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Ambiguity Revealed

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Abstract: We derive necessary and sufficient conditions for data sets composed of state-contingent prices and consumption to be consistent with two prominent models of decision making under ambiguity: variational preferences and smooth ambiguity. The revealed preference conditions for the maxmin expected utility and subjective expected utility models are characterized as special cases.

Keywords: ambiguity, revealed preference, maxmin, variational preferences, expected utility, smooth, uncertainty

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1. Introduction

In recent years, developments in behavioral economics and decision theory have led to a profusion of increasingly general models of decision making under risk and uncertainty, e.g., Choquet expected utility (Schmeidler, 1989), maxmin expected utility (Gilboa and Schmeidler, 1989), cumulative prospect theory (Tversky and Kahneman, 1992), biseparable preferences (Ghirardato, Maccheroni, and Marinacci, 2004), smooth ambiguity (Klibanoff, Marinacci, and Mukerji, 2005), variational preferences (Maccheroni, Marinacci, and Rustichini, 2006), subjective expected uncertain utility (Gul and Pesendorfer, 2008), vector expected utility (Siniscalchi, 2009), uncertainty averse preferences (Cerreia-Vioglio *et al.*, 2011), to name just a few.[1] Now consider a social scientist who has collected data and wants to know whether these data could have been generated by maximizing one of the aforementioned preferences. What tests can he use? This paper addresses the issue of testability for two popular models of decision making under uncertainty: variational preferences (Maccheroni, Marinacci, and Rustichini (2006)) and smooth ambiguity (Klibanoff, Marinacci, and Mukerji (2005)).[2]

More precisely, this paper follows the revealed preference approach pioneered by Samuelson (1938, 1948) and Afriat (1967). We assume that we observe data sets composed of state-contingent prices and consumption and ask whether these data are consistent with the hypothesis of preference maximization. The main contribution of the paper is to derive necessary and sufficient conditions for the data to be consistent with the models of variational preferences and smooth ambiguity.

We illustrate our problem with the help of a simple example, inspired by the experiment in Ahn *et al.* (2011). There are three states of the world, s_1, s_2, and s_3, and a single consumption good (say, money) in each state. The

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[1] See Gilboa and Marinacci (2011) and Wakker (2010) for recent surveys.

[2] As special cases, these models include subjective expected utility (Savage, 1954), maxmin expected utility (Gilboa and Schmeidler, 1989), multiplier preferences (Hansen and Sargent, 2001), and mean-variance preferences (Markowitz, 1952, 1959; Tobin, 1958).

probability of state s_1 is known to be 1/3, while the probabilities of the other two states are unknown. An individual has to choose a contingent consumption plan, (x_1, x_2, x_3), where x_i refers to consumption of the good in state s_i, from two different budget sets. The first budget set is $B := \{(x_1, x_2, x_3) : x_1 + (4/5)x_2 + 2x_3 \leq 1\}$, while the second is $B' := \{(x_1, x_2, x_3) : 2x_1 + 2x_2 + x_3 \leq 3\}$. Suppose now that we observe the individual choosing $(1, 0, 0)$ from budget set B and $(0, 1, 1)$ from budget set B'.

As a preliminary remark, note that the consumption plan $(0, 1, 1)$ is chosen when the plan $(1, 0, 0)$ is affordable, while the plan $(0, 1, 1)$ is not affordable when the plan $(1, 0, 0)$ is chosen. The data set therefore obeys the generalized axiom of revealed preference, and by Afriat's Theorem, there exists a monotonic preference ordering that, when maximized subject to each budget constraint, could have generated the observed choices.

The question we ask is whether these observations are consistent with particular forms of preference maximization. For example, are they consistent with a model of subjective expected utility? Or with an ambiguity averse preference? In other words, can we construct models from different classes of preferences in such a way that they exactly generate the observed behavior? Alternatively, can we rule out certain models as never having been able to generate the data?

We first argue that the data set in our example is *inconsistent* with a model of subjective expected utility. Notice that the plans $(0, 1, 0)$ and $(1, 0, 1)$ are in B and B', respectively. So it must be that $(1, 0, 0)$ is preferred to $(0, 1, 0)$ and $(0, 1, 1)$ is preferred to $(1, 0, 1)$ for the data to be rationalizable. Moreover, assuming that preferences are monotonic, $(1, 0, 0)$ is *strictly* preferred to $(0, 1, 0)$.[3] Therefore, we have that $(1, 0, 0)$ is strictly preferred to $(0, 1, 0)$, while $(0, 1, 1)$ is preferred to $(1, 0, 1)$. This is a violation of the sure-thing principle, and therefore the data cannot be consistent with a model of subjective expected utility. There is no utility function and probability over the state s_3 (and hence of state s_2) that rationalizes the observed choices.

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[3] To see this, note that the plan $(1/20, 1 + 1/20, 1/20) \gg (0, 1, 0)$ is in B. Consequently, if the decision maker is indifferent between $(1, 0, 0)$ and $(0, 1, 0)$, we would have that $(1/20, 1 + 1/20, 1/20)$ is strictly preferred to $(1, 0, 0)$ by monotonicity, a contradiction.

We next argue that the data set in our example is *consistent* with a model of maxmin expected utility. To see this, consider the utility function $u(x) = x$ and the set of priors $\Pi = [0, 2/3]$, where $\pi \in \Pi$ is the probability of state s_3. It is straightforward to verify, using linear programming techniques, that $(1, 0, 0)$ is a solution to the optimization problem $\max_{x\in B} \min_{\pi\in\Pi}(1/3)x_1+(2/3-\pi)x_2+\pi x_3$, while $(0, 1, 1)$ is a solution to the optimization problem $\max_{x\in B'} \min_{\pi\in\Pi}(1/3)x_1+(2/3 - \pi)x_2 + \pi x_3$.

