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TERM STRUCTURE OF INFLATION FORECAST UNCERTAINTIES AND SKEW NORMAL DISTRIBUTIONS

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ABSTRACT

Empirical evaluation of macroeconomic uncertainties and their use for probabilistic forecasting are investigated. A new weighted skew normal distribution which parameters are interpretable in relation to monetary policy outcomes and actions is proposed. This distribution is fitted to recursively obtained forecast errors of monthly and annual inflation for 38 countries. It is found that this distribution fits inflation forecasts errors better than the two-piece normal distribution, which is often used for inflation forecasting. The new type of ‘fan charts’ net of the epistemic (potentially predictable) element is proposed and applied for UK and Poland.

1. INTRODUCTION

Although the concept of uncertainty has been widely used in macroeconomics, there is not much of consensus about its measurement. On the one hand, there is currently substantial development in measures of uncertainty understood in the Knightian sense, that is as unobservable *ex-post* phenomenon (see e.g. Bloom, 2009, Baker, Bloom and Davis, 2013**,** Jurado, Ludvigson and Ng, 2013). On the other hand, the concept of uncertainty has been also used in non-Knightian sense in relation to particular macroeconomic indicators of macroeconomics, like inflation and output growth, where the uncertainty can be checked *ex-post*, by evaluation of point forecast errors (Clements, forthcoming). In Knight's (1921) terminology this type of uncertainty should be called *risk* rather than *uncertainty*. However, following the tradition we call these Knightian risks inflation uncertainties, output uncertainties, *etc*..

In this paper we develop from Clements (forthcoming) classification of indicators' forecast uncertainties (the Knightian risks). He distinguishes between *ex-post* forecast uncertainty, derived from differences between point forecasts of individual forecasters and corresponding realisations, and *ex-ante* uncertainty, understood as a disagreement measure between the forecasters or dispersion of consensus forecasts (see also Patton and Timmermann, 2010, 2011). Clements' implicit assumptions are that (*i*) data on distributions of forecast uncertainty come from the distribution of forecasts surveys or consensus forecasts (*ii*) distributions of *ex-post* and *ex-ante* uncertainties are independent and (*iii*) forecast uncertainties are independent from forecast-induced policy actions, e.g. monetary policy decisions based on the grounds of forecasts' knowledge. Assumption (*i*) is somewhat restrictive, if it comes to the analysis of uncertainties for countries where data on systematic forecasts surveys are either not available, or not reliable. Assumption (*ii*) might be not be fully realistic and the validity of assumption (*iii*) has already been questioned, in context of inflation, by Clements (2004) and Granger and Pesaran (2000). Regarding (*iii*), if forecast of inflation is taken seriously by the monetary authorities and happens to be unfavourable (that is, inflation is to be too high or too low, according to the inflation targeters), they would impose an anti-inflationary or pro-inflationary action, as the result of which inflation would miss the level originally forecasted and the forecast would prove to be inaccurate.

Limiting our interest to inflation uncertainties, we aim at relaxing assumptions (*ii*) and (*iii*) and substituting assumption (*i*) by another one, that the distribution of non-Knightian uncertainty can be derived from series of recursive observations of *ex-post* uncertainties. This approach does not require an access to forecast survey data, but rather a reasonably long time series data of the forecasted indicator (and possible related series) which enables recursive estimation and forecasting. Such approach is often made by the practitioners preparing so-called fan charts in most central banks, with a notable exception of UK (see e.g. Kowalczyk, 2013; for UK fan charts see Clements, 2004, Elder *et al*., 2005). In Patton and Timmerman (2010) terminology, fan charts are visual representations of forecast term structure, that is differences in distributions of forecast uncertainty for different horizons. We base our analysis on a new simple statistical distribution, called weighted skew normal herein, introduced in Section 2. It explicitly identifies the elements of ontological uncertainty, that is related to pure randomness (unpredictability) of future events and epistemic uncertainty, expressing incomplete or potentially biased knowledge of the forecasters. The parameters of this distribution can be interpreted as (1) outcomes of pro- and anti- inflationary economic policy actions undertaken on the basis of experts' forecast signals, assuming an infinite number of such forecasters who are making their predictions on the basis of the common baseline point forecast, (2) accuracy of experts' forecasts aimed at improving on the baseline forecast, (3) thresholds that define the presumed 'safe' region for inflation, that is where monetary intervention is not required, and (4) dispersion of the unobserved 'net' uncertainties, free from monetary policy effects.

We have checked to what extent a measure of dispersion of econometric point forecast~~s~~ errors is related to the Knightian measures of aggregate macroeconomic uncertainty. There is a reasonably high and significant rank correlation of a simple dispersion measure of inflation forecast errors with the recently proposed Knightian aggregate uncertainty measures, namely the Economic Policy Uncertainty index (see Baker, Bloom and Davis 2013) and the measure developed by Jurado, Ludvigson and Ng (2013); see Section 3. In Section 4 we apply the weighted skew normal distribution to approximation of forecast errors derived from simple time series models of monthly and annual inflation for 38 countries. We have found that the weighted skew normal distribution fits the forecast errors better than two-piece normal distribution which is usually applied in central banking inflation forecasting (see e.g. Wallis, 2004). Moreover, interpretation of the estimated parameters of the weighted skew normal distribution is often in line with general monetary policy characteristics. In Section 5 we propose new types of fan charts representing forecast uncertainty term structure based on decompositions of uncertainties derived from the weighted skew normal distribution. These term structures show, for UK and Poland, some interpretational and statistical (predictive) advantages of the proposed approach. Section 6 contains conclusions and suggestions for further development.

2. MODEL OF INFLATION UNCERTAINITES

We consider that π_{t+h}, inflation in time $t+h$, is a random variable that can be split into two parts: predictable and nonpredictable from the past. However, the component nonpredictable from the past can still be forecastable by methods other than those of time series analysis ('fine tuning', or experts' corrections based on some sort of inside information). We follow central banks' tradition of two-stage probabilistic forecasting, which consists of first conducting past-related econometric forecast and then assessing the uncertainty relatively to this forecast. (see e.g. Pinheiro and Esteves, 2012). Consequently, we decompose π_{t+h} as:

$$\pi_{t+h} = \hat{\pi}_{t+h} + U_{t+h} \quad , \tag{1}$$

where $\hat{\pi}_{t+h}$ is the baseline point forecast, usually obtained from a time series econometric model and U_{t+h} is a random variable representing *ex-post* uncertainties revealed in time t regarding inflation in time $t+h$, so that h is the forecast horizon. To avoid confusion with other concepts (see e.g. Fountas *et al.* 2006) we refer to U_{t+h} as U-uncertainties. It is further shown that realisations of U-uncertainties can be recovered from data (see Section 3 below). Further on we regard each U_{t+h} separately for each time period and forecast horizon and, for simplicity of notation, we drop the subscripts, so that $U \overset{def}{=} U_{t+h}$.

Let us consider the following specification of U:

$$U = X + \alpha \cdot Y \cdot I_{Y>\bar{m}} + \beta \cdot Y \cdot I_{Y<\bar{k}} \quad , \tag{2}$$

where $I_{Y>\bar{m}} = \begin{cases} 1 & \text{if } Y > \bar{m} \\ 0 & \text{otherwise} \end{cases}$, $I_{Y<\bar{k}} = \begin{cases} 1 & \text{if } Y < \bar{k} \\ 0 & \text{otherwise} \end{cases}$,

and

$$(X,Y) \sim N\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\sigma^2 & \rho\sigma^2\\ \rho\sigma^2 & \sigma^2\end{bmatrix}\right) \quad ,$$

$\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\bar{m} \in \mathbb{R}$, $\bar{k} \in \mathbb{R}$ and $-1 < \rho < 1$.

Below we provide interpretation to all six parameters in (2). Random variable *X* contains two elements usually regarded as characteristics (or types) of uncertainties: *ontological uncertainties*, related to the purely random (unpredictable in mean) nature of future inflation, and *epistemic uncertainties*, related to fragmentary and incomplete knowledge of the forecaster (for general discussion of these concepts see e.g. Walker *et al*., 2003, and for application in inflation forecasting context see Kowalczyk, 2013).[1] The epistemic elements in *X* can in fact be predictable, e.g. by experts who based their judgements on the analysis of non-quantitative data, expected effects of current political decisions, etc. That is why we refer to *X* as to *quasi-uncertainties*. For interpretation of *Y*, we assume that an infinite number of such expert forecasters deliver individual forecasts based on individually obtained information. These forecasters have the common knowledge of the baseline forecast, so that their forecasts are formulated in relation to $\hat{\pi}_{t+h}$. As these individual forecasts differ from each other and each forecaster is supposed to have own sources of information, it also create uncertainties, represented by *Y*, called in the literature *uncertainties by disagreement.* The intuition is simple here: if the forecasters don't agree, they are uncertain. In the context of forecasting inflation the concept of uncertainty by disagreement has been introduced by Bomberger (1996) and developed further, in particular, by Diebold, Tay and Wallis (1999), Giordani and Söderlind (2003), Lahiri and Liu (2006), Lahiri and Sheng (2010), Patton and Timmermann (2010) and Siklos (2013).

To what extent these experts' forecasts represented by *Y* are 'educated', or accurate, is expressed by the correlation coefficient ρ between *X* and *Y*. If either *X* is totally unpredictable (that is, if quasi-uncertainty becomes fully ontological) or if the experts are ignorant, then $\rho = 0$. The higher is the value of ρ, the more epistemic becomes *X* and/or the experts become more competent. For this reason we refer to *Y* as to *imperfect knowledge* (the knowledge becomes perfect if $\rho = 1$). It is reasonable to assume that the variances of *X* and *Y* are identical. They are denoted as σ^2. This assumption is grounded within the conjecture that, in the absence of epistemic element in quasi-uncertainties, disagreement between the experts has the same variability as ontological uncertainty. We also assume that the experts' forecasts cannot be negatively related to quasi-uncertainties *X*, that is $0 \le \rho < 1$.

Further four parameters of the model (2), that is $\bar{m}$, $\bar{k}$, α, and β, can be interpreted in the light of actions and outcomes of some sort of monetary policy. There is no need to assume anything specific regarding this policy except for the fact that it is based on experts' forecasts *Y* regarding quasi-uncertainty *X* and that the policy undertaken in time *t* might affect inflation in time *t*+*h*. The model requires rather strong assumption that the baseline forecast $\hat{\pi}_{t+h}$ does not stimulate monetary policy outcomes and that the monetary authorities react to information passed to them through *Y* only. In another words, the baseline forecast $\hat{\pi}_{t+h}$ is assumed to be a policy-neutral part of inflation.

[1] Walker *et al*. (2003) and Kowalczyk (2013) talk of variability uncertainty rather than of ontological uncertainty. Walker's *et al*. classification has been criticised for incompleteness and tautology (Norton, Brown and Mysiak, 2006). Other definitions and classifications, also often criticised, are frequently used in different sciences. It is also important to acknowledge the relative and time-varying nature of ontological uncertainty in economics (see e.g. Lane and Maxfield, 2005).

The parameters $\bar{m}$ and $\bar{k}$ denote respectively 'high' and 'low' thresholds for imperfect knowledge Y which, if breached, signal to the monetary authorities the necessity of undertaking an anti-inflationary decision (if $\bar{m}$ is breached from below) or pro-inflationary (if $\bar{k}$ is breached from above). While $\bar{m}$ and $\bar{k}$ decide on signals for monetary policy actions, α and β describe actual effectiveness (outcomes) of these actions. The parameter α tells to what extent anti-inflationary decisions undertaken on the basis of inflation signals are transmitted into the change in inflationary uncertainties and β tells the same for output-stimulating pro-inflationary decisions. Rational behaviour of the policy makers and forecasters implies that $\alpha \le 0$, $\beta \le 0$, $\bar{m} \ge 0$, $\bar{k} \le 0$ and $0 < \rho < 1$.

