *Assume that the decision maker has an inverse-S shaped probability weighting function (Definition 7) with $p \in (0, \widetilde{p}]$. Then the hyperbolic punishment function is an upper bound for the optimal punishment function.*

Proof: Since $p \in (0, \widetilde{p}]$ it follows, from Definition 7, that $w(1-p) \leq 1-p$ and, hence, $1 - w(1-p) \geq 1 - (1-p)$. From (5.8) it then follows that

$$\beta(p) \geq p \tag{5.10}$$

To simplify the algebraic formulae we will write β for $\beta(p)$. From (5.9) we get

$$\Phi'(\beta F) = -(h - \beta F) z(\beta F), \tag{5.11}$$

$$\Phi'(0) = -hz(0) < 0, \tag{5.12}$$

$$\Phi''(\beta F) = z(\beta F) - (h - \beta F) z'(\beta F). \tag{5.13}$$

From (5.8) and (5.12) we see that $\Phi'(\alpha(p)) = \Phi'(0) < 0$. Hence (4.4) holds. From (5.11), we see that $\Phi'(\beta F) = 0 \Leftrightarrow \beta F = h$. Moreover, from (5.13), we see that $\Phi''(\beta F) = z(h) > 0$. So, (4.5) holds. Thus, all the conditions of Corollary 3 hold. Hence, the hyperbolic punishment function is an upper bound for the optimal punishment function. ■.

Remark 7 : *Recall, from Remark 3, that $(0, \widetilde{p}]$ is the empirically relevant range.*

Proposition 6 demonstrates the usefulness of the HPF. For a decision maker who uses RDU, under the relevant assumptions, if a HPF cannot deter the decision maker from crime nor can the optimal punishment function.

5.4. Cumulative prospect theory (CP).

Cumulative prospect theory (Tversky and Kahneman, 1992), henceforth, CP, is a more radical departure from EU than is RDU. RDU is a special case of CP. All phenomena that can be explained by RDU (e.g., the Allais paradoxes) can also be explained by CP. But the converse is not the case. There are many important economic phenomena that can be explained by CP but cannot be explained by EU or RDU. The reason for this is that CP captures a number of robust psychological findings that are absent from EU and RDU. These are:

1. Reference dependence.
2. Loss aversion.
3. Declining sensitivity.
4. Cumulative transformation of probability.

Reference dependence states that people are sensitive not to final levels of wealth, income, assets or bundles of goods but to the deviations of these from a reference point.[17] In other words, people are sensitive to gains and losses relative to a reference point, rather than absolute levels. By convention, the reference point is taken to be at the origin and the utility at the reference point is taken to be zero. Tversky and Kahneman (1992) proposed the following utility function, to capture reference dependence:

$$v(x) = \begin{cases} x^{\gamma} & \text{if} \quad x \geq 0 \text{ (domain of gains)} \\ -\lambda(-x)^{\theta} & \text{if} \quad x < 0 \text{ (domain of losses)} \end{cases}, \tag{5.14}$$

where γ, θ, λ are constants. The parameters satisfy $0 < \gamma \leq 1$, $0 < \theta \leq 1$. $\lambda > 1$ is known as the *coefficient of loss aversion*. Tversky and Kahneman (1992) estimated that $\gamma \simeq \theta \simeq 0.88$ and $\lambda \simeq 2.25$. These parameter values are used to plot (5.14) in figure 5.2, below.