In general, however, it is impossible to consider *all* utility functions and sets of priors, to derive the optimal choices, and then to verify that they are consistent with the observed choices. The contribution of the paper is to derive non parametric tests for the data to be consistent with the maximization of variational preferences or "smooth ambiguity" preferences.

The rest of the paper is organized as follows. Section 2 presents a brief outline of well-established revealed preference conditions for the most general preferences over state-contingent consumption. Section 3 contains the main results for models of decision making under uncertainty when agents are ambiguity averse. Section 4 concludes.

2. Revealed Preference

Consider an economy with commodity space $X = \mathbb{R}^L_+$ and finite state space S, with generic elements x and s, respectively. Trading takes place before the realization of a state in a complete market. Let $x(s)$ denote the consumption of the L goods in state s, with corresponding state-contingent price vector $p(s) \gg 0$. Let $\Delta(S)$ denote the set of probability distributions over S, with generic element π. In such an economy, the choice of a consumer corresponds to the choice of an act $x : S \to X$. Let $\mathbb{X}$ denote the set of all acts, and let $\succcurlyeq \subseteq \mathbb{X} \times \mathbb{X}$ be the consumer's preference ordering over acts.

Suppose that we observe the consumer's choices over a finite number of periods, that is, we have the data set $(x_t, p_t)_{t\in T}$. The main question we address is whether the data set is consistent with preference maximization? In other words, does there exist a preference relation $\succcurlyeq$ such that for all $t \in T$, $x_t \succcurlyeq x$ for all $x \in B(x_t, p_t) := \{x : p_t \cdot x \leq p_t \cdot x_t\}$, the budget set at (x_t, p_t)?

Naturally, without further assumptions, *any* data set is consistent with preference maximization. To see this, simply assume that the consumer is indifferent between all acts. Throughout the paper, we make a series of assumptions on the consumer's preferences: completeness, transitivity, continuity, monotonicity, and non-degeneracy (axioms A.1, A.3, A.4 and A.6 in Gilboa and Schmeidler (1989)). These assumptions are standard and imply that the data set $(x_t, p_t)_{t \in T}$ is consistent with preference maximization if and only if it satisfies the Afriat inequalities, namely, if and only if there exist $(U_t, \lambda_t)_{t \in T}$, with $(U_t, \lambda_t) \in \mathbb{R} \times \mathbb{R}_{++}$, such that $U_{t'} \leq U_t + \lambda_t p_t \cdot (x_{t'} - x_t)$ for all $(t, t') \in T \times T$.[4] Thus, if the data set satisfies the Afriat inequalities, there exists a utility function $U : \mathbb{X} \to \mathbb{R}$ that rationalizes the observed choices. The purpose of this paper is to establish similar revealed preference characterizations for specific functional forms of U, e.g., when there exists a Bernouilli utility function $u : X \to \mathbb{R}$ and a prior $\pi \in \Delta(S)$ such that $U(x) = \sum_s u(x(s))\pi(s)$ for all x.

3. Ambiguity Revealed

3.1 Variational Preferences

The class of variational preferences (Maccheroni, Marinacci, and Rustichini, 2006) is a broad class that incorporates ambiguity and generalizes the popular class of multiple prior preferences (Gilboa and Schmeidler, 1989). It also includes, as special cases, the multiplier preferences of Hansen and Sargent (2001) and the mean-variance preferences of Markowitz (1952, 1959) and Tobin (1958), two popular models in macroeconomics and finance. According to this model, a decision maker evaluates the act $x \in \mathbb{X}$ as $\min_{\pi \in \Delta(S)} \sum_s u(x(s))\pi(s) + c(\pi)$, where $u : X \to \mathbb{R}$ is a utility function and $c : \Delta(S) \to [0, +\infty]$ a grounded,

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[4] See Afriat (1967), Diewert (1973), Varian (1982), and Fostel, Scarf, and Todd (2004) for proofs of Afriat's Theorem. The revealed preference approach has been adopted in a variety of settings, for example, firm production (Hanoch and Rothschild, 1972; Varian, 1984), consumer demand (Varian, 1983a), investor behavior (Varian, 1983b), choice under risk (Green and Srivastava, 1986; Varian 1988), intertemporal allocation (Browning, 1989; Crawford, 2010), collective decision making (Cherchye, De Rock, and Vermeulen, 2007), and preferences over characteristics (Blow, Browning, and Crawford, 2008), to name a few.

convex, and lower semi-continuous function. The function c can be viewed as an index of ambiguity aversion: the lower is c, the higher is the ambiguity aversion. Multiple prior preferences and multiplier preferences then correspond to specific choices of c, namely, the indicator function of a closed convex set and the relative entropy. The axioms of weak certainty independence and uncertainty aversion are central for the representation of variational preferences. The weak certainty independence axiom is a weakening of the classical independence axiom and essentially requires independence only when acts are mixed with *constant* acts for fixed mixing coefficients. Uncertainty aversion states that if a decision maker is indifferent between two acts, then he prefers mixtures of these two acts over either of them. A decision maker might therefore prefer to hedge against ambiguity.

Accordingly, we say that the data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of variational preferences if there exist an increasing and concave utility function $u : X \to \mathbb{R}$ and a grounded, semi-continuous, and convex function $c : \Delta(S) \to [0, +\infty]$ such that $\min_{\pi\in\Delta(S)} \left(\sum_{s\in S} \pi(s)u(x_t(s)) + c(\pi)\right) \geq \min_{\pi\in\Delta(S)} \left(\sum_{s\in S} \pi(s)u(x(s)) + c(\pi)\right)$ for all $x \in B(x_t, p_t)$, for all $t \in T$. It is worth stressing that we assume a concave utility function to reflect risk aversion. While this assumption is standard, it is with loss of generality, as we shall prove in the conclusion.