Random variable U defines a family of distributions which we name the *weighted skew-normal* and abbreviated as $\mathrm{WSN}_\sigma\left(\alpha, \beta, \bar{m}, \bar{k}, \rho\right)$. For operational simplicity it is convenient to normalize WSN in such way that σ=1 and define U^* as:

$$U^* = \frac{U}{\sigma} \sim \mathrm{WSN}_1\left(\alpha, \beta, m, k, \rho\right) \quad , \tag{3}$$

where $m = \bar{m}/\sigma$ and $k = \bar{k}/\sigma$. The probability density function (*pdf*) of U^* is given by:

$$\begin{aligned} f_{WSN_1}(t) = {} & \frac{1}{\sqrt{A_\alpha}}\varphi\left(\frac{t}{\sqrt{A_\alpha}}\right)\Phi\left(\frac{B_\alpha t - mA_\alpha}{\sqrt{A_\alpha(1-\rho^2)}}\right) + \frac{1}{\sqrt{A_\beta}}\varphi\left(\frac{t}{\sqrt{A_\beta}}\right)\Phi\left(\frac{-B_\beta t + kA_\beta}{\sqrt{A_\beta(1-\rho^2)}}\right) \\ & + \varphi(t)\cdot\left[\Phi\left(\frac{m-\rho t}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{k-\rho t}{\sqrt{1-\rho^2}}\right)\right] \quad , \end{aligned} \tag{4}$$

where φ and Φ denote respectively the density and cumulative distribution functions of the standard normal distribution, and:

$$A_\tau = A(\tau) = 1 + 2\tau\rho + \tau^2, \qquad B_\tau = B(\tau) = \tau + \rho \quad .$$

If, in (2), $\alpha = -2\rho$ and $\beta = m = 0$, the distribution of U coincides with the Azzalini (1985, 1986) skew-normal $\mathrm{SN}(\xi)$ distribution with *pdf* $f_{\mathrm{SN}}(t;\xi) = 2\varphi(t)\Phi(\xi t)$, where $\xi = \frac{-\rho}{\sqrt{1-\rho^2}}$.

It follows from (4) that the *pdf* of the weighted skew-normal variable $\mathrm{WSN}_1(\alpha, \beta, m, k, \rho)$ can be interpreted as a weighted sum of *pdf*'s for two Azzalini-type skew normal densities with different ξ's and a *pdf* of conditional distribution $\left.\frac{X}{\sigma}\right| k \le \frac{Y}{\sigma} \le m$; hence the name for the distribution. Basic characteristics and properties of WSN, generalisations and moment generating function are given in Appendix A, Part I.

As the main moments, especially variance, of U can be interpreted as aggregate characteristics of uncertainties, we can evaluate their dependence on the parameters of the monetary policy: decision thresholds (k and m), strength of forecast-induced anti-inflationary (α) and output-stimulative (β) monetary outcomes, and the degree of predictability of X from imperfect knowledge, measured by ρ.

It immediately follows from (2) that the WSN distribution is symmetric only if $\alpha = \beta = 0$ or, if $\bar{k} = -\bar{m}$ and $\alpha = \beta$; otherwise it is asymmetric. This is in line with a general consensus that distributions of inflation uncertainties might be skewed (for recent advances see

Demetrescu and Wang, 2012). The type of skew distribution usually applied for central banks for constructing fan charts is the two-piece skew normal (see e.g. Wallis, 2004). Explanation in the literature or interpretation of skewness in this distribution is usually not detailed. It is described by Wallis (2004) that: ‘the degree of skewness shows their collective assessment of the balance of risks on the upside and downside of the forecast’. We argue that skewness might result from widely defined asymmetries in monetary policy actions and outcomes. Our interpretation of WSN distribution implies that, under the assumption of normal distributions of quasi-uncertainties X and imperfect knowledge Y, inflation uncertainties become skewed if either the strength of anti- and pro-inflationary policy differ from each other and/or thresholds defining the expected inflationary ‘danger zones’ are not symmetric.

Ontological uncertainty, that is the non-predictable component in X, can be extracted as:

$$Z = X - E(X \mid Y) = X - \rho Y \quad ,$$

so that:

$$(Z, Y) \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} (1-\rho^2)\sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \right) \quad .$$

In another words, Z is the part of the quasi-uncertainty X which is net of epistemic element, that is of imperfect knowledge. Similarly we can retrieve the epistemic part of X from U as:

$$V = U - E(X \mid Y) = U - \rho Y \quad . \tag{5}$$

Although V does not contain the epistemic element of X, it is contaminated by it through possible monetary policy outcomes, as:

$$V = U - E(X \mid Y) = U - \rho Y = Z + \alpha \cdot Y \cdot I_{Y > \bar{m}} + \beta \cdot Y \cdot I_{Y < \bar{k}} \quad .$$

Further in the text we refer to V as *V-uncertainties* or, not very precisely (because of this contamination), as *net uncertainties*. Distribution of V is also related to WSN, as:

$$\frac{1}{\sigma\sqrt{1-\rho^2}} V \sim \mathrm{WSN}_1\left(\frac{\alpha}{\sqrt{1-\rho^2}}, \frac{\beta}{\sqrt{1-\rho^2}}, \frac{\bar{m}}{\sigma}, \frac{\bar{k}}{\sigma}, 0 \right) \quad .$$

If parameters in (2) can be estimated or calibrated for each forecast horizon h, both U-uncertainties and V-uncertainties can be used for constructing probabilistic forecasts in the form of fan charts, that is by retrieving and then plotting equiprobable quantiles. Each of these fan charts would have different interpretation. The fan chart based on U-uncertainties would incorporate the possible forecast-induced monetary policy outcomes, and also epistemic uncertainty, and the fan chart based on V-uncertainties would be free of them. The former would be of interest for ‘end users’ who do not have any influence on monetary policy, or the monetary policy body after the decision was made, and the latter to the monetary policy body before the decision is made, as it excluded the effects of its own decision and possible effects of imperfect forecasting. In another words, V-uncertainties can be interpreted as such where information from the forecasters do not have any effect on uncertainty. Comparing both could provide an idea of the influence that the epistemic element has on the distribution of inflation forecasts.

A straightforward way of doing this is by comparing variances of V- and U-uncertainties through their variance ratio, denoted as VRUV:

$$\text{VRUV} = \frac{\text{var}(V)}{\text{var}(U)} = 1 + 2\rho \frac{-(\alpha D_m + \beta D_k) - \rho/2}{Var(U^*)} \quad , \tag{6}$$

where $D_a = D(a) = \int_{|a|}^{+\infty} t^2 \varphi(t) dt = 1 - \Phi(|a|) + |a| \varphi(a)$.

VRUV is equal to unity, that is $\text{var}(U) = \text{var}(V)$, if $\rho = 0$ or $\rho = -2(\alpha D_m + \beta D_k)$. It should be noted that VRUV does not depend on σ, but rather on the ratios $m = \bar{m}/\sigma$ and $k = \bar{k}/\sigma$. Deviation of VRUV from unity represents the effect of the epistemic element on the uncertainties (through ρ) and the effect of monetary policy (through $\alpha D_m + \beta D_k$), where α and β reflect the strength of individual monetary policy actions, and D_m and D_k reflect the frequency of such actions. Let us denote the compound monetary policy effect as $S = |\alpha| D_m + |\beta| D_k$. Appendix A, Part II, gives a general representation of VRUV as a function of S and forecast bias $E(U)$. In the case of unbiased forecast, that is when $E(U) = 0$, it is convenient to represent VRUV as a function of *S*:

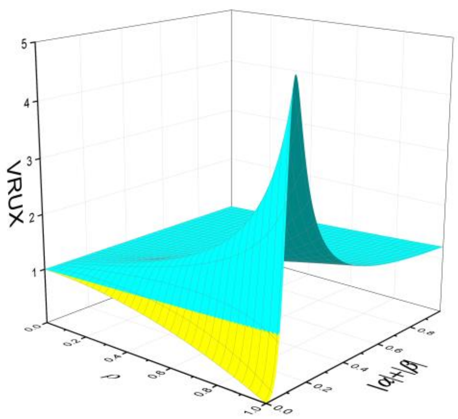
$$\text{VRUV} = \frac{\text{var}(V)}{\text{var}(U)} = 1 + \rho \frac{2S - \rho}{1 - 2\rho S + W_{m,k} \cdot S^2} \quad , \tag{7}$$

where $W_{m,k} = \left[D_m \varphi^2(k) + D_k \varphi^2(m)\right] / \left[D_m \varphi(k) + D_k \varphi(m)\right]^2$.

In is clear from (7) that $\text{VRUV}|_{S=0} = 1 - \rho^2$, $\text{VRUV}|_{\rho=0} \equiv 1$ and $\text{VRUV} \xrightarrow[S \to +\infty]{} 1$. It is also shows that, as a function of S and ρ, VRUV is (conditionally) unimodal for fixed $\rho > 0$. Let us denote its corresponding conditional maximum as $\text{VRUV}_{\max}(\rho)$ and the value of the argument *S* for which it is achieved as $S_{\max}$ (see Appendix A, Part II, for the derivation of this maximum).

Figure 1 plots VRUV for the fully symmetric case, that is for $\alpha = \beta$, $\sigma^2 = 1$, $\bar{m}/\sigma = -\bar{k}/\sigma = 1$, and for different values of ρ, against $|\alpha| + |\beta| = S / D_1$, representing the strength of forecast-induced monetary policy. For the very low strength of the forecast-

Figure 1: VRUV for the case where $\sigma^2 = 1$, $\alpha = \beta$, $\bar{m} = -\bar{k} = 1$ and for different values of ρ . Values of VRUV smaller than one are in a lighter shade



induced monetary policy VRUV is smaller than one and is decreasing with the increase in ρ, that is where the degree of imperfect knowledge is increasing. In this case variance of *U* increases in relation to variance of *V*. This means that, in such case, an increase in epistemic elements is raising inflation uncertainty. If, on the other hand, the epistemic element is utilised in the monetary policy, so that its forecast-induced strength increases up to the optimal point where VRUV reaches its maximum, inflation uncertainty represented by variance of *U* decreases in relation to variance of *V*. In the case shown by Figure 1, $S_{\max}(1)/D_1 = 1$, ($\alpha = \beta = -1/\left[2D_1\right]$) and $\mathrm{VRUV}_{\max}(1) = \left(1-2D_1\right)^{-1} \approx 5.03$.

Results given above suggest a practical application for the uncertainties defined as such which maximizes VRUV. As in practice the strength and future effects of the current monetary policy are unknown, in some cases it might be plausible to safeguard against the worst possible case, where the uncertainty also incudes that of the strength and frequency of monetary policy actions on the evaluation of *V*-uncertainties. Such maximum *V*-uncertainties are called herein the *M*-uncertainties. If $VRUV_{\max}(\rho)$ is known (as shown in Appendix A, Part II, it can be easily computed analytically or, for more complicated cases, numerically), *M*-uncertainties can be recovered from *U*-uncertainties as $M(\rho) = U\sqrt{VRUV_{\max}(\rho)}$, without the need of any additional estimation. An empirical application of $\mathrm{VRUV}_{\max}$ for constructing fan charts for UK and Poland is given in Section 5.

