Can Expected Utility Theory Explain Gambling?

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We investigate the ability of expected utility theory to account for simultaneous gambling and insurance. Contrary to a previous claim that borrowing and lending in perfect capital markets rules out a demand for gambles, we show that expected utility theory with non-concave utility functions can still explain gambling. When the rates of interest and time preference are equal, agents will seek to gamble unless income falls in a finite set of exceptional values. When these rates differ, there will be a range of incomes for which gambles are desired. In both cases repeated gambling is not explained but market imperfections such as different borrowing and lending rates can account for persistent gambling provided the rates span the rate of time preference.

Accounting for gambling presents a significant challenge to theories of decision making under uncertainty, particularly in a dynamic setting. If expected utility theory is to be used to model decision-making under uncertainty, the only way to explain simultaneous gambling and insurance is to introduce non-concave segments into the utility function. This approach was first taken by Friedman and Savage [8] who used a utility function with a single convex segment accompanied by a justification of this shape. They demonstrated that a utility function which included a section with increasing marginal utility could account for the existence of consumers who purchase both insurance and lottery tickets.

The explanatory power of the Friedman-Savage approach was challenged by Bailey, Olson and Wonnacott [1] who argued that non-concave utility functions could not, in principle, explain gambling. The intuition behind their argument is simple. Consider the Friedman-Savage utility function v shown in Figure 1 together with the common tangent to the curve at the points $\underline{c}$ and $\bar{c}$. We write Cv for the concave hull of v in which the graph of v is bridged by the common tangent between $\underline{c}$ and $\bar{c}$. An agent at c can move up from $v(c)$ to $Cv(c)$ by buying a fair gamble between $\underline{c}$ and $\bar{c}$. When there are two periods the agent has an alternative possibility: save by consuming $\underline{c}$ in the initial period to finance consumption of $\bar{c}$ in the second, or borrow to support consumption of $\bar{c}$ in the first period and $\underline{c}$ in the second period.

When the rates of interest and time preference are equal this does just as well as gambling. When they differ, one of these alternatives is strictly preferred to gambling.

Unfortunately, this argument encounters two difficulties. First, the required pattern of saving or borrowing is only feasible if income is chosen appropriately. For example, when the rates of interest and time preference are both zero, the amount saved in the first period must equal the increase in consumption in the second period. This requires that income be equal to $(\underline{c} + \bar{c})/2$. For all other income levels there will be gambles strictly preferred to the optimal pattern of saving and borrowing. This conclusion continues to hold when the rates of interest and time preference are equal and positive although there are now two exceptional income levels corresponding to saving or borrowing. The second difficulty is that the model of Bailey et al. does not allow for the possibility that an agent may wish to save or borrow and gamble. Permitting gambling as well as saving and borrowing can restore a demand for gambles even when pure saving or borrowing is strictly preferred to pure gambling. This follows from the observation that optimal saving and borrowing without gambling will typically lead to a consumption level different from $\underline{c}$ and $\bar{c}$ in at least one period. In any period in which the consumption lies strictly between $\underline{c}$ and $\bar{c}$ total expected utility can be increased by gambling in that period as this shifts expected utility upwards on to the common tangent. Hence a demand for gambles is restored.

In this paper, we extend the model of Bailey et al. by allowing agents to gamble as well as save and borrow. With this extension, the analysis shows that expected utility with non-concave utility functions can explain the desire to gamble even with perfect capital markets and time-separable utility functions. A demand for gambles will persist in our model when the rates of interest and time preference are equal unless income happens to take one of a finite set of exceptional values. When the rates differ, there will be a range of income levels for which there is a demand for gambles. However, as in Bailey et al., repeated gambling can not be explained in the model without invoking market failure.

Discomfort with the notion of increasing marginal utility of market goods has led several authors to offer a foundation for non-concavities of the Friedman-Savage type using indivisibilities in markets such as labor supply (Dobbs[4]) and education (Ng[13]) or capital market imperfections (Kim [11]). Jullien and Salanié[10] show that a sample of racetrack bettors exhibit local risk aversion similar to that arising from Friedman-Savage utility functions, within the context of cumulative prospect theory. These explanations and observations imply non-concave functions of wealth but are vulnerable to the idea that borrowing and saving can transform them into a concave function. In direct response to the Bailey et al. critique, Dowell and McLaren[5] show

how a model in which wage rates increase with work experience can lead to a Friedman-Savage function of nonhuman wealth without invoking market imperfections.