Theorem 1 *The following statements are equivalent:*

1. *The data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of variational preferences.*

2. *There exist $(u_t, \pi_t, c_t, \lambda_t)_{t\in T}$, with $(u_t, \pi_t, c_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times [0, +\infty] \times \mathbb{R}_{++}$ for each $t \in T$, such that*

$$u_{t'}(s') \leq u_t(s) + \lambda_t \frac{p_t(s)}{\pi_t(s)}(x_{t'}(s') - x_t(s)), \tag{V.1}$$

for all $(s, s', t, t') \in S \times S \times T \times T$, and

$$c_{t'} \geq c_t - \sum_{s\in S} u_t(s)(\pi_{t'}(s) - \pi_t(s)), \tag{V.2}$$

for all $(t, t') \in T \times T$.

The intuition behind Theorem 1 is simple. If the data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of variational preferences, it must be that x_t is a solution to the problem $\max_{x \in B(p_t, x_t)} \min_{\pi \in \Delta(S)} \sum_{s \in S} u(x(s))\pi(s) + c(\pi)$. From Fan's minimax theorem, there exists $\pi_t \in \Delta(S)$ such that (x_t, π_t) is a saddle point, i.e.,

$$\sum_{s \in S} u(x_t(s))\pi(s) + c(\pi) \geq \sum_{s \in S} u(x_t(s))\pi_t(s) + c(\pi_t) \geq \sum_{s \in S} u(x(s))\pi_t(s) + c(\pi_t),$$

for all $x \in B(p_t, x_t)$, for $\pi \in \Delta(S)$. The first inequality directly implies the condition (V.2). As for the second inequality, it implies that x_t maximizes the consumer's expected utility given the belief π_t, and therefore condition (V.1) follows from the classical optimality conditions for convex problems.[5] Conversely, if the conditions (V.1) and (V.2) hold true, we construct an increasing continuous concave utility function u and a continuous, grounded, convex function c such that (u, c) rationalize the data, as in Afriat (1967).

The conditions (V.1) and (V.2) exhaust all of the observational implications of maximizing variational preferences. To test whether a data set satisfies (V.1) and (V.2) requires that we make use of nonlinear programming techniques, which are computationally difficult. To circumvent this difficulty, we can fix a set $\{\pi_t : t \in T\}$, solve for $(u_t, \lambda_t, c_t)_{t \in T}$ given the set $\{\pi_t : t \in T\}$, and then perform a grid or random search over the space $\Delta(S)^{|T|}$, thereby simplifying the computational problem to a linear program, the techniques for which are well established.

We next consider the special cases of multiple prior preferences and multiplier preferences. To obtain the multiple prior preferences, we specify the ambiguity index c to be the indicator function of a non-empty, closed, and convex set $\Pi \subseteq \Delta(S)$ of priors, i.e., $c(\pi) = 0$ if $\pi \in \Pi$ and $c(\pi) = +\infty$ if $\pi \notin \Pi$. It immediately follows that a data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of multiple prior preferences if and only if there exist $(u_t, \pi_t, c_t, \lambda_t)_{t \in T}$, with $(u_t, \pi_t, c_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times [0, +\infty] \times \mathbb{R}_{++}$ for each $t \in T$, such that (V.1) and (V.2) hold, with the additional requirement that $c_t = 0$ for each $t \in T$.

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[5] A similar condition already appeared in Green and Srivastava (1986).

It is worth noting that we construct the set Π of priors as the convex hull of $\{\pi_t : t \in T\}$. This suggests that if there is a unique prior π that satisfies (V.1), then the data set is consistent with the maximization of Savage preferences, i.e., subjective expected utility. And indeed it is the case: the data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of Savage preferences if and only if there exist $(u_t, \pi_t, \lambda_t)_{t\in T}$, with $(u_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_{++}$ for each $t \in T$, such that (V.1) holds, with the additional requirement that $\pi_t = \pi_{t'}$ for each $(t, t') \in T \times T$.

To obtain multiplier preferences, we specify the ambiguity index c to be a positive multiple of the relative entropy of π with respect to some prior π^*, i.e., $c(\pi) = \theta R(\pi||\pi^*)$ with $\theta > 0$, $\pi^* \in \Delta(S)$ and $R(\pi||\pi^*) = \sum_{s\in S} \pi(s)[\ln \pi(s) - \ln \pi^*(s)]$. We have that a data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of multiplier preferences only if there exist $(u_t, \pi_t, \lambda_t)_{t\in T}$, with $(u_t, \pi_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_{++}$, and $(\theta, \pi^*) \in \mathbb{R}_{++} \times \Delta(S)$, such that (V.1) holds and

$$\pi_t(s) = \frac{\pi^*(s) e^{-\frac{u_t(s)}{\theta}}}{\sum_{\tilde{s}} \pi^*(\tilde{s}) e^{-\frac{u_t(\tilde{s})}{\theta}}},$$

for all $(s, t) \in S \times T$. We next show that this implies that the data set is consistent with the maximization of Savage preferences. To see this, define λ_t^* as $(1/\theta)\lambda_t \sum_{\tilde{s}} \pi^*(\tilde{s}) e^{-\frac{u_t(\tilde{s})}{\theta}}$, so that the inequality (V.1) reads:

$$u_{t'}(s') \leq u_t(s) + \frac{\lambda_t^*}{\pi^*(s)} \frac{\theta}{e^{-\frac{u_t(s)}{\theta}}} p_t(s)(x_{t'}(s') - x_t(s)).$$

Multiplying by $-(1/\theta)$ and taking exponential on both sides, we obtain

$$\begin{aligned} \hat{u}_{t'}(s') &\geq \hat{u}_t(s) \exp\left(-\frac{\lambda_t^*}{\pi^*(s)} \frac{1}{\hat{u}_t(s)} p_t(s)(x_{t'}(s') - x_t(s))\right), \\ &\geq \hat{u}_t(s) \left(1 - \frac{\lambda_t^*}{\pi^*(s)} \frac{1}{\hat{u}_t(s)} p_t(s)(x_{t'}(s') - x_t(s))\right), \end{aligned}$$

with $\hat{u}_{\tilde{t}}(\tilde{s}) = e^{-u_{\tilde{t}}(\tilde{s})/\theta}$ for all $(\tilde{s}, \tilde{t})$, where the second inequality follows from the convexity of the exponential function. Finally, defining $u_t^*(s)$ as $-e^{-u_t(s)/\theta}$, we obtain

$$u_{t'}^*(s') \leq u_t^*(s) + \frac{\lambda_t^*}{\pi^*(s)} p_t(s)(x_{t'}(s') - x_t(s)).$$

The data set is therefore consistent with the maximization of Savage preferences. It is easy to see that the converse is also true, so that a data set is consistent with the maximization of multiplier preferences if and only if it is consistent with the maximization of Savage preferences. This result is not entirely new and was already observed by Maccheroni, Marinacci and Rustichini (2006) and Strzalecki (2011). Yet we have chosen this more indirect proof to illustrate how seemingly different Afriat inequalities might actually be equivalent.