3. MEASUREMENT OF INFLATION UNCERTAINTIES

Assuming that $\hat{\pi}_{t+h}$ in (1) can be estimated, observations on *U*-uncertainties can simply be identified as baseline *ex-post* point forecast errors, that is as the differences between inflation actually observed in time *t*+*h* and the corresponding forecast, that is $\hat{\pi}_{t+h}$. Distributions of such *uncertainties by error* have been widely applied, with some variations, for constructing probabilistic forecasts of inflation in most of central banks, among others in Chile, Czech Republic, Norway, Poland, Slovakia and Sweden, with notable exceptions of Bank of England, Israel and US where parameters of the distributions used to construct fan-charts are derived differently. In these central banks where the uncertainties by error are used, their empirical distributions are usually smoothed, often subjectively adjusted and fitted with a theoretical distribution, for which the probabilities of inflation being within particular intervals (bands) are computed.

We have computed such baseline point forecast errors for 38 countries, that is for 32 OECD countries and for Brazil, China, India, Indonesia, South Africa and the Russian Federation. We have alternatively used non-deseasonalised monthly data on monthly inflation (that is on changes in CPI in relation to previous month, called further in this paper *monthly inflation*), and monthly data on annual inflation (changes in CPI in relation to the corresponding month of the previous year, called *annual inflation*). The data series are of various lengths and end at January or February 2013. The longest series starting in January 1949 is for Canada (770 observations) and the shortest are for Estonia (182 observations) and China (242 observations). The raw CPI data can be downloaded from: http://stats.oecd.org/. It is conjectured that if these countries conducted some sort of effective monetary policy, it might in turn affect the distribution of uncertainties. Some countries (e.g. the members of the European Monetary Union, EMU) do not have autonomous monetary policy since the creation of the Euro. Nevertheless, for EMU countries, the policy of European Central Bank affects inflationary uncertainties in a similar way an autonomous monetary policy might. In our approach it is not relevant how the monetary policy decisions are made; their effect on uncertainties is what matters.

The forecast errors have been computed separately for each series in the following way.

1. Orders of seasonal and non-seasonal integration have been identified using the Taylor (2003) test which takes into account the possibility of the presence of unit roots at frequencies other than tested.
2. Initial (for first recursion) period for estimation has been defined as a maximum of the first 80 observations and the 20% of the series length.
3. Point forecasts (that is, estimates of $\hat{\pi}_{t+h}$) have been made recursively using the estimated seasonal autoregressive moving average model (*SARMA*) for the first recursion period and then by updating it by one observation at a time and re-estimating the model. Orders of the lag polynomials are obtained in each recursion by the Gómez and Maravall (1998) procedure which is based on the automatic lag selection that minimises the Bayesian Information Criteria (BIC). Forecasts have been made for up to 12 periods ahead. These forecasts have not been adjusted or manipulated. As the result, for each country we have obtained reasonably long series of forecasts for different forecasts horizons, and then forecast errors, with the maximum number of sample observations for Canada being 613 and the smallest for Estonia at 99.

Apart from *SARMA*, we have also tried different univariate forecasting models, most notably the bilinear autoregressive-moving average model. These alternative models produced different uncertainties, but the final outcomes, in terms of interpretation, have been similar to those presented below and obtained from *SARMA*.

At the next stage we have checked whether such simple *ex-post* forecast errors can indeed be used for constructing non-Knightian components of an aggregate measure of Knightian uncertainty. If this is to be the case, a measure of uncertainty based on dispersion of the *ex-post* forecast errors should correlate with available measures of macroeconomic uncertainty computed independently and with the use of different types of data. As such measures of macroeconomic uncertainty we have selected the *economic policy uncertainty index* (*EPU*), developed by Baker, Bloom and Davis (2013) on the basis of Bloom (2009), available at http://www.policyuncertainty.com/ and another recently proposed measure called *aggregate macroeconomic uncertainty* (*AMU*) developed by Jurado, Ludvigson and Ng (2013), available at http://www.nyu.edu/user/ludvigsons/data.htm. The *EPU* is an aggregate index, based on (*a*) the frequency of the use of word ‘uncertainty’ in leading newspapers, (*b*) tax code provisions and (*c*) disagreement between the forecasters (that is, uncertainty by disagreement, as discussed in Section 2 above). Monthly data on *EPU* are available for Canada, China, France, Germany, India, Italy, Spain, UK and US and are constantly updated. The *AMU* is available only for US, also in monthly data, for the period July 1960 – December 2011. It is a complex aggregate obtained from the residuals’ covariance matrix of a factor augmented forecasting model using 279 individual series (for details see also http://www.econ.nyu.edu/user/ludvigsons/jln_supp.pdf).

As the purpose of this paper is to examine the distribution which fits the inflation forecast errors rather than to develop and alternative measure of uncertainty, we have limited ourselves to introducing a rather rudimentary measure of inflation uncertainty. It is called herein *econometric forecast uncertainty* (abbreviated as *EFU*) and is defined as a simple 24-months moving standard deviation of the smoothed SARMA forecast errors. Table 1 contains Spearman’s rank correlation coefficients of *EFU* with the logarithms of *EPU* and with *AMU* (for US only) for the forecast horizons 1,3,6,9 and 12.[2]

[2] The logarithms are used in order to milder the effect of so called informational cascades; see e.g. Baltag *et al*. (2013). Results which used levels rather than logarithms are similar.

All correlation coefficients in Table 1 for all countries except for Canada and China are significant at 1% level (using bootstrapped standard errors). For Canada and China they are mostly insignificant or at the margins of significance. For Canada, the lack of correlation is likely caused by unpredictable (by the univariate SARMA model) fall in inflation in the first half of 1990's, where the decline in Canadian inflation was preceded by an earlier inflation drop in US and therefore foreseen by the media. As media information constitute relevant component of *EPU*, this was not reflected by this index. For China, *EFU* index reflects two autoregressively unforecastable slowdowns in Chinese inflation (in 2007 due to anti-inflationary measures undertaken in order to cool off the economy and in 2009 due to a burst in the real estate bubble) while, on the other hand, *EPU* index mirrors the widespread discussion in professional press in 2008 about the possible breakdown of the Chinese economy and the aftermath of natural disasters in the same year.

For US, correlation of *EFU* with *AMU* is higher than that with *EPU*. There is no surprise here, as the econometric forecasts errors of inflation, although obtained by different techniques than SARMA modelling, constitute a substantial part of *AMU*. It is interesting to note that for US the correlation of *EFU* with the log of *EPU* seems to raise with the increase in forecast horizon, the correlation with *AMU* actually decreases.

Table 1: Spearman's rank correlation coefficients of *EFU* with logs of *EPU* and with *AMU*. *EFU* is computed as moving standard deviation of forecasts errors. *EPU* is economic policy uncertainty index reported at www.PolicyUncertainty.com and described in Baker, Bloom and Davis (2013) *AMU* is the aggregate macroeconomic uncertainty as in Jurado, Ludvigson and Ng (2013). Insignificance is marked by underlining; the coefficients with *p*-values>0.10 are doubly underlined, with *p*-values between 0.05 and 0.10 singly underlined by a solid line and with *p*-values between 0.01 and 0.05 underlined singly by a dotted line.

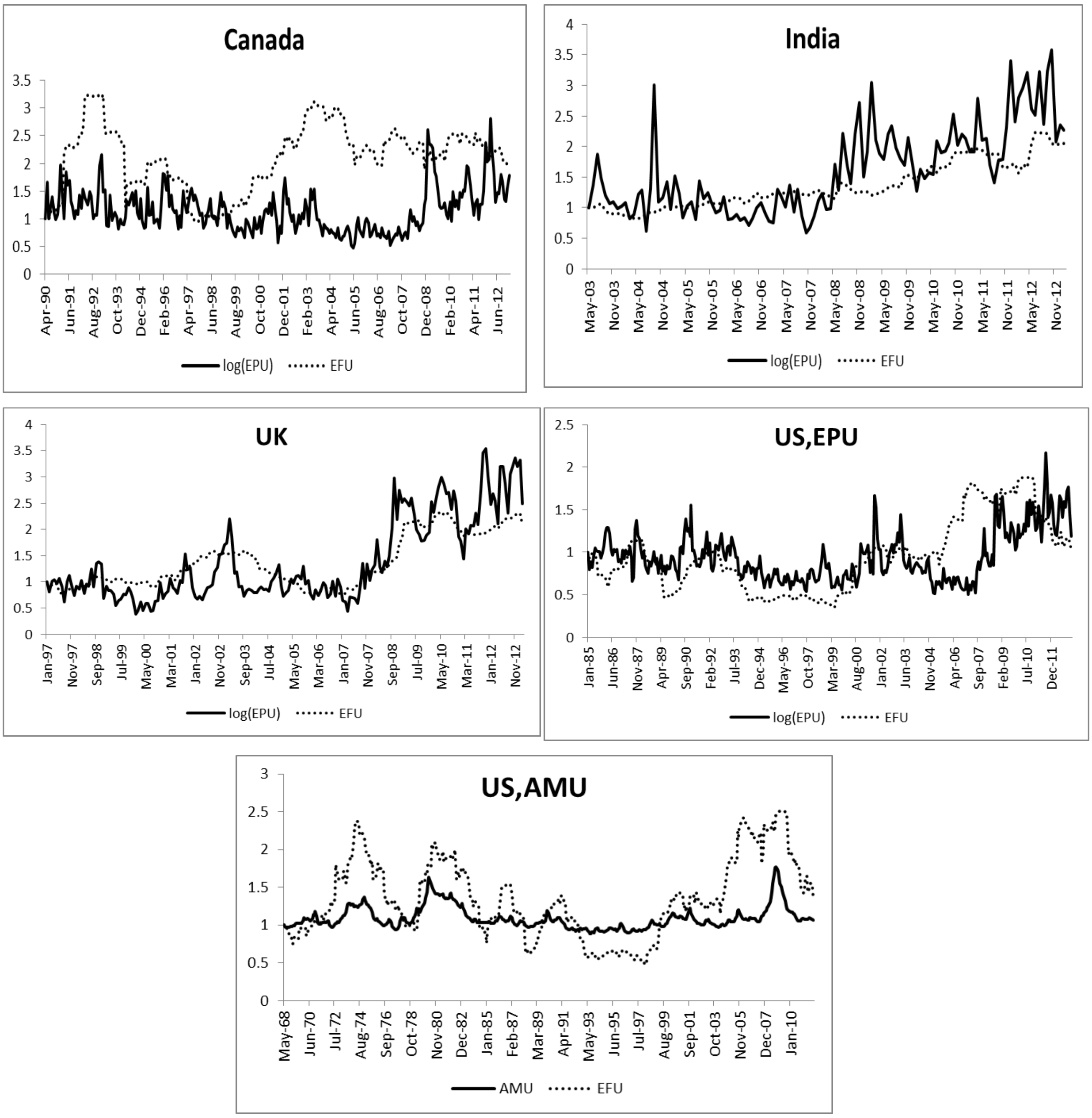
	with logs of *EPU*									with *AMU*
f.hor	Canada	China	France	Germany	India	Italy	Spain	UK	US	US
1	-0.07	0.06	0.28	0.20	0.69	0.41	0.29	0.73	0.42	0.77
3	0.05	0.15	0.37	0.29	0.73	0.54	0.34	0.81	0.50	0.69
6	0.18	0.21	0.42	0.32	0.74	0.51	0.36	0.78	0.60	0.63
9	0.15	0.25	0.41	0.36	0.69	0.49	0.46	0.74	0.61	0.53
12	0.13	0.26	0.46	0.30	0.68	0.47	0.49	0.77	0.57	0.54

Figure 2 presents time series of *EFU*'s and logarithms of *EPU*'s for Canada, India, UK and US and also the series of *AMU* for US compared with *EFU*. All series are normalised by the first observation. The similarities in dynamic development of both measures are striking. It is not the purpose of this paper to decide which one is better. The relevant conclusion is such that econometric forecast errors can be regarded as an adequate measure of uncertainty and, therefore, used for estimation of parameters of the distribution of *U* as defined by (2).