The principal alternative explanation of gambling is that it offers direct consumption value. It is useful to distinguish two forms of this assumption. Firstly, and most simply, the utility of non-monetary activities associated with gambling such as attending a race meeting or viewing a lottery-related television program when one has a stake in the outcome, could be included directly in the calculations. Historically, this approach has consisted of little more than informal comments, but more recently Simon [15] has used an explicit `dream' function to model demand for lottery tickets. Johnson and Shin [9] have estimated such a function for betting on horse races using data from bookmakers. These authors also point out punters' behavior which is hard to rationalize without invoking such a function.

The other form of the assumption modifies expected utility theory by supposing that the money values and probabilities in any risky prospect have direct value beyond that included in the expression for expected utility. A particularly elegant version was presented by Conlisk [3] who demonstrated that adding an arbitrarily small function of the money values and probabilities to an otherwise concave utility function could explain risk preferring behavior such as the purchase of lottery tickets. Other non-expected utility theories may explain features of gambling, such as the nature of the prizes in lottery games, which are hard to justify using expected utility theory. (See Quiggin [14].)

However, these approaches are not without difficulties. It is unclear whether dream functions should be applied to all risky decision-making as in Conlisk or only to, say, unfair gambles with very long odds such as are found in lottery games as in Simon. The latter possibility leaves many other forms of gambling unexplained. However, a universally applied dream function only partially determines how the characteristics of the gamble, such as the size of prizes, probability of winning, time at which uncertainty is resolved etc., could be explained. Without a clear prescription of the nature of the function, it becomes a difficult task to compare the demand for related gambles such as one gamble which is a mean-preserving risk spread of another, or to analyze the portfolio effects of activities such as laying fixed odds and spread bets on the same sporting event. The flexibility in functional form means that, rather than explain gambling, it is all too easy to impose observed behavior by suitable choice of a dream function. Furthermore, the dynamic consistency of such models is controversial [12] which makes their application in inter-temporal models problematic.

The rest of the paper describes our extension of the model of Bailey et al. and analyses its properties. In Section I we formulate the consumer's optimization problem when gambles are available and demonstrate how this problem may be solved in terms of a related deterministic problem. This construction allows us to relate the indifference maps when gambling is possible and when it is excluded and these results are applied to an analysis of two-period problems in Section II. In Section III, we outline results for more than two periods. In Section IV we show that the model cannot explain repeated gambling without introducing some market imperfection and investigate how different borrowing and lending rates may overcome this problem. Our conclusions are stated in Section V.

I. Solving the multi-period problem

A. Methodology

Our approach is in three steps.

1. We write down the multi-period optimization problem facing a consumer who can borrow and save in a perfect capital market and has a separable utility function in which intra-period preferences are reflected in a non-concave utility function. We refer to the optimal solution of this problem, when no gambles are available, as the no-gambling so-

lution.

2. We extend the previous optimization problem by allowing consumers access to fair gambles with any pattern of payoffs. This is our extension of the model of Bailey et al. The solution to this problem is simply referred to as optimal.

3. We ask whether the optimal objective values of the two problems are the same i.e. is the no-gambling solution optimal?

A negative answer to the final question implies a positive demand for fair gambles and, by continuity, for some unfair gambles. Whether this will actually result in gambling depends on the supply side of the gambling market which is not analyzed here[1]. We therefore interpret a negative answer to 3. as support for the explanatory power of Friedman-Savage or more general non-concave von Neumann-Morgenstern utility functions.