3.2 Smooth Ambiguity

The smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005) is another important class of preferences for decision making under uncertainty. This model captures the idea that a decision maker may form a range of predictions about future events and is uncertain about those predictions. As a concrete example, monetary authorities frequently use an array of models to predict future inflation and form assessments about the likelihood of each model to be "correct." The main feature of the smooth ambiguity model is to relax the reduction axiom of first and second order probabilities. According to this model, a decision maker evaluates an act $x \in \mathbb{X}$ as $\sum_{\pi_n} \phi(\sum_s u(x(s))\pi_n(s))\mu(\pi_n)$, where u is a utility function, π_n is a probability measure over S, μ is a probability measure over $\Delta(S)$, and ϕ is a real-valued function. We may think of $\mu(\pi_n)$ as the decision maker's subjective belief that the correct "model" is π_n. Much in the same way that the function u captures the attitude of the decision maker towards risk, the function ϕ captures the attitude towards ambiguity. In particular, a concave ϕ characterizes ambiguity aversion. Smooth ambiguity includes, as a limiting case under infinite ambiguity aversion, the maxmin expected utility model of Gilboa and Schmeidler (1989).

Accordingly, we say that the data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of "smooth ambiguity" preferences if there exist an increasing and concave function $u : X \to \mathbb{R}$, an increasing and concave function $\phi : \mathbb{R} \to \mathbb{R}$, a finite set of probability distributions $\Pi \subseteq \Delta(S)$ and a measure $\mu \in \Delta(\Pi)$, such that $\sum_{\pi \in \Pi} \phi\left(\sum_{s \in S} \pi(s) u(x_t(s))\right) \mu(\pi) \geq \sum_{\pi \in \Pi} \phi\left(\sum_{s \in S} \pi(s) u(x(s))\right) \mu(\pi)$ for all $x \in B(x_t, p_t)$, for all $t \in T$.

Theorem 2 *Let $(x_t, p_t)_{t \in T}$ be the data set. The following statements are equivalent:*

1. *The data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of "smooth ambiguity" preferences.*

2. *There exist a non-empty finite set N, $\Pi := \{\pi_n : n \in N\} \subset \Delta(S)$, $\mu \in \Delta(\Pi)$, and $(u_t, \phi_t, \rho_t, \lambda_t)_{t\in T}$, with $(u_t, \phi_t, \rho_t, \lambda_t) \in \mathbb{R}^{|S|} \times \mathbb{R}^{|N|} \times \mathbb{R}^{|N|}_{++} \times \mathbb{R}_{++}$ for each $t \in T$, such that*

$$u_{t'}(s') \leq u_t(s) + \lambda_t \frac{p_t(s)}{\sum_{n\in N} \pi_n(s)\rho_t(n)\mu(n)}(x_{t'}(s') - x_t(s)), \qquad \text{(S.1)}$$

for all $(s, s', t, t') \in S \times S \times T \times T$, and

$$\phi_{t'}(n') \leq \phi_t(n) + \rho_t(n) \sum_{s\in S}(\pi_{n'}(s)u_{t'}(s) - \pi_n(s)u_t(s)), \qquad \text{(S.2)}$$

for all $(n, n', t, t') \in N \times N \times T \times T$.

The intuition behind Theorem 2 is again simple. If the data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of "smooth ambiguity" preferences, then x_t is a solution to the problem $\max_{x\in B(p_t,x_t)} \sum_{\pi\in\Pi} \phi\left(\sum_{s\in S} \pi(s)u(x(s))\right)\mu(\pi)$, and consequently (S.1) results from the standard optimality conditions for convex problems. As for (S.2), it simply results from the concavity of ϕ, with the interpretation that $\rho_t(n)$ is the "derivative" of ϕ at $\sum_s u(x_t(s))\pi_n(s)$, the expected utility of x_t given the belief π_n. Conversely, if the conditions (S.1) and (S.2) hold true, we construct increasing, continuous, and concave functions u and ϕ such that $(u, \phi, \rho, \Pi, \mu)$ rationalize the data, as in Afriat (1967).

Note that if we define the probability π_t as

$$\pi_t(s) := \frac{\sum_{n\in N} \pi_n(s)\rho_t(n)\mu(n)}{\sum_{s\in S}\sum_{n\in N} \pi_n(s)\rho_t(n)\mu(n)},$$

for all $(s, t) \in S \times T$, and let

$$\lambda^*_t = \frac{\lambda_t}{\sum_{s\in S}\sum_{n\in N} \pi_n(s)\rho_t(n)\mu(n)},$$

then (S.1) is identical to (V.1), so that the observational differences between the smooth ambiguity model and the variational model are solely given by the conditions (V.2) and (S.2). Moreover, if the parameters ρ_t are independent of t (and thus the parameters π_t defined above are independent of t), then (V.2) and (S.2) trivially hold. This special case corresponds to the maximization of Savage preferences.