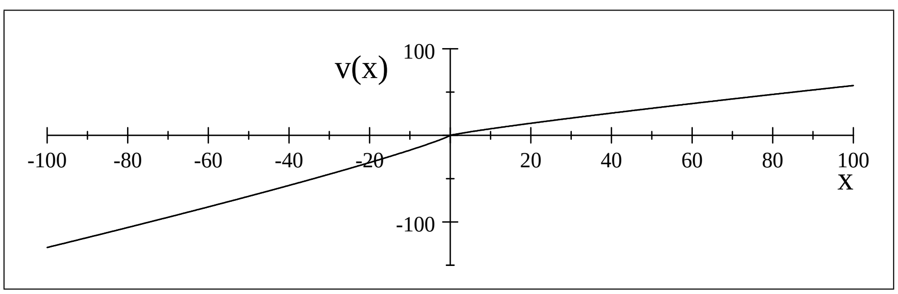

Figure 5.2: The utility function under CP

Figure 5.2 also illustrates *loss aversion*: A loss of, say, \$100 is more painful than a gain of \$100 is pleasurable. In addition the utility function is concave for gains but convex for losses; however, this barely perceptible (because $\gamma \simeq \theta \simeq 0.88$).

Remark 8 : *That estimated utility functions under CP are almost piecewise linear is important. It means that attitudes to risk are almost entirely captured by loss aversion*

[17]The following simple experiment illustrates the point. Fill three bowls with water: Fill the one on the left with hot water, the one in the middle with lukewarm water and the one on the right with cold water. Put your left hand in the hot water and your right hand in the cold water. After a few minutes put both hands in the lukewarm water. The left hand will feel the lukewarm water to be cold while the right hand will feel the lukewarm water to be hot.

and non-linear transformation of probability, rather than the shape of the value function. This is diametrically opposite to EU, where attitudes to risk are entirely captured by the shape of the utility function.

Reference dependence, loss aversion and declining sensitivity are all absent from EU and RDU. Cumulative transformation of probability is absent from EU but is, of course, present in RDU (recall subsection 5.2). However, it is applied differently in CP, as will be apparent from the following definition and example.

Definition 10 *(Decision weights under CP): Consider ranked outcomes, $y_{-m} \leq y_{-m+1} \leq ... \leq y_{-1} \leq y_0 = 0 \leq y_1 \leq y_2 \leq ... \leq y_n$. Let the probability of outcome y_i be $p_i \geq 0$, $i = -m, -m+1, ..., n$, $\sum_{i=-m}^{n} p_i = 1$. Let w^+, w^- be a probability weighting functions (Definition 6) for gains and losses, respectively. The decision weights, $\pi_{-m}, \pi_{-m+1}, ..., \pi_{-1}, \pi_0, \pi_1, \pi_2, ..., \pi_n$, are defines as follows.*[18]
$\pi_i = w^+\left(\sum_{j=i}^{n} p_j\right) - w^+\left(\sum_{j=i+1}^{n} p_j\right)$, $i = n, n-1, ..., 2, 1$ *(domain of gains),*
$\pi_{-i} = w^-\left(\sum_{j=-m}^{-i} p_j\right) - w^-\left(\sum_{j=-m}^{-i-1} p_j\right)$, $i = m, m-1, ..., 2, 1$ *(domain of losses).*

Example 3 *(Decision weights under CP, $n = m = 1$): Consider ranked outcomes, $y_{-1} \leq 0 \leq y_1$. Let the probability of outcome y_i be $p_i \geq 0$, $i = -1, 1$, $p_{-1} + p_1 = 1$. Let w^+, w^{-1} be a probability weighting functions (Definition 6) for gains and losses, respectively. The decision weights, π_{-1}, π_1, are then: $\pi_1 = w^+(p_1)$, $\pi_{-1} = w^-(p_{-1})$.*

Remark 9 : *In general, Definition 10 does not yield a cumulative transformation of probabilities (this is most obvious from Example 3). For that to be the case, we would need $w^-(p) = 1 - w^+(1-p)$. However, the empirical evidence suggests $w^-(p) = w^+(p)$; see Prelec (1998). In fact, for the case of exactly one outcome in each of the domain of gains and losses, CP reduces to the original prospect theory of Kahneman and Tversky (1979).*