B. The no-gambling solution

Since we wish to demonstrate that non-concave utility functions can explain gambling even when utility functions are separable, we will follow Bailey et al. in assuming a von Neumann-Morgenstern utility function of the form

$$U(c_1, \ldots, c_T) = \sum_{t=1}^{T} \frac{v(c_t)}{(1+\delta)^t} \tag{1}$$

where c_t is consumption in period $t(= 1, \ldots, T)$ and $\delta > 0$. We assume that v is strictly increasing but not necessarily concave[2]. In Figure 1, we graph both v and its concave hull Cv for the classic Friedman-Savage utility function. The non-concavity of v means that there will be consumption levels c satisfying $v(c) < Cv(c)$ and we write $(\underline{c}, \bar{c})$ for the set of all such consumption levels[3]. For such a c, the consumer will prefer to the status quo a gamble in which the ex post wealth is either $\underline{c}$ or $\bar{c}$ and the probability of winning is chosen to make the gamble fair. Indeed, there will be unfair gambles giving an expected utility greater than $v(c)$. It is also convenient to assume that for $c < \underline{c}$ or $c > \bar{c}$ the consumer is risk-averse: the Friedman-Savage function contains no linear sections[4].

Assuming perfect capital markets with rate of interest r, the optimal solution in the absence of gambling is found by maximizing U subject to

$$\sum_{t=1}^{T} \frac{c_t}{(1+r)^t} = y^* \sum_{t=1}^{T} \frac{1}{(1+r)^t}, \tag{2}$$

where y^* is permanent income.

C. Consumer's optimization problem

We now introduce the possibility of gambling by allowing the consumer to increase her wealth in period t by adding any random variable X_t satisfying $EX_t = 0$ for $t = 1; \ldots; T$. We also permit the consumption decision in period t to depend on the outcome of the gamble X_t and random events in previous periods. This makes consumption in any period a random variable and we place no restrictions on the joint distribution[5] of $(X_1; C_1; \ldots; X_T; C_T)$. We also require the budget constraint (2) to be satisfied for every sample path. Thus, the consumer's optimization problem for T periods, which we abbreviate to CP^T, becomes

$$\max E \sum_{t=1}^{T} \frac{v(C_t)}{(1+\delta)^t}$$

$$\text{subject to} \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} = \sum_{t=1}^{T} \frac{y^* + X_t}{(1+r)^t}; \text{ and } EX_1 = \cdots = EX_T = 0$$

where the maximization is with respect to $X_1; C_1; \ldots; X_T; C_T$ or, equivalently with respect to the joint distribution of these random variables.

D. Solving the consumer's problem

This problem can be solved by an indirect approach. Since the best choice of gamble moves the consumer from v to Cv, we start by solving a modification of the no-gambling problem of the previous subsection in which v in the objective function is replaced by Cv. Whenever the optimal solution of this problem requires consumption c between $\underline{c}$ and $\bar{c}$ in a certain period, an optimal solution of CP^T is found by choosing the gamble required to obtain expected utility $Cv(c)$ in that period. More formally, we proceed as follows.

Substituting Cv for v in CP^T yields an upper bound to the original problem since $Cv \geq v$. Furthermore, the concavity of Cv and linearity of the constraint allows us to replace the random variables with their expected values without reducing the value of the objective function[6]. This shows that the following deterministic problem, which we shall refer to as the deterministic equivalent of CP^T,

$$\max \sum_{t=1}^{T} \frac{Cv(c_t)}{(1+\rho)^t} \text{ subject to (2)},$$

yields an upper bound for CP^T.

We can construct a solution $(\hat{X}_1, \hat{C}_1, \ldots, \hat{X}_T, \hat{C}_T)$ of CP^T which achieves this upper bound, and is therefore optimal, as follows. Let $(\hat{c}_1, \ldots, \hat{c}_T)$ be the optimal solution of the deterministic equivalent and write I_x for the degenerate random variable which takes the value x with certainty. For each $t = 1, \ldots, T$, two cases are possible.

Case 1: $v(\hat{c}_t) = Cv(\hat{c}_t)$:

Let $\hat{X}_t = I_0$ and $\hat{C}_t = I_{\hat{c}_t}$.

Case 2: $v(\hat{c}_t) < Cv(\hat{c}_t)$:

Let $\hat{X}_t$ take the value

$$\begin{aligned} &\hat{c}_t - \underline{c}, \text{ with probability } 1 - \pi \text{ and} \\ &\bar{c} - \hat{c}_t, \text{ with probability } \pi, \end{aligned}$$

where

$$\pi = \frac{\hat{c}_t - \underline{c}}{\bar{c} - \underline{c}}$$

and let $\hat{C}_t = \hat{c}_t + \hat{X}_t$.