In general, suppose that we have found parameters that satisfy (S.1) (and hence (V.1)) and (S.2). This implies that there exist $(\phi_t)_t$ such that $\phi_{t'} \leq \phi_t + \chi_t \pi_t (u_{t'} - u_t)$ for all (t, t').[6] From Green and Srivastava (1986), a necessary and sufficient condition for the existence of such $(\phi_t)_t$ is that for any sequence $(t_0 = t, \ldots, t_{n+1} = t)$, $\sum_{i=1}^{n+1} \chi_{t_{i+1}} \pi_{t_{i+1}} (u_{t_i} - u_{t_{i+1}}) \geq 0$. Now, to satisfy (V.2) requires that we find $(c_t)_t$ such that $c_{t'} \geq c_t - (\pi_{t'} - \pi_t) u_t$ for all (t, t'). Again, from Green and Srivastava, this is equivalent to $\sum_{i=0}^{n} (\pi_{t_{i+1}} - \pi_{t_i}) u_{t_i} \geq 0$ for all sequences $(t_0 = t, \ldots, t_{n+1} = t)$. Noting that $\sum_{i=0}^{n} (\pi_{t_{i+1}} - \pi_{t_i}) u_{t_i} \geq 0$ is equivalent to $\sum_{i=1}^{n+1} \pi_{t_{i+1}} (u_{t_i} - u_{t_{i+1}}) \geq 0$, it follows that if we find parameters that satisfy (S.1) and (S.2) with the additional property that $\sum_{i=1}^{n+1} (\chi_{t_{i+1}} - 1) \pi_{t_{i+1}} (u_{t_i} - u_{t_{i+1}}) \leq 0$ for all sequences $(t_0 = t, \ldots, t_{n+1} = t)$, then we have found parameters that satisfy (V.1) and (V.2).

4. CONCLUDING REMARKS

This paper offers necessary and sufficient conditions for data sets composed of state-contingent prices and consumption to be consistent with two prominent models of decision making under ambiguity: variational preferences and smooth ambiguity. We emphasize that these restrictions are non-trivial and testable. We plan to implement our tests on the experimental data of Ahn *et al.* (2011), as soon as these data become available. We are currently working on the extension of our results to specific observational settings, e.g., markets for insurance and financial assets, and preliminary results suggest that our characterizations extend almost verbatim. We conclude with a series of remarks.

Concavity. We have assumed throughout that the utility function is concave. This is not without loss of generality. To see this, consider again the introductory example, but assume now that $B := \{(x_1, x_2, x_3) : x_1 + (4/5)x_2 + x_3 \leq 1\}$, i.e., the price of the consumption good in state s_3 has changed to 1. As in the

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[6] To see this, define $\chi_t := \sum_{s \in S} \sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)$, and $\phi_t := \sum_n \phi_t(n) \mu(n)$, multiply inequalities (S.2) by μ_n and sum over n.

introduction, the data set is inconsistent with a model of subjective expected utility. We further argue that it is also inconsistent with a model of maxmin expected utility if the utility function u is assumed to be concave. Without loss of generality, we normalize $u(0)$ to 0. Recall that since $(1,0,0)$ is chosen from B, we must have that $(1/3)u(1) \geq (1/3)u(x_1) + (2/3 - \pi)u(x_2) + \pi u(x_3)$ for any (x_1, x_2, x_3) such that $x_1 + (4/5)x_2 + x_3 = 1$. However, by monotonicity and concavity of u, for $(5/14, 5/14, 5/14)$, we have that $u(5/14) > u(1/3) \geq (1/3)u(1)$, and we obtain a contradiction. Yet, the data set is consistent with maxmin expected utility if we relax the concavity assumption. To see this, consider the piecewise linear utility function $u(x) = (9/10)x$ if $x \leq 5/9$ and $u(x) = (9/8)x - 1/8$ if $x > 5/9$. It is straightforward to verify, using linear programming techniques, that $(1,0,0)$ is a solution to the maximization problem $\max_{x \in B} \min_{\pi \in [0,2/3]} (1/3)u(x_1) + (2/3 - \pi)u(x_2) + \pi u(x_3)$, while $(0,1,1)$ is a solution to the maximization problem $\max_{x \in B'} \min_{\pi \in [0,2/3]} (1/3)u(x_1) + (2/3 - \pi)u(x_2) + \pi u(x_3)$. An open issue is to derive necessary and sufficient conditions without specific assumptions on the utility function u.

Errors. The conditions (V.1) and (V.2) (resp., (S.1) and (S.2)) provide *exact* tests for the hypothesis of maximization of variational preferences (resp., "smooth ambiguity" preferences), and do not allow for measurement errors, optimization errors, or computational limitations.[7] If the test fails, a simple method to evaluate the seriousness of the violations consists of finding the largest subsets of the data that are consistent and comparing their cardinality with the cardinality of the data set. Another method, suggested by Varian (1985), is to assume that the observed data set $(x_t, p_t)_{t \in T}$ is the true data set $(x_t^*, p_t^*)_{t \in T}$ perturbed by classical error terms $(\varepsilon_{x_t}, \varepsilon_{p_t})_{t \in T}$, i.e., $x_t^* = x_t + \varepsilon_{x_t}$ and $p_t^* = p_t + \varepsilon_{p_t}$ for each $t \in T$, and to minimize

$$\frac{1}{\sigma^2} \sum_{t \in T} ||\hat{x}_t - x_t||^2 + ||\hat{p}_t - p_t||^2,$$

subject to $(\hat{x}_t, \hat{p}_t)_{t \in T}$ satisfying conditions (V.1) and (V.2) (resp., (S.1) and

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[7] See Echenique, Golovin, and Wierman (2011) for a revealed preference approach to computational complexity.