Definition 11 *(The value function under CP): Consider a decision maker with a strictly increasing utility function, v, over outcomes relative to a reference point and satisfying $v(0) = 0$. Consider ranked outcomes, $y_{-m} \leq y_{-m+1} \leq ... \leq y_{-1} \leq y_0 = 0 \leq y_1 \leq y_2 \leq ... \leq y_n$. Let the probability of outcome y_i be $p_i \geq 0$, $i = -m, -m+1, ..., n$, $\sum_{i=-m}^{n} p_i = 1$. Let w^+, w^{-1} be a probability weighting functions (Definition 6) for gains and losses, respectively. Let $\pi_{-m}, \pi_{-m+1}, ..., \pi_n$, be the decision weights as in Definition 10. Then his/her value function, V, is*

$$V = \sum_{i=-m}^{n} \pi_i v(y_i). \tag{5.15}$$

[18] π_0 is not defined. This will be explained in Remark 4, below.

Example 4 *: Note that π_0 is not defined. This does not matter, since $v(0) = 0$, we can choose to give π_0 any convenient value.*

Axiom 3 *: The objective of a decision maker under CP is to maximize his/her value function (Definition 5.15), given the constraints he/she faces.*

5.5. Crime under cumulative prospect theory (CP).

We reformulate the Polinsky and Shavell model of crime under CP, then use Corollary 3 to show that the hyperbolic punishment function is an upper bound for the optimal punishment function for the Polinsky and Shavell model under CP. A similar construction and a similar result can be derived for the alternative model of subsection 4.2.

As before, we consider an individual who can earn either y_0 from a legal activity or y_1 from an illegal activity. Hence, the benefit from crime is $b = y_1 - y_0$. If the individual engages in the illegal activity he is caught with probability, p, and forced to pay a punishment, $F \geq 0$. Clearly, the ranking of the two outcomes of the illegal activity is $y_1 - F \leq y_1$.

We need to specify the reference point(s). Let y_{nc} be the reference point for income if the individual decides not to engage in the illegal activity. Let y_c be the reference point for income if the individual decides to engage in the illegal activity. There are three cases that could be considered:

1. $y_1 - F$ is in the domain of gains ($y_1 - F - y_c \geq 0$). Hence, y_1 is also in the domain of gains.
2. y_1 is in the domain of losses ($y_1 - y_c < 0$). Hence, $y_1 - y_c - F$ is also in the domain of losses.
3. $y_1 - F$ is in the domain of losses ($y_1 - F - y_c < 0$) but y_1 is in the domain of gains ($y_1 - y_c \geq 0$).

The analysis in cases 1 and 2 is very similar to that under RDU. So, in what follows, we will concentrate on the third case only. Let the individual's utility function be v. Hence, his utility from not committing the crime is $v(y_0 - y_{nc})$. Let his probability weighting functions for gains and losses be, respectively, w^+ and w^-. His utility from committing the crime is $w^-(p)\, v[y_1 - F - y_c] + w^+(1-p)\, v[y_1 - y_c]$. This individual will engage in the illegal activity if, and only if,

$$w^-(p)\, v[y_1 - F - y_c] + w^+(1-p)\, v[y_1 - y_c] \geq v(y_0 - y_{nc}). \tag{5.16}$$

We make three simplifying assumptions:

1. $w^{-} = w^{+} = w$. This assumption is, in fact, consistent with the evidence.

2. $x \geq 0 \Rightarrow v(x) = x$, $x < 0 \Rightarrow v(x) = \lambda x$, where $\lambda > 1$ is the coefficient of loss aversion.[19]

3. $y_c = y_{nc} = y_0$, i.e., the reference point is not state dependent.[20]

With the above simplifying assumptions, and recalling that $v(0) = 0$ and $y_1 - y_0 = b$, (5.16) becomes

$$\lambda [b - F] w(p) + bw(1-p) \geq 0, \tag{5.17}$$

which yields

$$b \geq \frac{\lambda w(p)}{\lambda w(p) + w(1-p)} F. \tag{5.18}$$

As in the Polinsky and Shavell model, punishments are ideal fines, so the cost of detection and punishment, $C(p)$, is independent of F. On account of (5.18), the damage to society from crime is slightly more general and is given by

$$D(p, F) = \int_{\frac{\lambda w(p)}{\lambda w(p)+w(1-p)} F}^{\infty} (h - b) z(b) \, db. \tag{5.19}$$