Note that, in Case 2, $E\hat{X}_t = 0$ as required, and

$$Ev(\hat{C}_t) = \pi v(\bar{c}) + (1 - \pi)v(\underline{c}) = Cv(\hat{c}_t).$$

These results are also trivially true in Case 1, so the constructed solution achieves the upper bound. Furthermore, since $(\hat{c}_1, \ldots, \hat{c}_T)$ is feasible in the deterministic equivalent, $(\hat{X}_1, \hat{C}_1, \ldots, \hat{X}_T, \hat{C}_T)$ is feasible in the original problem on every sample path. We refer to this construction as the Standard Construction and conclude that an optimal solution to CP^T may be obtained by first solving the deterministic equivalent and then using the standard construction to generate a solution of CP^T. Furthermore, the optimal objective values of CP^T and its deterministic equivalent are the same.

II. Two-period problems

A. Indifference maps

In this section we describe a graphical approach to problems with two periods. The starting point is the utility function for the problem with no gambling

$$U(c_1, c_2) = \frac{v(c_1)}{(1+\delta)} + \frac{v(c_2)}{(1+\delta)^2}. \tag{3}$$

The argument in the previous section shows that CP^2 has the same optimal objective function value as its deterministic equivalent and solving the latter involves substituting Cv for v in (3). Thus, for any reference level of utility, we can draw a corresponding pair of v- and Cv-indifference curves. In Figure 2, we display a pair of indifference curves[7] corresponding to the same utility level, where v has the shape shown in Figure 1. Indifference curve I, drawn as a solid line, is for v and I^*, drawn dashed where it differs from I, is for Cv. We note that indifference curve I does not 'fill in' the indentation in I^*.

We also include (drawn dotted) the four lines $c_t = \underline{c}$ and $c_t = \bar{c}$ for $t = 1, 2$. These lines divide the positive quadrant of the plane into nine regions. The central square includes all consumption vectors corresponding to gambling in both periods. In this region, Cv is linear in both periods so that all Cv-indifference curves have the same slope: $-(1+\delta)$ throughout the square. In the four corner regions there is no gambling in either period and indifference curves of v and Cv for the same level of utility coincide. The East ($c_1 > \bar{c}; \underline{c} < c_2 < \bar{c}$) and West regions correspond to gambling only in the second period and the North and South regions to gambling only in the first period. If I passes through $(c_1; c_2)$ where $\underline{c} < c_1 < \bar{c}$, then $v(c_1) < Cv(c_1)$ and there is a section of I^* lying closer to the origin than $(c_1; c_2)$. Similar conclusions hold if $\underline{c} < c_2 < \bar{c}$ proving Observation 1. Except in the four corner regions, including their boundaries, a Cv-indifference curve lies strictly below (i.e. on the origin side of) the v-indifference curve corresponding to the same utility level.

We have also included in Figure 2 (marked with dots and dashes) the iso-slope locus[8], L, of all points $(c_1; c_2)$ for which $v'(c_1) = v'(c_2)$. L is also the set of points at which the slope of the v-indifference curves is $-(1+\delta)$ and therefore where the first-order conditions for maximizing U subject to the inter-temporal budget constraint:

$$\frac{c_1}{(1+\delta)} + \frac{c_2}{(1+\delta)^2} = \frac{(2+\delta)y^*}{(1+\delta)^2} \tag{4}$$

are satisfied. This gives Observation 2. All no-gambling solutions for $r = \delta$ lie on L[9].

Since v has a common tangent at $\underline{c}$ and $\bar{c}$ (see Figure 1), the iso-slope locus must include the four vertices of the central square of Figure 2. Otherwise, the only part of L which can enter the four corner regions is the 45° line. This can be seen by examining the marginal utility function v' for a Friedman-Savage v, which we have graphed in Figure 3 and in which we have marked $\underline{c}$ and $\bar{c}$. If (c_1, c_2) is a point of L where $c_1 \neq c_2$, then $v'(c_1)$ and $v'(c_2)$ lie on the same horizontal line. Since this is also true of $v'(\underline{c})$ and $v'(\bar{c})$, at most one of c_1 and c_2 can fall outside the interval $(\underline{c}, \bar{c})$.

Combining this result with Observation 2 gives Observation 3. If $v(y^*) < Cv(y^*)$, no-gambling solutions for $r = \delta$ cannot lie in the interior of a corner region.