(S.2)), where σ^2 is the variance of the error terms.[8] If the true data set $(x_t^*, p_t^*)_{t \in T}$ is consistent with (V.1) and (V.2) (resp., (S.1) and (S.2)) (the null hypothesis), then the resulting value of the minimization problem is a lower bound for the "true" statistic $(1/\sigma^2) \sum_{t \in T} ||x_t^* - x_t||^2 + ||p_t^* - p_t||^2$, and therefore provides a conservative test.[9]

General Choice Sets. In this paper, the decision maker chooses from classical (linear) budget sets. However, the analysis is not limited to budget sets and extends to more general choice sets. More precisely, suppose that the data set consists of $(x_t, X_t)_{t \in T}$, where $x_t \in X_t$, and X_t is a non-empty and convex subset of $\mathbb{R}^L$ for each t. From standard arguments in convex analysis, condition (V.1) becomes

$$u_{t'}(s') \leq u_t(s) + g_t(x_t)(x_{t'}(s') - x_t(s)),$$

with $g_t(x_t)$ an element of the normal cone of X_t at x_t, while (V.2) remains the same. We refer the reader to Forges and Minelli (2009) for more on this issue.

Axioms of Revealed Preference. This paper derives Afriat-like inequalities for a data set to be consistent with the maximization of variational or "smooth ambiguity" preferences. However, these inequalities are not stated purely in terms of the observables, i.e., state-contingent prices and consumption. An open issue is to find testable implications that only involve observables, as in the generalized axiom of revealed preference.

APPENDIX

A.1 Proof of Theorem 1

Proof: We first show that (1) $\implies$ (2). Suppose that the data set $(x_t, p_t)_{t \in T}$ is consistent with a model of variational preferences. Since u is increasing, this is equivalent to x_t being a solution to the following constrained optimization program: $\max_{x \in \mathbb{X}} \left(\min_{\pi \in \Delta(S)} \left(\sum_{s \in S} \pi(s) u(x(s)) + c(\pi) \right) \right)$ subject to $p_t \cdot x \leq \omega_t$

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[8] Here, $||\cdot||$ denotes the Euclidean norm.

[9] Since we assume classical error terms, the statistic has a chi-squared distribution.

for some $\omega_t \geq 0$ for each $t \in T$ (choose $\omega_t = p_t \cdot x_t$). For any $x \in \mathbb{X}$, let $U(x)$ denote the minimum over $\pi \in \Delta(S)$ of $\sum_{s\in S} \pi(s)u(x(s)) + c(\pi)$, i.e., $U(x) := \min_{\pi\in\Delta S} \left(\sum_{s\in S} \pi(s)u(x(s)) + c(\pi)\right)$, and let $\Pi^{\min}(x)$ denote the set of minimizers. Notice that $\Pi^{\min}(x) \neq \emptyset$ for all $x \in \mathbb{X}$ and that U is concave. Note that the maximization program is equivalent to $\max_{x\in\mathbb{X}} U(x) - \mathbf{1}_{B(p_t,x_t)}(x)$, where $\mathbf{1}_{B(p_t,x_t)}$ is the indicator function of the convex and closed set $B(p_t, x_t)$. Let $\partial U(x)$ be the super-differential of U at x. From the optimality of x_t, it follows that there exists a scalar $\lambda_t > 0$ and a vector $\delta_t \in \mathbb{R}_+^{L|S|}$ such that $\lambda_t p_t - \delta_t \in \partial U(x_t)$ for each $t \in T$, with $\delta_t^\ell(s) = 0$ if $x_t^\ell(s) > 0$ (i.e., if the quantity consumed of the ℓ-th good in state s is strictly positive).

For each $s \in S$, define $u_s : \mathbb{X} \to \mathbb{R}$ with $u_s(x) = u(x(s))$. We have that $U(x) = \sum_{s\in S} u_s(x)\pi_s$ with $\pi \in \Pi^{\min}(x)$. From Theorem 4.4.2 in Hiriart-Urruty and Lemaréchal (2004, p. 189), we have that

$$\partial U(x) = \text{co}\left\{\sum_{s\in S} \pi(s) \cdot g_s(x) : g_s(x) \in \partial u_s(x), \pi \in \Pi^{\min}(x)\right\},$$

where $\partial u_s(x)$ denotes the super-differential of u_s at x. It follows that $\lambda_t p_t(s) - \delta_t(s) \in \{\pi_t(s) \cdot g(x_t(s)) : g(x_t(s)) \in \partial u(x_t(s)), \pi_t \in \Pi^{\min}(x_t)\}$. Concavity of u then implies that

$$u(x_{t'}(s')) \leq u(x_t(s)) + \lambda_t \frac{p_t(s)}{\pi_t(s)}(x_{t'}(s') - x_t(s)), \tag{1}$$

for all $(s, s', t, t') \in S \times S \times T \times T$, with π_t an element of $\Pi^{\min}(x_t)$ at x_t. Lastly, since π_t is an element of $\Pi^{\min}(x_t)$ at x_t, we have that $\sum_{s\in S} \pi_t(s)u(x_t(s)) + c(\pi_t) \leq \sum_{s\in S} \pi_{t'}(s)u(x_t(s)) + c(\pi_{t'})$ for all $(t, t') \in T \times T$. Letting $u_t(s) := u(x_t(s))$ for any $(s, t) \in S \times T$ and $c_t := c(\pi_t)$ for any $t \in T$ completes the first part of the proof.