This model can be recast in the form of Proposition 2 with the following choices[21]:

$$\Pi(p) = C(p), \pi(p) = 1, \alpha(p) = 0, \beta(p) = \frac{\lambda w(p)}{\lambda w(p) + w(1-p)}, \Psi(F) = F, \tag{5.20}$$

$$\Phi\left(\frac{\lambda w(p)}{\lambda w(p) + w(1-p)} F\right) = \int_{\frac{\lambda w(p)}{\lambda w(p)+w(1-p)} F}^{\infty} (h - b) z(b) \, db. \tag{5.21}$$

Proposition 7 : *Assume that the decision maker has an inverse-S shaped probability weighting function (Definition 7) with $p \in (0, \widetilde{p}]$. Then the hyperbolic punishment function is an upper bound for the optimal punishment function.*

Proof: Since $p \in (0, \widetilde{p}]$ it follows, from (5.20) and Lemma 2, that

$$\beta(p) \geq p \tag{5.22}$$

To shorten the algebraic formulae, write β for $\beta(p)$. From (5.21) we get

$$\Phi'(\beta F) = -(h - \beta F) z(\beta F), \tag{5.23}$$

[19] The piecewise linear approximation to the value function is found to be empirically a good one (recall Remark 8).

[20] The assumption $y_c = y_{nc} = y_0$ is made purely to simplify the exposition. Dropping this assumption only results in longer formulas.

[21] More generally, $\alpha(p) = \frac{y_0 - y_{nc}}{\lambda w(p) + w(1-p)} - (y_0 - y_c)$. Setting $y_c = y_{nc} = y_0$ gives $\alpha = 0$.

$$\Phi'(0) = -hz(0) < 0, \tag{5.24}$$

$$\Phi''(\beta F) = z(\beta F) - (h - \beta F) z'(\beta F). \tag{5.25}$$

From (5.20) and (5.24) we see that $\Phi'(\alpha(p)) = \Phi'(0) < 0$. Hence (4.4) holds. From (5.23), we see that $\Phi'(\beta F) = 0 \Leftrightarrow \beta F = h$. Moreover, from (5.25), we see that $\Phi''(h) = z(h) > 0$. So, (4.5) holds. Thus, all the conditions of Corollary 3 hold. Hence, the hyperbolic punishment function is an upper bound for the optimal punishment function. ■.

Remark 10 : *Under the conditions of Proposition 7 (but not using the simplification $y_c = y_{nc} = y_0$), using the Prelec probability weighting function (5.3) and using Proposition 2, it follows that the optimal punishment function is*

$$\varphi(p) = \left\{ y_{nc} - y_0 + (c + y_0 - y_c) \left[\lambda e^{-\beta(-\ln p)^\alpha} + e^{-\beta(-\ln(1-p))^\alpha} \right] \right\} \lambda^{-1} e^{\beta(-\ln p)^\alpha}.$$

From the above, we can see how intricate an optimal punishment function can be compared to the HPF, $H(p) = \frac{c}{p}$, thus illustrating the utility of the latter. However, under the conditions of Proposition 7, $\varphi(p) \leq H(p)$ and, for many purposes we can work with the much more tractable, $H(p)$, rather than $\varphi(p)$.

6. Conclusions

The hyperbolic punishment function (HPF) is popular in the law and economics literature. A typical justification for its usage is that it is very tractable. Under risk-neutrality and certain classical assumptions about the objective function, the HPF is an *optimal punishment function*; see, e.g., Polinsky and Shavell (2007). We demonstrate the optimality of the HPF for a much wider class of models. Furthermore, we show that the HPF is a lower (upper) bound for the optimal punishment functions for an even wider class of models. Hence, if the HPF can (cannot) deter crime then all (none) of the optimal punishment functions in that class can deter crime.