4. ESTIMATION OF DISTRIBUTION OF *EX-POST* UNCERTAINTIES

With the use of the forecast errors computed in the way described in Section 3, we have estimated the parameters of the WSN distribution (4). In order to reduce the computational burden we have assumed that the decision thresholds are fixed (relatively to σ) and identical for all countries so that $m = \bar{m}/\sigma = -k = -\bar{k}/\sigma = 1$ (that is, the thresholds are equal to one standard deviation of the uncertainties) and the correlation coefficient ρ = 0.75, that is that

Figure 2: Monthly time series of *EFU* and log of *EPU* for selected countries. For definitions and sources of EPU and EFU see the header of Table 1.



the level of imperfect knowledge and quality of forecasts made by individual forecasters, who constitute Y, are reasonably good. Hence, we are left with three parameters to be estimated: α, β and σ. Computations also have been repeated for different thresholds and correlation coefficients and the results seem to be relatively robust to changes of these parameters.

We have compared the fit of WSN with that of two-piece skew normal distribution, TPN, often used for constructing fan charts of inflation (for its statistical properties see John, 1982 and Kimber, 1985; for wider discussion and use in the context of fan-chart modelling see e.g. Tay and Wallis, 2000). One of the representations of its *pdf* is:

$$f_{TPN}(t) = \begin{cases} A\exp\left\{-(t-\mu)^2 / 2\sigma_1^2\right\} & \text{if } t \le \mu \\ A\exp\left\{-(t-\mu)^2 / 2\sigma_2^2\right\} & \text{if } t > \mu \end{cases} \quad ,$$

where $A = \left(\sqrt{2\pi}(\sigma_1 + \sigma_2)/2\right)^{-1}$. Three parameters to be estimated are σ_1^2 , σ_2^2 and μ .

Maximum likelihood estimation of parameters of various types of skew normal distributions, albeit formally straightforward, as the density functions are expressed in closed form, is usually numerically awkward, with possible bias and convergence problems (see e.g. Pewsey, 2000, Monti, 2003). For this reason we have decided to apply the minimum distance estimators (*MDE's*) rather than maximum likelihood. Appropriately defined *MDE*'s are asymptotically efficient and asymptotically equivalent to the maximum likelihood estimators (see Basu, Shioya and Park, 2011). Additional advantage is the ease of their interpretation, as the distance measures tell of the fit of the theoretical to empirical distribution, and the possibility of comparison, in order to search for the one which gives the best fit.

The minimum distance criteria can be defined in different ways. In this paper we have used the Hellinger twice squared distance criterion, defined as (see e.g. Basu, Shioya and Park, 2011):

$$HD(d_n\ ,\ f_\theta) = 2\sum_{i=1}^{m}[d_n(i)^{1/2} - f_\theta(i)^{1/2}]^2 \quad ,$$

where n is the sample size, m is number of disjoint intervals, $d_n(i)$ is the empirical frequency of data falling into the i^{th} interval and $f_\theta(i)$ is the corresponding theoretical probability for this interval, that is an integral of the density function over the interval. Properties of estimators based on Hellinger distances have been well researched in the context of other skew normal distributions (see Greco, 2011), and it is known that the estimates are reasonably robust to the presence of outliers, which might appear in a large sample of inflation forecast errors, especially for longer forecast horizons. Other distance measures belonging to the Cressie and Read (1984) family of power divergence disparities, have also been used leading to similar results.

We made a deviation from the established tradition of computing the f_θ analytically and obtained the estimates of the theoretical probabilities by simulation, that is, by Monte Carlo approximation of the theoretical probabilities. Random number generators of the distributions considered here are straightforward (for WSN see Appendix A, Part III, and for TPN see Nakatsuma, 2003). Details of this estimation procedure, called the *simulated minimum distance estimator*, *SMDE*, are given in Charemza *et. al* (2012); similar approach have been used by Dominicy and Veredas (2013). The version of *SMDE* applied here can be defined as:

$$\hat{\theta}_n^{SMDE} = \arg\min_{\theta\in\Theta}\left\{\xi\left\{HD(d_n\ ,\ f_{r,\theta}\right\}_{r=1}^{R}\right\} \quad ,$$

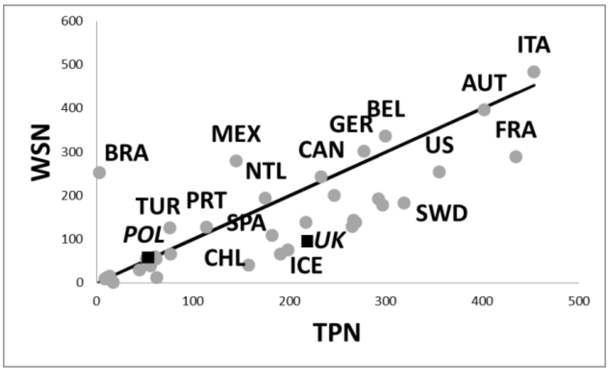
where $f_{r,\theta}$ is the Monte Carlo approximation of the theoretical probabilities, f_θ, of a random variable obtained by generating r = 1,2,…,R replications (drawings) from a distribution with parameters θ ($\theta \in \Theta \subset \mathbb{R}^k$), d_n denotes the density of empirical sample of size n, and ξ is an aggregation operator based on R replications, which deals with the problem of the 'noisy' criterion function (median, in this case).

Figure 3 illustrates the differences between the Hellinger distances obtained for the estimated WSN and TPN distributions for the forecasts horizons 1 and 4. Explanation of country labels (abbreviations) is given in Appendix B. Detailed estimation results are available on request. In each panel a straight 45 degree line marks the points for which the Hellinger distances for the WSN and TPN distribution would be identical. If the dot representing a particular country is below this line, WSN distribution has a better fit (that is, smaller distance measure) than TPN distribution and *vice versa*. For monthly inflation, better fit of WSN in comparison to TPN is evident for most countries, and for annual inflation for nearly all countries. It appears that the advantage of WSN over TPN is increasing with the increase in forecast horizon. Figure 4 depicts the comparison between estimated α and β parameters (multiplied by -1, for the clarity of graphs). Deviations from the 45 degree line downwards denote the dominance of the forecast-induced anti-inflationary effects on uncertainty (that is, $-\alpha > -\beta$) and *vice versa*. As forecasts of Poland and UK are analyzed further in a greater detail, they are denoted by italics and squares as country symbols.

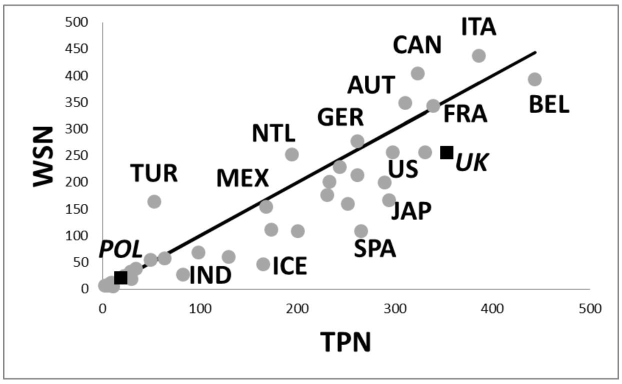
Figure 3: WSN and TPN Hellinger distances. Each point represents the pair of minimum Hellinger distances obtained for fitting the weighted skew normal (WSN) and two-piece normal (TPN) distributions to 1 and 4-step ahead forecasts errors for 32 countries. Number of observations varies from 182 to 770. Country symbols are explained in Appendix B. Poland and UK are marked by squares and italics.

Forecasts made using monthly data on monthly inflation

3.a: 1-step ahead forecasts

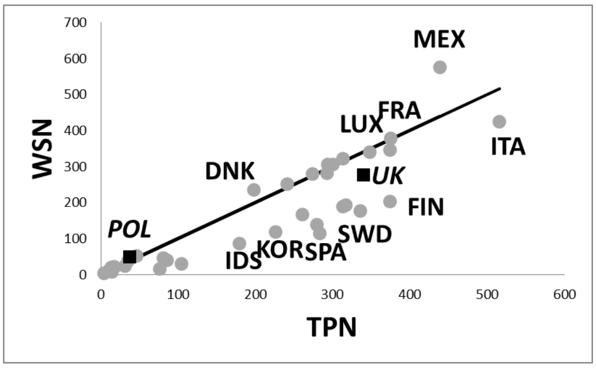


3.b: 4-steps ahead forecasts

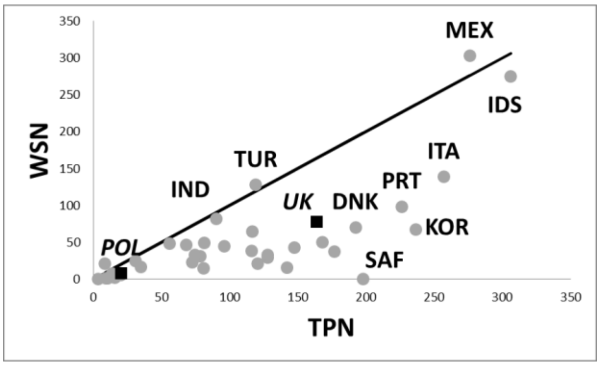


Forecasts made using monthly data on annual inflation

3.c: 1-step ahead forecasts



3.d: 4-steps ahead forecasts



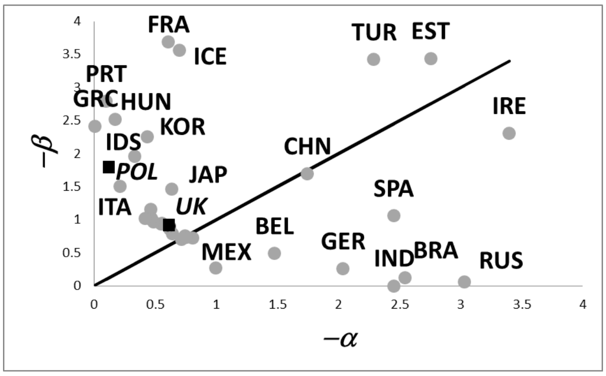
Results in Figure 4 differ markedly for different forecast horizons and for different types of inflation forecasted. For monthly inflation points on the graphs are widely scattered and hence difficult to interpret. Generally, the results seem to be smoother for the annual

Figure 4: Estimated α and β parameters. Each point represents the corresponding α and β parameters which minimized the Hellinger distances obtained for fitting the weighted skew normal (WSN) distribution

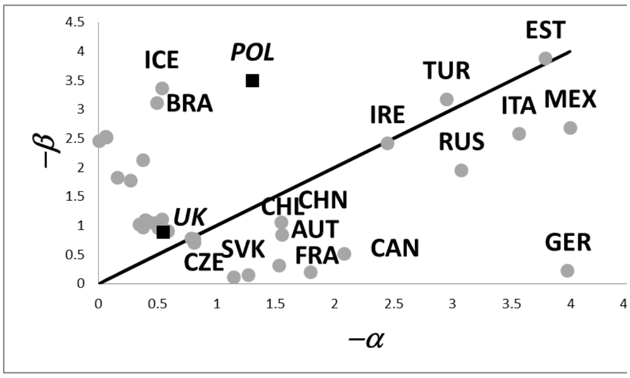
to 1 and 4-step ahead forecasts errors for 32 countries. Number of observations varies from 182 to 770. Country symbols are explained in Appendix B.