B. The optimality of no-gambling solutions

Throughout this subsection, we assume that $v(y^*) < Cv(y^*)$. We first examine the case $r = \delta$. Observation 3 implies that a tangency point between the budget line (4) and a v-indifference curve cannot lie in the interior of a corner region. Observation 1 allows us to conclude that, unless the tangency point happens to be a corner point of the central square, there are points on the Cv-indifference curve with the same utility level which lie closer to the origin than the tangency point. Thus, the same utility level may be achieved in the interior of the budget set when gambling is allowed so the no-gambling solution is sub-optimal.

This is illustrated in Figure 2 where the no-gambling solution is at the intersection of I and L in the East region whereas the set of tangency points between the budget line and the corresponding Cv-indifference curve is AB.

An exceptional case where the v-indifference curve passes through the point $(\underline{c}, \bar{c})$ is shown in Figure 4. Here, $A = (\underline{c}, \bar{c})$ is optimal but the slope of both curves at A is $-(1+\delta)$. The complete set of optimal solutions is the line-segment AB. Hence there is an optimal no-gambling solution although there are alternative optimal solutions which do involve gambling. These are the only exceptions and occur only if one of these corners happens to lie on the budget line which requires that

$$y^{*} = [(1+\delta)\bar{c} + \underline{c}]/(2+\delta) \text{ or } y^{*} = [(1+\delta)\underline{c} + \bar{c}]/(2+\delta). \tag{5}$$

These results establish the next theorem.

Theorem 1 *If $\underline{c} < y^{*} < \bar{c}$ and (5) does not hold, the no-gambling solution is sub-optimal*[10].

We now turn to the case $r \neq \delta$ and start from the case $r = \delta$ when the tangency set between the budget line, which has slope $-(1+\delta)$, and the optimal Cv-indifference curve is the set ADB in Figure 2. As r increases above [decreases below] δ, the budget line rotates [anti]clockwise. The tangency point with the Cv-indifference curve always lies above the 45° line and moves away from it. This is illustrated in Figure 5, where we have redrawn the indifference curves from Figure 2. For the budget line B_1; the optimal solution is A_1 and it is clear that the no-gambling solution is sub-optimal. The point A_2 is optimal for the budget line B_2: A_2 is also the no-gambling solution but only in a trivial sense: the optimal solution does not involve gambling in spite of the non-concavity of v. We may conclude that, provided r is not too different from η, there is a range of incomes for which forbidding gambling makes consumers worse off and thus for which there is a demand for unfair gambles. This remains true even for the exceptional cases, (5), identified above: an examination of the indifference curves from Figure 4 shows that if $r > \eta$; all globally optimal solutions lie on both curves whereas, if $r < \eta$; there are income levels for which the no-gambling solution is sub-optimal. We have established the following result.

Theorem 2 There is a $\delta > 0$ such that, if $r \neq \eta$ and $|r - \eta| < \delta$; there is range of income levels for which the no-gambling solution is sub-optimal.

C. Pure gambling

In this subsection, we look at the two-period problems studied by Bailey et al. who compared the no-gambling solution with pure gambling i.e. without inter-temporal substitution, and claimed that the former would be preferred (weakly if $r = \delta$).

When $r = \delta$, we can carry out the comparison in Figure 2. The budget line coincides with the optimal Cv-indifference curve in the central square so that the pure-gambling solution is found at the intersection of the indifference curve and the 45° line (point D in the figure). Unless this curve passes through $(\bar{c}, \underline{c})$ or $(\underline{c}, \bar{c})$, it lies below the v-indifference curve with the same utility level by Observation 1, in which case D is preferable to the no-gambling solution. Hence, unless income happens to satisfy (5), pure gambling is strictly preferred to borrowing and saving[11].

When $r \neq \delta$; the results are ambiguous. In Figure 6 we have drawn Cv- and v-indifference curves for the same utility level as well as two possible budget lines passing through the point D, where the Cv-indifference curve crosses the 45° line. For B_1B_1, pure gambling is preferable to borrowing and saving whereas, for B_2B_2, the converse is true. Indeed, as the budget line through D rotates clockwise beginning at a low angle with the horizontal axis, it starts by crossing the corresponding v-indifference curve. Then, after reaching a critical slope, where it is a tangent, it ceases to cross the v-indifference curve. This continues until a second tangency point is reached after which the v-indifference curve is crossed again. This means that there will be interest rates r_D and r_U ($> r_D$) such that, if $r_D < r < r_L$, then pure gambling is preferred to borrowing and saving whereas, if $r < r_D$ or $r > r_U$, preferences are reversed[12].