Conversely, we next show that (2) $\implies$ (1). Assume that there exist $(u_t, \pi_t, c_t, \lambda_t)_{t\in T} \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_+ \times \mathbb{R}_{++}$, such that (V.1) and (V.2) are satisfied. Define $u : X \to \mathbb{R}$ as follows:

$$u(x) := \min_{(s,t)\in S\times T} \left(u_t(s) + \lambda_t \frac{p_t(s)}{\pi_t(s)}(x - x_t(s))\right). \tag{2}$$

Note that u is increasing and concave. Also, we have that $u(x_t(s)) = u_t(s)$ for each $(s, t) \in S \times T$. Clearly, we have that $u(x_t(s)) \leq u_t(s)$. Assume that

$u_t(s) > u(x_t(s))$. Since $u(x_t(s)) = u_{t^*}(s^*) + \lambda_{t^*}\frac{p_{t^*}(s^*)}{\pi_{t^*}(s^*)}(x_t(s) - x_{t^*}(s^*))$ for some $(s^*, t^*) \in S \times T$, we have a contradiction with (V.1). Similarly, define $c : \Delta(S) \to \mathbb{R}$ as follows:

$$c(\pi) := \max_{t\in T}\left(c_t - \sum_{s\in S} u_t(s)(\pi(s) - \pi_t(s))\right). \tag{3}$$

Note that c is convex and continuous. Also, we have that $c(\pi_t) = c_t$ for each $t \in T$. Clearly, we have that $c(\pi_t) \geq c_t$. Assume that $c(\pi_t) > c_t$. Since $c(\pi_t) = c_{t^*} - \sum_{s\in S} u_{t^*}(s)(\pi_t(s) - \pi_{t^*}(s))$ for some $t^* \in T$, we have a contradiction with (V.2). To ground c, subtract $\min_{\pi\in\Delta(S)} c(\pi)$, which is well-defined by continuity of c and compactness of $\Delta(S)$. With a slight abuse of notation, we also denote by c the grounded function.

Finally, we want to show that for each $t \in T$, when $p_t \cdot x_t \geq p_t \cdot x$, it then follows that $\min_{\pi\in\Delta(S)}\left(\sum_{s\in S}\pi(s)u(x_t(s)) + c(\pi)\right) \geq \min_{\pi\in\Delta(S)}\left(\sum_{s\in S}\pi(s)u(x(s)) + c(\pi)\right)$. We have that for each $t \in T$,

$$\begin{aligned}
\min_{\pi\in\Delta(S)}\left(\sum_{s\in S}\pi(s)u(x(s)) + c(\pi)\right) &\leq \sum_{s\in S}\pi_t(s)u(x(s)) + c(\pi_t) &(4)\\
&\leq \sum_{s\in S}\pi_t(s)u_t(s) + c(\pi_t)\\
&\quad + \lambda_t\sum_{s\in S}p_t(s)(x(s) - x_t(s)) &(5)\\
&\leq \sum_{s\in S}\pi_t(s)u_t(s) + c(\pi_t) &(6)\\
&= \sum_{s\in S}\pi_t(s)u_t(s) + c_t &(7)\\
&\leq \min_{\pi\in\Delta(S)}\left(\sum_{s\in S}\pi(s)u_t(s) + c(\pi)\right) &(8)\\
&= \min_{\pi\in\Delta(S)}\left(\sum_{s\in S}\pi(s)u(x_t(s)) + c(\pi)\right). &(9)
\end{aligned}$$

Inequality (4) follows from minimization; inequality (5) follows from the definition of u in (2); inequality (6) follows from the assumption that for each $t \in T$, $p_t \cdot x_t \geq p_t \cdot x$, which is equivalent to $\sum_{s\in S} p_t(s)(x(s) - x_t(s)) \leq 0$, and $\lambda_t > 0$; equality (7) follows from the previous argument that $c(\pi_t) = c_t$ for each $t \in T$; inequality (8) follows since the definition of c in (3) implies that

$\sum_{s\in S}\pi_t(s)u_t(s)+c_t \le \sum_{s\in S}\pi(s)u_t(s)+c(\pi)$ for all $\pi\in\Delta(S)$ and $t\in T$; and equality (9) follows from the previous argument that $u_t(s)=u(x_t(s))$ for each $(s,t)\in S\times T$. This completes the proof. **QED**

A.2 Proof of Theorem 2

Proof: We first show that (1) $\implies$ (2). Suppose that the data set $(x_t,p_t)_{t\in T}$ is consistent with the smooth ambiguity model. Since u and ϕ are increasing, this is equivalent to x_t being a solution to the following constrained optimization program: $\max_{x\in\mathbb{X}}\left(\sum_{\pi\in\Pi}\phi\left(\sum_{s\in S}\pi(s)u(x(s))\right)\mu(\pi)\right)$ subject to $p_t\cdot x\le\omega_t$ for some $\omega_t\ge 0$ for each $t\in T$ (choose $\omega_t=p_t\cdot x_t$). Let $U_\pi(x):=\sum_{s\in S}\pi(s)u(x(s))$ for each $x\in\mathbb{X}$ and $\pi\in\Pi$. Since u is concave, U_π is also concave for each $\pi\in\Pi$. Since ϕ is concave, it follows that the super-differential of $\phi\circ U_\pi$ at x is given by $\{\rho\cdot g:\rho\in\partial\phi(U_\pi(x)),g\in\partial U_\pi(x)\}$. (See Theorem 4.3.1 in Hiriart-Urruty and Lemaréchal (2004, p. 186).) From the optimality of x_t, it follows that there exist $\lambda_t>0$ for each $t\in T$, $g(x_t(s))\in\partial u(x_t(s))$ for each $s\in S$ and $t\in T$, and $\rho_\pi(x_t)\in\partial\phi(U_\pi(x_t))$ for each $\pi\in\Pi$ and $t\in T$, such that $\lambda_t p_t(s)=g(x_t(s))\sum_{\pi\in\Pi}\pi(s)\rho_\pi(x_t)\mu(\pi)$ for each $s\in S$ and $t\in T$. (Here, we implicitly assume that $x_t>>0$. The general case is treated as in the proof of Theorem 1.) Concavity of u then implies that

$$u(x_{t'}(s'))\le u(x_t(s))+\lambda_t\frac{p_t(s)}{\sum_{\pi\in\Pi}\pi(s)\rho_\pi(x_t)\mu(\pi)}(x_{t'}(s')-x_t(s)),\tag{10}$$

for all $(s,s',t,t')\in S\times S\times T\times T$. Similarly, concavity of ϕ implies that

$$\phi(U_{\pi'}(x_{t'}))\le\phi(U_\pi(x_t))+\rho_\pi(x_t)(U_{\pi'}(x_{t'})-U_\pi(x_t)),\tag{11}$$

for all $(n,n',t,t')\in N\times N\times T\times T$. Letting $u_t(s):=u(x_t(s))$ for any $(s,t)\in S\times T$, $\pi_n(s):=\pi(s)$ for any $(\pi,s)\in\Pi\times S$, $\phi_t(n):=\phi(U_\pi(x_t))$ and $\rho_t(n):=\rho_\pi(x_t)$ for any $(\pi,t)\in\Pi\times T$, and $\mu(n):=\mu(\pi)$ for any $\pi\in\Pi$ completes the first part of the proof.