While our contribution is technical in nature it is potentially, as we noted above, of much practical use. For many problems of interest in Law and Economics, the focus of the paper need not necessarily be about optimal punishment functions. Rather, the researcher might wish to use a punishment function that is tractable, in order to address other important issues. This should justify the existing popularity of the HPF and also underpin its greater usage.

References

[1] Allais, M., 1953, "Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Américaine", Econometrica 21, 503-546.

[2] Allingham, M.G., Sandmo A. (1972) Income tax evasion: A theoretical analysis. Journal of Public Economics. 1: 323-38.

[3] Becker, G. (1968) Crime and Punishment: an Economic Approach. *Journal of Political Economy.* 76: 169-217.

[4] Bentham, J. (1789), An Introduction to the Principles of Morals and Legislation, in: The Utilitarians (Anchor Books, Garden City, N.Y., 1973).

[5] Bernoulli, Daniel, Originally published in 1738; translated by Dr. Louise Sommer. (January 1954). "Exposition of a New Theory on the Measurement of Risk". Econometrica (The Econometric Society) 22 (1): 22–36. doi:10.2307/1909829. JSTOR 1909829. http://www.math.fau.edu/richman/Ideas/daniel.htm. Retrieved 2006-05-30.

[6] Camerer, C.F. and G. Loewenstein (2004), Behavioral Economics: Past, Present and Future. Chapter 1 in C.F. Camerer, G. Loewenstein and M. Rabin (eds.), *Advances in Behavioral Economics*, New York: Princeton University Press, 3-51.

[7] Dhami, S., al-Nowaihi, A. (2007) Why do people pay taxes? Prospect theory versus expected utility theory. *Journal of Economic Behavior and Organization.* 64: 171–192.

[8] Garoupa, N. (2001) Optimal Magnitude and Probability of Fines, *European Economic Review*, 45: 1765-1771.

[9] Kahneman, D. and Tversky, A. (2000) *Choices, Values and Frames.* Cambridge University Press.

[10] Kahneman, D. and Tversky, A. (1979). Prospect theory: an analysis of decision under risk. Econometrica 47(2) 263-291.

[11] Kolm, Serge-Christophe (1973) A Note on Optimum Tax Evasion. *Journal of Public Economics.* 2: 265-270.

[12] Kunreuther, Howard, et al. (1978), *Disaster insurance protection: public policy lessons*, New York: Wiley.

[13] Luce, R. D. and Raiffa, H. (1957) *Games and Decisions: Introduction and critical survey.* John Wiley and Sons, Inc.

[14] Machina, M. J. (2008). Non-expected utility theory. *International Encyclopedia of Social Science*, William Darity Jr. (ed.). 519-520.

[15] Polinsky, M., and Shavell, S., (2000) The Economic Theory of Public Enforcement of Law. *Journal of Economic Literature.* 38: 45-76.

[16] Polinsky, M., and Shavell, S. (2007) The theory of public enforcement of law. Chapter 6 in Polinsky, Mitchell and Steven Shavell (eds.) *Handbook of Law and Economics*, Vol. 1, Elsevier.

[17] Prelec D. (1998). The probability weighting function. *Econometrica* 60. 497-528.

[18] Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior and Organization*, 3, 323-343.

[19] Quiggin, J. (1993). *Generalized Expected Utility Theory.* Kluwer Academic Publishers.

[20] Rabin, M. (2000) Risk Aversion and Expected-Utility Theory: A Calibration Theorem. *Econometrica*, 68, 1281-1292.

[21] Rasmusen, E. (1994) *Games and Information. An Introduction to Game Theory* (2nd edition). Cambridge: Blackwell.

[22] Starmer, C. (2000) Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk. *Journal of Economic Literature*, 34, 332-382.

[23] Tversky, A., and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of risk and uncertainty* 5 297-323.

[24] Wakker, P. P. (2010) *Prospect Theory: For Risk and Ambiguity.* Cambridge University Press.