III. More than two periods

The results of the previous section extend to more than two periods. When income lies between $\underline{c}$ and $\bar{c}$ and the rates of interest and time preference are equal there will still be a demand for gambles. In particular, the no-gambling solution of CP^T, for $T > 2$, is suboptimal provided income does not fall in a finite set of exceptional values. However, this set grows exponentially larger as the number of periods increases, for exceptional income levels correspond to a consumption pattern equal to either $\underline{c}$ or $\bar{c}$ in each of the T periods. This leads to $2^T - 2$ such income levels between $\underline{c}$ and $\bar{c}$. Furthermore, as T increases the exceptional values fill in the interval $(\underline{c}; \bar{c})$ and the per-period value of the optimal no-gambling solution[13] approaches Cv. This accords with the intuition behind the analysis of Bailey et al. The more periods are available, the more closely the consumer can replicate the gamble which moves her from v onto Cv using a feasible pattern of deterministic consumption. Such a conclusion suggests that the demand for gambles will disappear if the number of periods is allowed to become infinite. Confirmation of this suggestion may be found in a detailed analysis of the infinite horizon case carried out in Farrell and Hartley [6].

The conclusions of the previous section also extend to more than two periods when the rates of interest and time preference differ. Provided this difference is not too great, the optimal solution of the deterministic equivalent of CP^T entails consumption at a level between $\underline{c}$ and $\bar{c}$ in some period for a range of incomes. Employing the standard construction we find that the optimal solution of CP^T requires the consumer to gamble in that period. Hence, there will be a range of incomes for which the no-gambling solution is sub-optimal and a demand for gambles will persist for $T > 2$. By contrast with the result when interest and time preference rates are equal, this demand does not go away as the number of periods approaches infinity. For a range of incomes, consumers will demand gambles even if the number of periods is unlimited.

IV. Repeated gambling

Although a positive demand for gambling is predicted for Friedman-Savage utility functions, when $r \neq \eta$, expected utility-theory still has difficulty in explaining repeated gambling. For a Friedman-Savage utility function, gambles will be demanded in at most one period in CP^T both for finite or infinite T. For $T = 2$, the fact that the budget line has slope $-(1 + r)$ whilst the Cv–indifference curves have slope $-(1 + \delta)$ in the central square means that the optimal solution cannot lie in the central square and this rules out gambling in both periods. For general T, the result follows from the first-order conditions for the deterministic equivalent of CP^T:

$$(Cv)^0(c_t) = \lambda \left(\frac{1 + \delta}{1 + r} \right)^t \tag{6}$$

for $t = 1; 2; \ldots;$ where λ is a multiplier. If $r \neq \delta$, there can be at most one value of t for which the right hand side of (6) is equal to the slope of Cv in the interval $(\underline{c}; \bar{c})$. Hence, $\underline{c} < c_t < \bar{c}$ for at most one t which, by the standard construction, leads to a demand for gambles in at most one period. Even when $r = \delta$, although there can be optimal solutions involving gambling in every period, the optimal solution is not unique and there will typically (e.g. for a Friedman-Savage utility function) be alternative optimal solutions that entail gambling in at most one period.

In contrast to these theoretical results, periodic gambling behavior seems to be widespread. For example, participants in lottery games typically purchase a small number of tickets each week rather than making a large purchase in a single week. The inability of the model to account for repeated gambling is a serious problem that can only be avoided by modifying the objective function or the constraint (or both). The latter involves dropping the assumption of a perfect market for borrowing and saving and we now show that an interest rate wedge can account for a demand for gambling in every period.