Forecasts made using monthly data on monthly inflation

4a: 1-step ahead forecasts

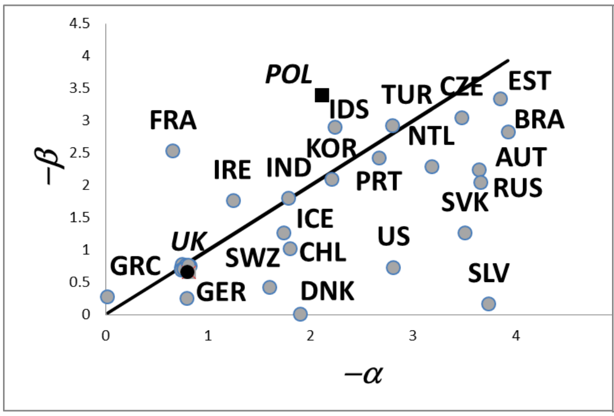


4b: 4-steps ahead forecasts

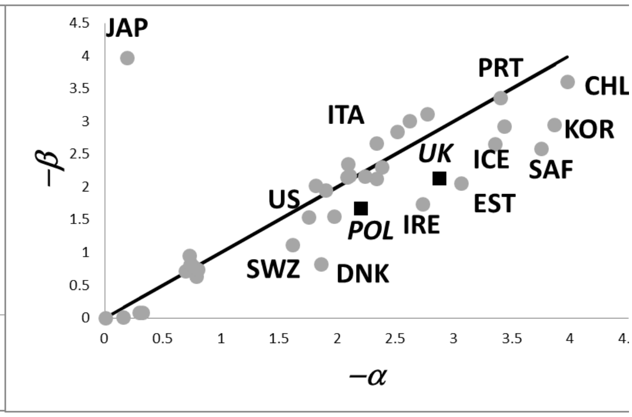


Forecasts made using monthly data on annual inflation

4c: 1-step ahead forecasts



4d: 4-steps ahead forecasts



inflation, and more symmetric, in terms of discrepancies of the corresponding values of α's and β's, for the longer forecast horizon. For annual inflation the anti-inflationary effects on uncertainty are clearly more evident than output-stimulating effects, as the majority of points are below the 45 degree line. It is interesting that the clear outlier here is Japan, with its prolonged output-stimulating and pro-inflationary policy. Also Russia, the biggest natural-resource economy which maintains nominal exchange rate control, resulting in 'dirty float' and inflationary pressure in times of oil price raises, is constantly marked below the 45 degree line, which is consistent with its persistent anti-inflationary policy.

5. INFLATION TERM STRUCTURES FOR UK AND POLAND

In order to compare WSN and TPN applied for computing forecasts term structures we have selected two countries, UK and Poland, for which we have evaluated the probabilistic forecasts for the last 12 months of our sample, that is for the period from March 2012 to February 2013. We have compared our results with these given by the probabilistic forecasts published as fan charts by the Bank of England and National Bank of Poland in their *Inflation Reports*.

For these countries we have used the same method of computing point forecasts errors as in Sections 3 and 4. Next, for each horizon from 1 to 12, we have estimated the parameters of

WSN and TPN distributions, using the simulated minimum distance estimation with the Hellinger criterion, as described in Section 4.

Figures 3a-3d indicate, that for Poland, for the forecasts horizons 1 and 4, the Hellinger distances are nearly identical for WSN and TPN, while for UK the fit of WSN is slightly better. In order to investigate the fit of the distributions closer, we have additionally applied tests based on the probability integral transform, *pit*. The results of *pit*'s uniformity testing (not reported here and available on request) confirm that the WSN is generally superior, in terms of fit, to TPN (for h=1 for both countries and for h=10 for Poland the results are inconclusive).

At the next stage the estimated parameters of WSN have been applied for simulating the distributions of *U*-uncertainties (100,000 replications for each forecast horizon). Then we have retrieved *V*-uncertainties using (5). We have also computed *M*-uncertainties, as described in Section 2 which corresponds to the maximum of VRUV.

Analogously to *U*-uncertainties, we have simulated and applied TPN uncertainties obtained by fitting TPN, rather than WSN, distribution to *ex-post* forecast errors. For all these types of uncertainties (*U, V, M, TPN*) and for each forecast horizon we have computed 6 equidistant quantiles, giving five finite intervals of identical probabilities of 14.29% and two infinite, at the upper and lower ends. The distributions are then centered around point forecasts, that is, the estimates of $\hat{\pi}_{t+h}$ in (1). The fan charts derived in this way are given at Figures 5 (for UK) and 6 (for Poland), where different darks of grey represent the equidistant quantiles and the dashed lines represent the point forecasts of inflation. For the sake of comparison, we keep the scale identical for all fan charts.

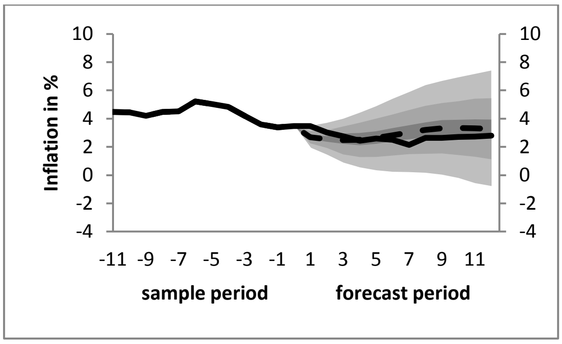
For both countries, the fan charts based directly on *ex-post* forecast errors, that are *U*-uncertainties obtained from WSN and *TPN*-uncertainties, do not differ much from each other. This is hardly surprising, as the both are based on the same set of empirical data, so that the empirical variance of forecast errors is the same. However, for *V*-uncertainties (Figures 5b and 6b), the bands are visibly wider than for *U*-uncertainties and *TPN*-uncertainties. Evidently, the widest are the fan-charts based on *M*-uncertainties.

The fact that the fan charts made with the use of *V*-uncertainties are, for UK and Poland, wider than these made with the use of *U*-uncertainties suggests that the ratio of variances of *V* and *U*, that is VRUV, as defined by (6), is greater than unity. It implies that for both countries the forecast-induced monetary policy reflected by *U* is effective for diminishing inflation uncertainty.

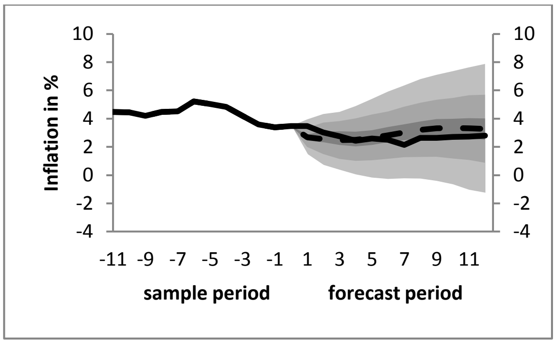
Potential confusion between concepts of *U*- and *V*-uncertainties might lead to misunderstandings. An example here is the critique of the Bank of England forecasts by Dowd (2007) who claimed that the Bank of England overestimated the inflation uncertainty in the sense that in the period 1997-1999 the observed inflation was within an interval which has a low probability according to the Bank of England fan chart assumptions. In fact the published Bank of England forecasts claim to be monetary policy neutral and are made under the assumption of the monetary policy being unaffected by the forecasts. If this is the case and (*a*) accuracy of experts' forecasting is good, that is, ρ in (2) is high, and (*b*) strength of forecast-induced monetary policy outcomes is reasonable, that is α and β are markedly negative, VRUV is greater than one and there is no surprise that the Bank of England uncertainties were high and, at the same time, inflation was often close to its target.

Figure 5: Fan charts for UK. The solid line represents recorded inflation; the dashed line is the point forecasted inflation, which is the mean of the distribution of uncertainties. Different darks of grey represent equiprobable intervals of 14.29%.

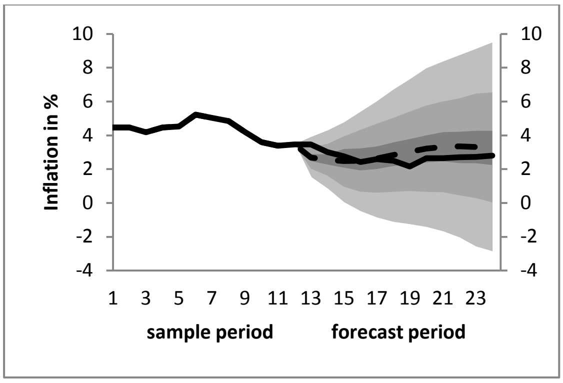
5a: Using *U*-uncertainties



5b: Using *V*-uncertainties



5c: Using *M*-uncertainties



5d: Using *TPN*-uncertainties

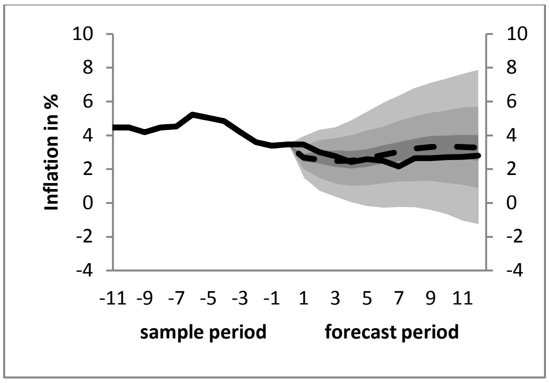
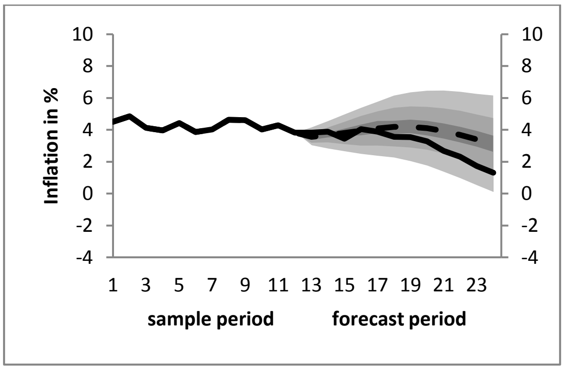
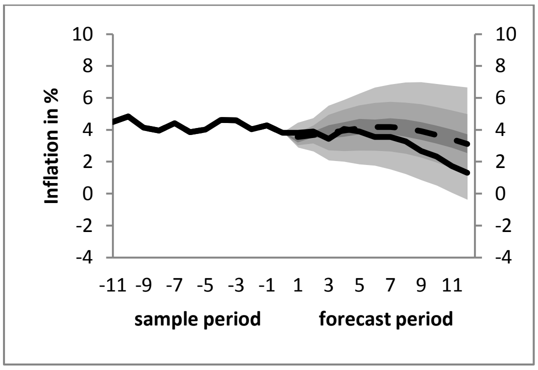


Figure 6: Fan charts for Poland (for description see Figure 5

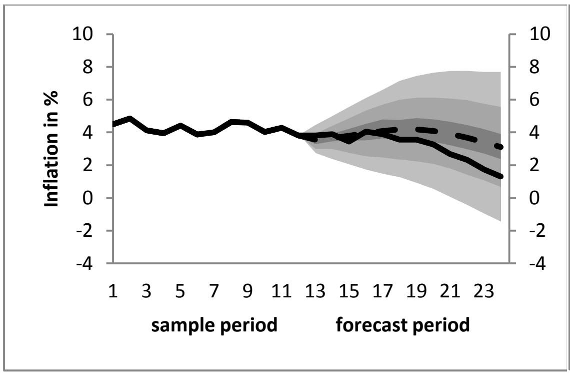
6a: Using *U*-uncertainties



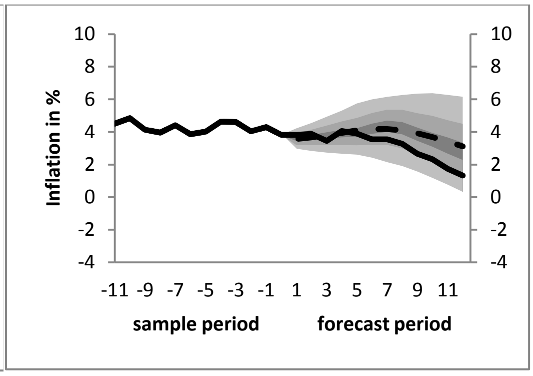
6b: Using *V*-uncertainties



6c: Using *M*-uncertainties



6d: Using *TPN*-uncertainties



In order to assess the accuracy of point forecasts with the use of the uncertainty distributions introduced above, we use the concept of forecast *p*-values which is, to an extent, the analogue to *p*-values used in testing statistical hypotheses. If recorded (observed) inflation is above the baseline point forecast, the forecast *p*-value is a probability of inflation being greater than the recorded inflation conditionally on inflation being greater than the expected value. If, on the other hand, recorded inflation is below the point forecast, it is the probability that inflation is below the recorded inflation, conditional on inflation being smaller than the expected value. Due to possible asymmetry in uncertainties we are using one-sided probabilities, so that an ideally accurate forecast would have a *p*-value of 1. We have computed the forecast *p*-values using, as a benchmark, point forecasts obtained with the use of *SARMA* model discussed in Section 3. These benchmark forecasts have been used as means for the uncertainty distributions. It is difficult to say what the ideal forecast *p*-values should be. If they are high (close to unity), the benchmark forecasts are relatively accurate and the distribution of the uncertainties is too wide. If they are low, the accuracy of the benchmark forecast is poor and the distribution is too narrow. It might be a convenient consensus to look at the forecast *p*-values close to that on one standard deviation as optimal. Tables 2 and 3 show values of benchmark point forecasts, recorded inflation and forecast *p*-values. The benchmark forecasts are as marked on Figures 5 and 6 by dashed lines.