Conversely, we next show that (2) $\implies$ (1). Assume that there exist $\Pi:=\{\pi_n\}_{n\in N}\subset\Delta(S)$, $\mu\in\Delta(\Pi)$, and $(u_t,\phi_t,\rho_t,\lambda_t)_{t\in T}\in\mathbb{R}^{|S|}\times\mathbb{R}^{|N|}\times\mathbb{R}^{|N|}_{++}\times\mathbb{R}_{++}$, such that (S.1) and (S.2) are satisfied. Define $u:X\to\mathbb{R}$ as follows:

$$u(x):=\min_{(s,t)\in S\times T}\left(u_t(s)+\lambda_t\frac{p_t(s)}{\sum_{n\in N}\pi_n(s)\rho_t(n)\mu(n)}(x-x_t(s))\right).\tag{12}$$

Note that u is increasing and concave. Also, we have that $u(x_t(s)) = u_t(s)$ for each $(s,t) \in S \times T$. Clearly, we have that $u(x_t(s)) \leq u_t(s)$. Assume that $u_t(s) > u(x_t(s))$. Since $u(x_t(s)) = u_{t^*}(s^*) + \lambda_{t^*} \frac{p_{t^*}(s^*)}{\sum_{n \in N} \pi_n(s^*)\rho_{t^*}(n)\mu(n)}(x_t(s) - x_{t^*}(s^*))$ for some $(s^*, t^*) \in S \times T$, we have a contradiction with (S.1). Similarly, define $\phi : \mathbb{R} \to \mathbb{R}$ as follows:

$$\phi(U) := \min_{(n,t) \in N \times T} \left(\phi_t(n) + \rho_t(n) \left(U - \sum_{s \in S} \pi_n(s) u_t(s) \right) \right). \tag{13}$$

Note that ϕ is increasing and concave. Also, we have that $\phi\left(\sum_{s \in S} \pi_n(s) u_t(s)\right) = \phi_t(n)$ for each $(n,t) \in N \times T$. Clearly, we have that $\phi\left(\sum_{s \in S} \pi_n(s) u_t(s)\right) \leq \phi_t(n)$. Assume that $\phi_t(n) > \phi\left(\sum_{s \in S} \pi_n(s) u_t(s)\right)$. Since $\phi\left(\sum_{s \in S} \pi_n(s) u_t(s)\right) = \phi_{t^*}(n^*) + \rho_{t^*}(n^*)\left(\sum_{s \in S}(\pi_n(s) u_t(s) - \pi_{n^*}(s) u_{t^*}(s))\right)$ for some $(n^*, t^*) \in N \times T$, we have a contradiction with (S.2). Finally, we want to show that for each $t \in T$, when $p_t \cdot x_t \geq p_t \cdot x$, it then follows that $\sum_{\pi \in \Pi} \phi\left(\sum_{s \in S} \pi(s) u(x_t(s))\right) \mu(\pi) \geq \sum_{\pi \in \Pi} \phi\left(\sum_{s \in S} \pi(s) u(x(s))\right) \mu(\pi)$. We have that for each $t \in T$,

$$u(x(s)) - u(x_t(s)) \leq \lambda_t \frac{p_t(s)}{\sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)} (x(s) - x_t(s)), \tag{14}$$

which follows from the definition of u in (12) and the previous argument that $u(x_t(s)) = u_t(s)$ for each $(s,t) \in S \times T$. We have that for each $n \in N$ and $t \in T$,

$$\begin{aligned} \phi\left(\sum_{s \in S} \pi_n(s) u(x(s))\right) \leq \phi & \left(\sum_{s \in S} \pi_n(s) u(x_t(s))\right) \\ & + \rho_t(n) \sum_{s \in S} \pi_n(s)(u(x(s)) - u(x_t(s))), \end{aligned} \tag{15}$$

which follows from the definition of ϕ in (13) and the previous arguments that $u(x_t(s)) = u_t(s)$ for each $(s,t) \in S \times T$ and $\phi\left(\sum_{s \in S} \pi_n(s) u_t(s)\right) = \phi_t(n)$ for each $(n,t) \in N \times T$. Together, (14) and (15) imply that

$$\begin{aligned} \phi\left(\sum_{s \in S} \pi_n(s) u(x(s))\right) \leq \phi & \left(\sum_{s \in S} \pi_n(s) u(x_t(s))\right) \\ & + \lambda_t \sum_{s \in S} \frac{\pi_n(s) \rho_t(n)}{\sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)} p_t(s)(x(s) - x_t(s)), \end{aligned} \tag{16}$$

for each $n \in N$ and $t \in T$. Multiplying both sides of (16) by $\mu(n)$ and summing over $n \in N$, we obtain

$$\sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x(s))\right)\mu(n) \leq \sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x_t(s))\right)\mu(n) + \lambda_t \sum_{s\in S} p_t(s)(x(s) - x_t(s)). \tag{17}$$

Finally, if for each $t \in T$, $p_t \cdot x_t \geq p_t \cdot x$, which is equivalent to $\sum_{s\in S} p_t(s)(x(s) - x_t(s)) \leq 0$, then

$$\sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x(s))\right)\mu(n) \leq \sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x_t(s))\right)\mu(n). \tag{18}$$

Letting $\mu(\pi_n) := \mu(n)$ for each $n \in N$ and $\Pi := \{\pi_n\}_{n\in N}$ completes the proof. **QED**

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