A. A model with an imperfect market

We suppose that r_B and r_L $(< r_B)$ are the borrowing and lending rates, respectively. The consumer's optimization problem with market failure, which we shall write $CMFP^T$, can then be written:

$$\max E \sum_{t=1}^{T} \frac{v(C_t)}{(1+\delta)^t}$$

$$\text{subject to} \qquad W_{t+1} = \begin{cases} (1+r_B)(W_t + y^* + X_t - C_t) \\ \qquad\qquad \text{if } W_t + y^* + X_t - C_t \leq 0 \\ (1+r_L)(W_t + y^* + X_t - C_t) \\ \qquad\qquad \text{if } W_t + y^* + X_t - C_t > 0 \end{cases}$$

$$\text{for } t = 1, \ldots, T$$

$$\text{and} \qquad W_1 = 0;\; W_{T+1} \geq 0;$$

where W_t represents accumulated wealth (or, if negative, debt) at the beginning of period t.

We will apply the method of Section I by first noting that, since v is strictly increasing and $r_L < r_B$, the equation for W_{t+1} can be replaced with

$$W_{t+1} \le (1 + r_B)(W_t + y^* + X_t - C_t), \text{ and}$$

$$W_{t+1} \le (1 + r_L)(W_t + y^* + X_t - C_t),$$

without changing the set of optimal solutions of $CMFP^T$.

Since the objective function can be regarded as a concave function of $(W_1, X_1, C_1, \ldots, W_T, X_T, C_T, W_{T+1})$ and the inequality constraints are linear, we can apply Jensen's inequality[14] and argue as before that an optimal solution of $CMFP^T$ problem can be obtained by solving the deterministic equivalent:

$$\begin{array}{ll} \max & \displaystyle\sum_{t=1}^{T} \frac{Cv(c_t)}{(1+\delta)^t} \\ \text{subject to} & \left\{ \begin{array}{l} w_{t+1} \le (1 + r_B)(w_t + y^* - c_t) \\ w_{t+1} \le (1 + r_L)(w_t + y^* - c_t) \end{array} \right\} \quad \text{for } t = 1, \ldots, T \\ \text{and} & w_1 = 0;\ w_{T+1} \ge 0 \end{array}$$

followed by the standard construction to obtain a solution to $CMFP^T$.

To illustrate the application of this result, consider Figure 7 in which we have drawn a budget line B_1B_1 for $CMFP^2$ which has a kink at D where it crosses the 45° line and a slope of $-(1 + r_B)$ below and $-(1 + r_L)$ above D. Then D is the optimal solution of $CMFP^2$ provided the slope of the Cv-indifference curve lies between the slopes of the two sections of the budget line which requires $r_L \le \eta \le r_B$. We have established, for $T = 2$, the following theorem which is proved for general T in the appendix.

Theorem 3 *If $r_L \le \eta \le r_B$ and $v(y^*) < Cv(y^*)$, then $(y^*, \ldots, y^*)$ is an optimal solution of the deterministic equivalent of $CMFP^T$ and corresponds to gambling in every period.*

If $r_L > \eta$ [or $\eta > r_B$], the optimal solution of $CMFP^2$, is the same as in Section II with $r = r_L$ [or $r = r_B$]. In this case (and for general T) there will be at most one period of gambling.

We note that the solutions referred to in Theorem 3 predict gambling or borrowing and saving but not both in each period. A more sophisticated model is required to explain both borrowing or saving and gambling in every, or at least more than one, period.

V. Conclusion

It has not been our intention in this study to deny the explanatory power of non-expected utility theories of decision-making or that gambling may offer direct consumption value. Rather, we have explored the extent to which expected utility theory with non-concave utility functions can account for gambling in an inter-temporal setting and have demonstrated that the theory can explain a desire for gambling even when capital markets are perfect and utility functions are separable. Our arguments have not exploited the fact that intra-period preferences are the same for all periods and we expect broadly similar conclusions to hold for more general preferences over consumption streams provided we maintain inter-period separability.

However, when the rates of interest and time preference differ, it is optimal to gamble in at most one period. Even when these rates are equal, consumers will prefer to gamble at most once, weakly if fair gambles are available and strictly if only unfair gambles can be bought. One way to account for repeated gambling using expected utility theory is to invoke market failure as in the preceding section[15]. An alternative approach is to permit inter-period interactions. This could change the results substantially. For example, if preferences in one period are positively related to previous consumption, as in Becker and Murphy's model of rational addiction [2], repeated gambling is possible. Nevertheless, it would seem unlikely that habituation is the sole explanation for repeated gambling. An empirical study of lotto participation by Farrell et al.[7] finds evidence of habit-formation, but its extent is small and appears inadequate as a complete model of repeated purchase of lotto tickets.