The tables also show the computed probabilities of the violation of inflation target bands. For Poland, these target bands are officially announced as being from 1.5% to 3.5%, with target inflation being equal to 2%. For UK, where only point target of 2% is given, we have assumed the usual ±1% hypothetical bands, from 1% to 3%. Tables 3 and 4 show that, at the moment of making the forecast, the probabilities of inflation actually recorded obtained with the use of *V*-uncertainties were higher than these obtained with the use of *U*-uncertainties. This is quite understandable, as for both countries and all forecast horizons VRUV is greater than unity, so that the variances of *V*-uncertainties are greater than the corresponding variances of *U*-uncertainties. Differences between the *p*-values obtained for *U*-uncertainties and for *TPN*-uncertainties are not substantial, with the *p*-values being usually higher for the former rather than for the latter. Forecast *p*-values for *M*-uncertainties are interpreted as a cautious analog to these for *V*-uncertainties, with additional doubt related to the possible effects of the forecast-induced monetary policy.

Table 2: UK. Observed inflation, forecasts, probabilities of violation of target bands (1%-3%) and *p*-values computed with the use of *U, V, M* and *TPN*-uncertainties for SARMA-based forecasts. Distributions and forecasts are evaluated using data up to February 2012.

f.hor	obs.infl	mean forecast	infl. target violation prob.				forecast *p*-values			
			U-unc	*V*-unc	*M*-unc	*TPN*-unc	*U*-unc	*V*-unc	*M*-unc	*TPN*-unc
Mar 12	3.5	2.7	0.35	0.46	0.46	0.35	0.30	0.52	0.49	0.29
Apr 12	3.0	2.5	0.42	0.64	0.56	0.40	0.66	0.81	0.77	0.65
May 12	2.8	2.5	0.59	0.71	0.75	0.49	0.88	0.91	0.92	0.84
Jun 12	2.4	2.5	0.66	0.72	0.78	0.57	0.98	0.98	0.98	0.97
Jul 12	2.6	2.6	0.69	0.73	0.78	0.63	0.99	0.99	0.99	0.98
Aug 12	2.5	2.8	0.72	0.75	0.81	0.68	0.90	0.91	0.93	0.89
Sep 12	2.2	3.0	0.74	0.76	0.83	0.72	0.75	0.78	0.84	0.74
Oct 12	2.6	3.2	0.76	0.78	0.84	0.75	0.85	0.87	0.90	0.84
Nov 12	2.6	3.3	0.77	0.79	0.84	0.76	0.83	0.85	0.89	0.83
Dec 12	2.7	3.3	0.78	0.80	0.85	0.78	0.86	0.87	0.90	0.84
Jan 13	2.7	3.3	0.79	0.81	0.86	0.80	0.87	0.88	0.91	0.87
Feb 13	2.8	3.3	0.80	0.82	0.87	0.81	0.90	0.91	0.93	0.90

Table 3: Poland. Observed inflation, forecasts, probabilities of violation of target bands (1.5%-3.5%) and *p*-values computed with the use of *U, V, M* and *TPN*-uncertainties for SARMA-based forecasts. Distributions and forecasts are evaluated using data up to February 2012.

f.hor	obs.infl	benchmark forecast	infl. target violation prob.				forecast *p*-values			
			U-unc	*V*-unc	*M*-unc	*TPN*-unc	*U*-unc	*V*-unc	*M*-unc	*TPN*-unc
Mar 12	3.8	3.6	0.55	0.54	0.53	0.53	0.64	0.73	0.75	0.67
Apr 12	3.9	3.7	0.58	0.59	0.60	0.58	0.79	0.83	0.87	0.79
May 12	3.4	3.8	0.60	0.63	0.64	0.62	0.79	0.88	0.86	0.74
Jun 12	4.0	3.9	0.61	0.65	0.69	0.66	0.95	0.96	0.97	0.94
Jul 12	3.9	4.1	0.65	0.70	0.75	0.68	0.93	0.94	0.95	0.90
Aug 12	3.6	4.2	0.67	0.71	0.76	0.71	0.78	0.81	0.86	0.73
Sep 12	3.6	4.2	0.70	0.75	0.79	0.74	0.79	0.82	0.86	0.75
Oct 12	3.3	4.1	0.71	0.76	0.80	0.74	0.74	0.78	0.83	0.71
Nov 12	2.7	3.9	0.73	0.76	0.80	0.71	0.62	0.68	0.75	0.58
Dec 12	2.3	3.7	0.73	0.77	0.82	0.69	0.62	0.68	0.75	0.55
Jan 13	1.7	3.4	0.74	0.77	0.82	0.70	0.55	0.60	0.69	0.50
Feb 13	1.3	3.1	0.73	0.77	0.82	0.72	0.53	0.59	0.67	0.49

Different patterns in evolution of *p*-values with the increase in the forecast horizon can be noticed for UK and Poland. For UK, the *p*-values systematically increase with the increase in forecast horizon, while for Poland they tend to decrease. Under ideal forecasting conditions, where uncertainty distributions are invariant in time, one would expect these probabilities not to evolve with the changes in forecast horizon. In fact, for UK, the computed SARMA-based point forecast tend to become more accurate with the increase in forecast horizon, while for Poland it becomes less accurate. For UK the realisations are closer to the benchmark forecast, the *p*-values tend to increase, and for Poland it is the other way around. A reverse interpretation is also possible. Assuming that the SARMA-based forecasts are the most efficient, the UK ideal fan charts (that is such in which the *p*-values are invariant to forecast horizon) should be narrower for short horizons and wider for longer horizons and for Poland it should be the other way around.

Results given above depend on the accuracy of benchmark point forecasts and are hence not fully comparable. The applied SARMA-based forecast is, to an extent, arbitrarily chosen and by no means the most efficient. It is, however, possible to compare uncertainties obtained from externally produced forecasts using the concepts of *p*-values described above. We have computed the *p*-values using the distributions described above for forecasts published by the Bank of England for UK and National Bank of Poland for Poland in their *Inflation Reports*. Data on quarterly inflation point forecasts (medians) have been collected from the websites (for Bank of England) and interpolated from fan charts (for Poland). Also the original *p*-values, obtained using probabilistic distributions of the Bank of England and National Bank of Poland forecasts have been computed on the basis of interpolations from the published fan charts. They might therefore be not very accurate. These point forecasts have been in turn interpolated into months and used as medians for the previously computed distributions of uncertainties and the *p*-values have been recomputed. Results are given in Tables 4 and 5.

For all uncertainties computed by us around the banks' own forecast medians the pattern is such that *p*-values increase with the increase in forecasts horizon. This would suggest that, in relation to the bank' own forecasts, our fan charts have been either too narrow for short horizons or too wide for long horizons. However, the probabilistic forecasts produced by the banks exhibit different pattern. The *p*-values tend to decrease (albeit not monotonically) with the increase in forecast horizon suggesting that their fan charts becomes too thin for long horizons. The banks' own uncertainties should be identified with *V*-uncertainties, as it is claimed that the forecasts have been made under assumption of no policy action. In comparison with *V*-uncertainties, for longer horizons the banks' own fan charts seem to be 'too thin', or overoptimistic, which is indicated by the relatively low *p*-values of 4-quarter forecasts.

Table 4: UK. Observed inflation at the end of quarter, forecasts, probabilities of violation of target bands (1%-3%) and *p*-values computed with the use of *U, V, M* and *TPN*-uncertainties for Bank of England forecasts as published in November 2011. Forecast *p*-values for bank own distribution are interpolated from bank fan chart.

f.hor	obs.infl	bank median forecast	forecast *p*-values				
			bank own	*U*-unc	*V*-unc	*M*-unc	*TPN*-unc
Q1 12	3.5	4.3	0.36	0.27	0.48	0.46	0.28
Q212	2.4	3.9	0.40	0.50	0.59	0.67	0.37
Q3 12	2.2	4.3	0.50	0.44	0.50	0.61	0.41
Q4 12	2.7	3.7	0.14	0.77	0.79	0.85	0.76

Table 5: Poland. Observed inflation at the end of quarter, forecasts, probabilities of violation of target bands (1.5%-3.5%) and *p*-values computed with the use of *U, V, M* and *TPN*-uncertainties for National Bank of Poland forecasts as published in November 2011. Bank point forecast and forecast *p*-values are interpolated from bank fan chart.

f.hor	obs.infl	bank point forecast	forecast *p*-values				
			bank own	*U*-unc	*V*-unc	*M*-unc	*TPN*-unc
Q1 12	3.8	3.4	0.90	0.48	0.62	0.62	0.50
Q212	4.0	3.0	0.58	0.46	0.66	0.67	0.43
Q3 12	3.6	2.5	0.68	0.65	0.71	0.77	0.56
Q4 12	2.3	1.9	0.52	0.87	0.88	0.92	0.86

6. CONCLUSIONS

Ex-post forecast errors of inflation might tell us more than just by how much the forecasters err. Firstly, they can be interpreted as a relevant component of aggregate macroeconomic uncertainties, as simple measures of their variability are highly correlated with more complex aggregate uncertainty measures. Secondly, they might reveal interesting stories about outcomes of some monetary actions, if the weighed skew normal distribution proposed in this paper is fitted to them. For most countries, this distribution fits to the inflationary forecast errors better than the widely used two-piece normal distribution.

We also conclude that the weighted skew normal distribution fitted to the empirical distribution of point forecast errors is an attractive tool for making inflation fan charts, which are the graphical representation of forecast uncertainty term structure. Nature of this distribution enables to construct different types of fan charts: (*i*) these which represent historical forecast errors, by direct fitting the weighted skew normal distribution to the observed forecast errors and (*ii*) these which are based on the uncertainties which are free from the epistemic element. Both types of fan charts could be used for different practical purposes, and possibly by different users; the former by the 'end users', who do not have direct influence on monetary policy and who do not really care of what is epistemic and what is not, and the latter by central bankers and other policy decision makers, aiming, among other things, at reducing uncertainty through policy action. In some cases yet another type of fan charts can be utilised, based on the safest possible (widest) distributions, covering for the possible errors in identification of the ontological element in the uncertainties.

Our results suggest that more has to be done on the evaluation of forecast error uncertainties and the role of uncertainty by disagreement in relation to uncertainty by error. Nevertheless, we feel that it is already possible to learn more about the different types and nature of inflation uncertainty, and also on the effects of monetary policy actions onto it.

APPENDIX A

Properties of weighted skew-normal distribution and VRUV

I. Weighted skew normal distribution and its properties.

General notation

For a random variable Y and a real number a, notation $I_{Y>a}$ (or $I_{Y<a}$) denotes an indicator of the event $\{Y>a\}$ (correspondingly $\{Y<a\}$), that is equal to unity if $Y>a$ and zero otherwise.

Definition 1. Let *X* and *Y* constitute a bivariate normal random variable such as:

$$(X,Y)\sim N\left(\begin{bmatrix}\mu_X\\ \mu_Y\end{bmatrix},\begin{bmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{bmatrix}\right),\ \text{with } |\rho|<1 \quad , \tag{A.1}$$

and

$$U = X+\alpha\cdot Y\cdot I_{Y>\bar{m}}+\beta\cdot Y\cdot I_{Y<\bar{k}}\ ,\ \text{where } \alpha,\beta,\bar{k}<\bar{m}\in\mathbb{R}\ . \tag{A.2}$$

The distribution of U defined by (A.1)-(A.2) is called the *weighted skew normal* so that $U\sim \mathrm{WSN}_{\sigma_X,\sigma_Y}^{(\mu_X,\mu_Y)}(\alpha,\beta,\bar{m},\bar{k},\rho)$.

Definition 2. Let

$$(X_0,Y_0)\sim N\left(\begin{bmatrix}0\\ 0\end{bmatrix},\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)\ \text{with } |\rho|<1, \tag{A.3}$$

and

$$U^* = X_0+\alpha\cdot Y_0\cdot I_{Y_0>m}+\beta\cdot Y_0\cdot I_{Y_0<k}\quad ,\quad \alpha,\beta,k<m\in\mathbb{R}\quad . \tag{A.4}$$

The distribution of U^* defined by (A.3)-(A.4) is called the *standard weighted skew normal* so that $U^*\sim \mathrm{WSN}_1(\alpha,\beta,m,k,\rho)$.

Proposition 1. The probability density function (*pdf*) of the standard weighted skew normal distribution $U^*\sim \mathrm{WSN}_1(\alpha,\beta,m,k,\rho)$ is given by:

$$f_{\mathrm{WSN}_1}(t)=\frac{1}{\sqrt{A_\alpha}}\varphi\left(\frac{t}{\sqrt{A_\alpha}}\right)\Phi\left(\frac{B_\alpha t-mA_\alpha}{\sqrt{A_\alpha(1-\rho^2)}}\right)+\frac{1}{\sqrt{A_\beta}}\varphi\left(\frac{t}{\sqrt{A_\beta}}\right)\Phi\left(\frac{-B_\beta t+kA_\beta}{\sqrt{A_\beta(1-\rho^2)}}\right)$$
$$+\varphi(t)\cdot\left[\Phi\left(\frac{m-\rho t}{\sqrt{1-\rho^2}}\right)-\Phi\left(\frac{k-\rho t}{\sqrt{1-\rho^2}}\right)\right]\qquad ,$$

where φ and Φ denote respectively the density and cumulative distribution functions of the standard normal distribution, and

$$A_\tau = A(\tau)=1+2\tau\rho+\tau^2,\qquad B_\tau=B(\tau)=\tau+\rho\quad . \tag{A.5}$$

Proof. In order to derive the cumulative distribution function F_{WSN_1} of U^* we integrate joint *pdf* of (X_0,Y_0), that is $\varphi_\rho(x,y)=e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}/2\pi\sqrt{1-\rho^2}$, over three disjoint areas as follows:

$$F_{WSN_1}(t) = \int_{-\infty}^{t-\alpha m} dx \int_{m}^{(t-x)/\alpha} \varphi_\rho(x,y)dy + \int_{-\infty}^{t-\beta k} dx \int_{(t-x)/\beta}^{k} \varphi_\rho(x,y)dy + \int_{-\infty}^{t} dx \int_{k}^{m} \varphi_\rho(x,y)dy =$$

$$= \int_{-\infty}^{t-\alpha m} \varphi(x)\left[\Phi\left(\frac{t-B_\alpha x}{\alpha\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{m-\rho x}{\sqrt{1-\rho^2}}\right)\right]dx$$

$$- \int_{-\infty}^{t-\beta k} \varphi(x)\left[\Phi\left(\frac{t-B_\beta x}{\beta\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)\right]dx$$

$$+ \int_{-\infty}^{t} \varphi(x)\left[\Phi\left(\frac{m-\rho x}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)\right]dx$$

Taking the first derivative $dF_{WSN_1}(t)/dt$ complete the proof.■

As:

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{A_\alpha}}\varphi\left(\frac{t}{\sqrt{A_\alpha}}\right)\Phi\left(\frac{B_\alpha t - mA_\alpha}{\sqrt{A_\alpha(1-\rho^2)}}\right)dt = \Phi(-m), \quad \int_{-\infty}^{+\infty} \frac{1}{\sqrt{A_\beta}}\varphi\left(\frac{t}{\sqrt{A_\beta}}\right)\Phi\left(\frac{-B_\beta t + kA_\beta}{\sqrt{A_\beta(1-\rho^2)}}\right)dt = \Phi(k),$$

$$\int_{-\infty}^{+\infty} \varphi(t)\cdot\left[\Phi\left(\frac{m-\rho t}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{k-\rho t}{\sqrt{1-\rho^2}}\right)\right]dt = \Phi(m) - \Phi(k),$$

and $\Phi(-m) + \Phi(k) + [\Phi(m) - \Phi(-k)] = 1$,

it follows from Proposition 1 that f_{WSN_1} can be interpreted as a weighted sum of three *pdf*'s as $f_{\mathrm{WSN}_1}(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_3(t)$, where

$$f_1(t) = \frac{1}{\Phi(-m)}\frac{1}{\sqrt{A_\alpha}}\varphi\left(\frac{t}{\sqrt{A_\alpha}}\right)\Phi\left(\frac{B_\alpha t - mA_\alpha}{\sqrt{A_\alpha(1-\rho^2)}}\right), \qquad \alpha_1 = \Phi(-m)$$

$$f_2(t) = \frac{1}{\Phi(k)}\frac{1}{\sqrt{A_\beta}}\varphi\left(\frac{t}{\sqrt{A_\beta}}\right)\Phi\left(\frac{-B_\beta t + kA_\beta}{\sqrt{A_\beta(1-\rho^2)}}\right), \qquad \alpha_2 = \Phi(k)$$

$$f_3(t) = \frac{1}{\Phi(m) - \Phi(k)}\varphi(t)\cdot\left[\Phi\left(\frac{m-\rho t}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{k-\rho t}{\sqrt{1-\rho^2}}\right)\right], \quad \alpha_3 = \Phi(m) - \Phi(k)$$

The *pdf* f_3 is a *pdf* of conditional variable $\left(X_0 \,|\, k \le Y_0 \le m\right)$.

Relations between f_1 and f_2 and skew normal distribution are as follows. Simple Azzalini (1985, 1986) skew normal distribution $SN(\lambda,\omega)$ can be defined by its *pdf* as $f_{SN}(t;\lambda,\omega) = \frac{2}{\omega}\varphi(t/\omega)\Phi(\lambda x/\omega)$. Hence, for $m = k = 0$ and $\alpha = -2\rho$ functions f_1 and f_2 reduce to *pdf*'s of $SN(\lambda_1,\omega_1)$ and $SN(\lambda_2,\omega_2)$ with $\omega_1 = \sqrt{A_\alpha}$, $\lambda_1 = -\rho/\sqrt{1-\rho^2}$ and $\omega_2 = \sqrt{A_\beta}$, $\lambda_2 = \rho/\sqrt{1-\rho^2}$ respectively. This representation allows for yet another interpretation of simple skew normal distribution, as $U^{SN} \sim \mathrm{WSN}_1(-2\rho,0,0,0,\rho)$, or $U^{SN} = X_0 - 2\rho Y_0 \cdot I_{Y_0>0}$, where $X_0, Y_0 \sim N(0,1)$ and $corr(X_0,Y_0)=\rho$.

Representation $f_{\mathrm{WSN}_1}(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_3(t)$ can be now interpreted as a weighted sum of conditional *pdf* of $\left(X_0 \middle| k \le Y_0 \le m\right)$ and two *pdf*'s that, under some restrictions on parameters, coincide with that of Azzalini skew normal (hence the name of the WSN distribution).

Proposition 2. Moment generating function (MGF) of $U^* \sim \mathrm{WSN}_1(\alpha,\beta,m,k,\rho)$ is given by:

$$R_{\mathrm{WSN}_1}(u) = e^{\frac{u^2}{2}A_\beta}\Phi(k - B_\beta u) + e^{\frac{u^2}{2}} \cdot \left[\Phi(m-\rho u) - \Phi(k-\rho u)\right] + e^{\frac{u^2}{2}A_\alpha}\Phi(B_\alpha u - m), \quad \text{(A.6)}$$

Proof. By definition of MGF and U^* we get:

$$R_{WSN_1}(u) = E\left(e^{u\cdot U^*}\right) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty} dx \left[\int_{-\infty}^{k} e^{u[x+\beta\cdot y]} + \int_{k}^{m} e^{ux} + \int_{m}^{+\infty} e^{u[x+\alpha\cdot y]}\right] \cdot e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}} dy\ .$$

Changing order of integration and in each of the integrals above, substituting $z = (x-\rho y)/\sqrt{1-\rho^2}$ and noting that MGF of standard normal distribution is $e^{u^2/2}$, complete the proof.■

Corollary. Moment generating function R_{WSN} of $U \sim \mathrm{WSN}_{\sigma_X,\sigma_Y}^{(\mu_X,\mu_Y)}(\alpha,\beta,\bar m,\bar k,\rho)$ defined by (A.1)-(A.2) is given by:

$$R_{\mathrm{WSN}}(u) = e^{u\mu_X}\left\{ e^{\alpha\mu_Y u + \frac{(u\sigma_X)^2}{2}A_{\alpha\frac{\sigma_Y}{\sigma_X}}}\Phi\left(B_{\alpha\frac{\sigma_Y}{\sigma_X}}u\sigma_X - \frac{\bar m - \mu_Y}{\sigma_Y}\right) + e^{\beta\mu_Y u + \frac{(u\sigma_X)^2}{2}A_{\beta\frac{\sigma_Y}{\sigma_X}}}\Phi\left(\frac{\bar k - \mu_Y}{\sigma_Y} - B_{\beta\frac{\sigma_Y}{\sigma_X}}u\sigma_X\right) \right.$$
$$\left. + e^{\frac{(u\sigma_X)^2}{2}} \cdot \left[\Phi\left(\frac{\bar m - \mu_Y}{\sigma_Y} - \rho u\sigma_X\right) - \Phi\left(\frac{\bar k - \mu_Y}{\sigma_Y} - \rho u\sigma_X\right)\right]\right\} \quad \text{(A.7)}$$

Proof A weighted skew normal random variable $U \sim \mathrm{WSN}_{\sigma_X,\sigma_Y}^{(\mu_X,\mu_Y)}(\alpha,\beta,\bar m,\bar k,\rho)$ can be expressed through $U^* \sim \mathrm{WSN}_1\left(\alpha\frac{\sigma_Y}{\sigma_X}, \beta\frac{\sigma_Y}{\sigma_X}, m, k, \rho\right)$ as:

$$U = \sigma_X U^* + \mu_X + \mu_Y \cdot \left(\alpha I_{Y_0>m} + \beta I_{Y_0<k}\right) \quad ,$$

where $m = \dfrac{\bar m - \mu_Y}{\sigma_Y}$, $k = \dfrac{\bar k - \mu_Y}{\sigma_Y}$, and U^* is constrained with the used of $(X_0,Y_0) \sim N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$. Therefore

$$R_{\mathrm{WSN}}(u)=E\left(e^{u\cdot U}\right)=$$

$$=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}dx\left[\int_{-\infty}^{k}e^{u\left[\sigma_X\left\{x+\beta\frac{\sigma_Y}{\sigma_X}\cdot y\right\}+\mu_X+\beta\mu_Y\right]}+\int_{k}^{m}e^{u\{\sigma_X x+\mu_X\}}+\int_{m}^{+\infty}e^{u\left[\sigma_X\left\{x+\alpha\frac{\sigma_Y}{\sigma_X}\cdot y\right\}+\mu_X+\alpha\mu_Y\right]}\right]\cdot e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}dy$$

Applying Proposition 2 to the representation above, complete the proof.■

In order to calculate moments of $\mathrm{WSN}_{\sigma_X,\sigma_Y}^{(\mu_X,\mu_Y)}(\alpha,\beta,\overline{m},\overline{k},\rho)$ it is enough to calculate derivatives at zero of corresponding MGF and, due to (A.7), calculate derivatives at zero of MGF for $\mathrm{WSN}_1(\alpha,\beta,m,k,\rho)$.