Appendix

Proof of Theorem 3

We will show that the proposed solution satisfies the Kuhn-Tucker conditions which, given the concave objective function and linear constraints are necessary and sufficient for optimality. We are thus assuming differentiability of v (and therefore of Cv).

We can eliminate the constraint $w_1 = 0$ in the deterministic equivalent of $CMFP^T$ by substitution. Write $\phi_t \geq 0$ [$\psi_t \geq 0$] for the Kuhn-Tucker multiplier associated with the upper [lower] constraint having w_{t+1} on its left hand side in the resulting problem and $\varphi \geq 0$ for the multiplier associated with $w_{T+1} \geq 0$. The optimality conditions at the proposed solution can be written

$$
\begin{aligned}
\frac{[Cv]'(y^*)}{(1+\eta)^t} &= (1+r_B)\phi_t + (1+r_L)\psi_t \text{ for } t = 1,\ldots,T; \\
\phi_{t-1} + \psi_{t-1} &= (1+r_B)\phi_t + (1+r_L)\psi_t \text{ for } t = 2,\ldots,T; \\
\phi_T + \psi_T &= \varphi.
\end{aligned}
$$

We also have the requirement that any multiplier associated with a non-binding constraint must be zero, but, at the proposed solution $w_2 = \cdots = w_{T+1} = 0$, so all constraints bind. It is readily verified that the optimality conditions are satisfied if we set

$$\psi_t = \frac{(\delta - r_L)[Cv]'(y^*)}{(r_B - r_L)(1+\delta)^{t+1}} \geq 0;$$

$$\chi_t = \frac{(r_B - \delta)[Cv]'(y^*)}{(r_B - r_L)(1+\delta)^{t+1}} \geq 0;$$

for $t = 1, \ldots, T$ and $\varphi = \psi_T + \chi_T$. ■

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[1]However, it can be shown that if enough consumers with identical preferences demand an unfair gamble they will be able to increase their individual expected utilities by betting with each other.

[2]We also assume that either v is defined for all c or there is a minimum acceptable consumption level (which we fix arbitrarily at 0) at which v approaches $-\infty$. This assumption, made for expositional convenience, avoids corner solutions which complicate but do not substantially modify our conclusions.

[3]For the standard Friedman-Savage function, the set of c for which $v(c) < Cv(c)$ is a connected set. The results in the paper do not depend on this property; the argument extends to the general case.

[4]This assumption avoids 'thick' indifference curves in the subsequent analysis.

[5]It is natural also to require independence of C_t and $X_{t+1}, \ldots, X_T$ but doing so has no effect on our conclusions.

[6]Formally, this is an application of Jensen's inequality.

[7]Although the curves drawn have a section bowed away from the origin, this is not necessarily the case for all indifference curves. Mathematica Notebooks containing complete indifference maps and other diagrams (including the locus L introduced below) based on specific functional forms are available from the authors.

[8]Although we have drawn L as a bounded, symmetric curve (plus the 45° line) only the symmetry is a universal property. It is quite possible for L to vary widely in shape and even be unbounded.

[9]However, not all points on L are no-gambling solutions. The 45° line is always part of the locus but, where the indifference curve is concave to the origin as it crosses this line, the second order conditions are not satisfied. Even points where the second order conditions are satisfied may be only local maxima.

[10]We establish this and the following theorem using graphical methods assuming a Friedman-Savage utility function. The result can be generalized (with an extended set of exceptional values) to functions with several non-concave segments and to more than two periods, using a more formal argument, which we omit. Proofs are available from the authors on request.

[11]Bailey et al. implicitly assumed (5) in their argument.

[12]If $r = r_D$ or r_U; the consumer is indifferent between the alternatives.

[13] I.e. the optimal no-gambling objective function divided by $\sum_{t=1}^{T}(1+\eta)^{-t}$.

[14] Convexity of the feasible region is essential. If this were false, we could use gambles to `fill in' indentations in the feasible set thereby potentially increasing the value of the objective function.

[15] See also the suggestion by Dowell and McLaren [5] that in their model an individual unable to borrow against future earnings may repeatedly accumulate small sums with which to wager.