Proposition 3. Let R_{WSN_1} be a moment generating function (MGF) given by (A.5), then

$$R'_{\mathrm{WSN}_1}(0)=\alpha\cdot\varphi(m)-\beta\cdot\varphi(k);$$

$$R''_{\mathrm{WSN}_1}(0)=A_\alpha+\left[1-A_\alpha\right]\Phi(m)+\left[B_\alpha^2-\rho^2\right]m\varphi(m)+\left[A_\beta-1\right]\Phi(k)-\left[B_\beta^2-\rho^2\right]k\varphi(k);$$

$$R^{(3)}_{\mathrm{WSN}_1}(0)=\varphi(m)\cdot\left\{B_\alpha\cdot\left[3A_\alpha+B_\alpha^2\left(m^2-1\right)\right]-\rho\cdot\left[3+\rho^2\left(m^2-1\right)\right]\right\}+$$
$$\varphi(k)\cdot\left\{-B_\beta\cdot\left[3A_\beta+B_\beta^2\left(k^2-1\right)\right]+\rho\cdot\left[3+\rho^2\left(k^2-1\right)\right]\right\}\quad;$$

$$R^{(4)}_{\mathrm{WSN}_1}=3\cdot\left\{A_\alpha^2+\Phi(m)\cdot\left[1-A_\alpha^2\right]\right\}+m\cdot\varphi(m)\cdot\left\{\left(3-m^2\right)\cdot\left(\rho^4-B_\alpha^4\right)+6\cdot\left(A_\alpha B_\alpha^2-\rho^2\right)\right\}$$
$$+3\cdot\Phi(k)\cdot\left[A_\beta^2-1\right]-k\cdot\varphi(k)\left\{\left(3-k^2\right)\cdot\left(\rho^4-B_\beta^4\right)+6\cdot\left(A_\beta B_\beta^2-\rho^2\right)\right\}\quad.$$

Proof.

Let $g_{a,b,c}(u)=e^{\frac{au^2}{2}}\Phi(bu+c)$. (A.8)

By Taylor expansion we get: $e^{\frac{au^2}{2}}=1+\frac{au^2}{2}+\frac{a^2u^4}{8}+\ldots,$ (A.9)

and:

$$\Phi(bu+c)=\Phi(c)+b\varphi(c)\cdot u-\frac{b^2c}{2}\varphi(c)\cdot u^2+\frac{b^3(c^2-1)}{3!}\varphi(c)\cdot u^3+\frac{b^4c(3-c^2)}{4!}\varphi(c)\cdot u^4+\ldots$$
(A.10)

Substituting (A.9) and (A.10) to (A.8) yields:

$$g_{a,b,c}(u)=$$
$$=\left(1+\frac{au^2}{2}+\frac{a^2u^4}{8}+\ldots\right)\cdot\left(\Phi(c)+b\varphi(c)\cdot u-\frac{b^2c}{2}\varphi(c)\cdot u^2+\frac{b^3(c^2-1)}{3!}\varphi(c)\cdot u^3+\frac{b^4c(3-c^2)}{4!}\varphi(c)\cdot u^4+\ldots\right)$$
$$=\Phi(c)+b\varphi(c)\cdot u+\frac{1}{2}\left[a\cdot\Phi(c)-b^2c\cdot\varphi(c)\right]\cdot u^2+\frac{1}{3!}\left[3a+b^2(c^2-1)\right]b\varphi(c)\cdot u^3$$
$$+\frac{1}{4!}\left[3a^2\Phi(c)-6ab^2c\varphi(c)+b^4c(3-c^2)\varphi(c)\right]\cdot u^4+\ldots$$

Therefore:

$g'_{a,b,c}(0)=b\varphi(c)$, (A.11)

$$g''_{a,b,c}(0) = a \cdot \Phi(c) - b^2 c \cdot \varphi(c) \quad , \tag{A.12}$$

$$g^{(3)}_{a,b,c}(0) = \left[3a + b^2(c^2 - 1)\right] b\varphi(c) \quad , \tag{A.13}$$

$$g^{(4)}_{a,b,c}(0) = 3a^2\Phi(c) - 6ab^2 c\varphi(c) + b^4 c(3 - c^2)\varphi(c) \quad , \tag{A.14}$$

Bearing in mind that

$$R_{\mathrm{WSN}_1}(u) = g_{A_\alpha, B_\alpha, (-m)}(u) + g_{A_\beta, (-B_\beta), k}(u) + g_{1,(-\rho),m}(u) - g_{1,(-\rho),k}(u) \quad , \tag{A.15}$$

Taking derivative of the both sides of (A.15) and substituting to (A.11)-(A.14) complete the proof. ■

Note. For $m > 0$ and $k < 0$ it is convenient to simplify expression for $R''_{\mathrm{WSN}_1}(0)$ as:

$$R''_{\mathrm{WSN}_1}(0) = 1 + C_\alpha D_m + C_\beta D_k \quad , \tag{A.16}$$

where

$$C_\tau = C(\tau) = \tau(\tau + 2\rho) \text{ and } D_a = D(a) = 1 - \Phi(|a|) + |a|\varphi(a) = \int_{|a|}^{+\infty} t^2 \varphi(t)dt \quad . \tag{A.17}$$

II. Properties of VRUV

Definition 3. Let $U \sim \mathrm{WSN}^{(0,0)}_{\sigma,\sigma}(\alpha, \beta, \bar{m}, \bar{k}, \rho)$ be defined by (A.1)-(A.2) and $V = U - E\left(X \middle| Y\right) = U - \rho Y$. Define $\mathrm{VRUV} = \dfrac{\mathrm{var}(V)}{\mathrm{var}(U)}$.

Proposition 4.

1) Noting that $\mathrm{var}(U) = \sigma^2 \,\mathrm{var}(U^*)$ and:

$$\mathrm{var}(V) = \mathrm{var}(U) + \rho^2\sigma^2 - 2\rho E\left(X + \alpha \cdot Y \cdot I_{Y > \bar{m}} + \beta \cdot Y \cdot I_{Y < \bar{k}}\right)Y = \sigma^2 \,\mathrm{var}(U^*) - \rho^2\sigma^2 + \sigma^2\left(\alpha D_m + \beta D_k\right)$$

yields the following representation:

$$\mathrm{VRUV} = 1 - 2\rho \frac{\alpha D_m + \beta D_k - \rho/2}{\mathrm{var}(U^*)} \quad , \tag{A.18}$$

where $D_a = D(a) = \int_{|a|}^{+\infty} t^2 \varphi(t)dt = 1 - \Phi(|a|) + |a|\varphi(a)$.

2) For $m > 0$ and $k < 0$, applying Proposition 3 and (A.16) to U^* we get:

$$\mathrm{var}(U^*) = 1 + C_\alpha D_m + C_\beta D_k - \left[R'_{\mathrm{WSN}_1}(0)\right]^2 =$$

$$= 1 - 2\rho S + \left[\frac{S\varphi(k) - ZD_k}{D_m\varphi(k) + D_k\varphi(m)}\right]^2 D_m + \left[\frac{S\varphi(m) + ZD_m}{D_m\varphi(k) + D_k\varphi(m)}\right]^2 D_k \quad ,$$

where $Z = EU = \alpha\varphi(m) - \beta\varphi(k)$, $S = -\alpha D_m - \beta D_k$. This, together with (A.18), yields another convenient expression for VRUV:

$$\mathrm{VRUV} = 1 + \rho \frac{2S - \rho}{1 - 2\rho S + \left[\dfrac{S\varphi(k) - ZD_k}{D_m\varphi(k) + D_k\varphi(m)}\right]^2 D_m + \left[\dfrac{S\varphi(m) + ZD_m}{D_m\varphi(k) + D_k\varphi(m)}\right]^2 D_k} \quad . \tag{A.19}$$

3) For $Z = EU \equiv 0$, $\rho > 0, k < 0 < m$ and $\alpha < 0, \beta < 0$ the above simplifies for

$$\text{VRUV} = \frac{\text{var}(V)}{\text{var}(U)} = 1 + \rho \frac{2S - \rho}{1 - 2\rho S + W_{m,k} \cdot S^2},$$

where $W_{m,k} = \left[D_m \varphi^2(k) + D_k \varphi^2(m) \right] / \left[D_m \varphi(k) + D_k \varphi(m) \right]^2$.

By taking partial derivative with respect to S from the above we get:

$$S_{\max} = S_{\max}(\rho) = \text{argmax}\left\{ \text{VRUV}_{\max}(\rho) \right\} \equiv \text{argmax}\left\{ \max_{S>0} \text{VRUV} \right\} = \rho/2 + \sqrt{\rho^2/4 + (1-\rho^2)/W_{m,k}}$$

In particular, in a fully symmetric case when $\alpha = \beta$ and $k = -m$, it yields:

$$W_{m,m} = \left(2D_m\right)^{-1},\ S_{\max} = S_{\max}(1) = 1,\ \text{VRUV}_{\max}(1) = (1 - 2D_m)^{-1}. \tag{A.20}$$

■

III. Simulation

Formulae (A.3)-(A.4) suggest a convenient way of generating random numbers from $\text{WSN}_1(\alpha, \beta, m, k, \rho)$ distribution. A straightforward algorithm is:

Step 1: generate a pair of random numbers (x, y) from a bivariate normal distribution with zero means, unitary variance and covariance equal to ρ.

Step 2: (a) if $y \le m$ and $y \ge k$: return $z = x$,

(b) if $y > m$: return $z = x + \alpha y$,

(c) if $y < k$: return $z = x + \beta y$.

APPENDIX B

Country symbols

AUT	Austria	FRA	France	JAP	Japan	SLV	Slovenia
BEL	Belgium	GER	Germany	KOR	Korea	SAF	South Africa
BRA	Brazil	GRC	Greece	LUX	Luxembourg	SPA	Spain
CAN	Canada	HUN	Hungary	MEX	Mexico	SWD	Sweden
CHL	Chile	ICE	Iceland	NTL	Netherlands	SWZ	Switzerland
CHN	China	IND	India	NOR	Norway	TUR	Turkey
CZE	Czech Rep	IDS	Indonesia	POL	Poland	UK	United Kingdom
DNK	Denmark	IRE	Ireland	PRT	Portugal	US	United States
EST	Estonia	ISR	Israel	RUS	Russia		
FIN	Finland	ITA	Italy	SVK	Slovak Rep		

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