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Abstract

I model a single-club economy with heterogeneous consumers as an aggregative game. I give a sufficient condition, normality of demand for the club good in full income, for the existence and uniqueness of a Nash equilbrium by the Cornes-Hartley (2007) method. Then, confining attention to club quality functions that are homogeneous in the investment in the club facility and the aggregate usage of the club, I examine when the sufficient condition is satisfied. I show that, under common assumptions on the utility function, this occurs for all positive degrees of homogeneity.

Keywords: Nash equilibrium; heterogeneous clubs; aggregative game; homogeneous function; existence; uniqueness

JEL Classifications: C7; D1; D5; H4

1 Introduction

Many treatments of Nash equilibrium existence and uniqueness in the voluntary provision of a pure public good now exist in various contexts.[1] Allouch (2012), building on Bramoullé and Kranton (2007, 2011), extends this analysis to a public good on a network. Much less attention has been given to equilibrium existence and uniqueness in models of other shared goods: either impure or "joint product" public goods (Cornes and Sandler,1984, 1996), or clubs (Buchanan, 1965). However, Kotchen (2007) exploits the equivalence between the "joint product" model and Andreoni's (1986, 1990) "warm-glow giving" model, together with the Cornes and Hartley (2007) "aggregative games" approach, to show equilibrium existence and uniqueness simply in the impure public good model. I here show equivalence between the impure public good

*The idea for this paper arose from a conversation the author had with Richard Cornes and Roger Hartley when he presented a paper at the University of Keele circa 2002.

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[1] For example, see Bergstrom, Blume and Varian (BBV; 1986, 1992), Fraser (1992), Nett and Peters (1993), Andreoni and Bergstrom (1996), Cornes, Hartley and Sandler (1999) and Cornes and Hartley (2007).

model and a class of single-club models (Fraser 2000; Konishi, 2010). I use this and the aggregative games approach to show equilibrium existence and uniquess in a single-club model with heterogeneous consumers. As with all the existence and uniquess results with shared goods, my result follows from a normality assumption. I explore if this assumption might be satisfied in the club context. Confining attention to club models with quality (or "congestion") functions homogeneous in the investment in the club facility and total usage of the club, I show that, under plausible assumptions, normality holds if and only if such functions are homogeneous of positive degree.

2 The Model

There is a single numeraire private good and a single club supplying a single club good. The private good is essential, so anyone with income will consume it, but the club good is not and does not satisfy an Inada condition. There are a finite number, N, of consumers. Consumers have continuous quasi-concave utility functions, $u^i(x, v, c)$, $i = 1, ..., N$, defined over and strictly increasing in x, private good consumption, v, club good consumption, and c, the club's quality (or "congestion"). They have fixed incomes, m_i, $i = 1, ..., N$. The club's quality as perceived by an arbitrary individual i depends on the total spending on the club facility, y, and the total usage of the club, V (often called total visits), given by the quality function, $c^i(y, V)$, satisfying $c_1^i > 0$, $c_2^i < 0$. If v_i and x_i are, respectively, the arbitrary ith person's non-negative club usage and private good consumption, s/he has utility $u^i\left(x_i, v_i, c^i(y, V)\right)$. By the notational conventions of the public good literature, let $V = V_{-i} + v_i$, $V_{-i} = \sum_j v_j - v_i$ being the total club usage of all bar individual i. I follow that literature in assuming that I can choose units so that a unit of the club facility can be purchased at unit cost. I further assume that the club good provision breaks even. So, if V is the total club usage, then spending on the club facility is also $y = V$.[2]

I study price-taking equilibria in which individuals facing unit prices for the club and private goods choose utility maximising quantities to maximise utility subject to their budget, taking others' club consumption as given:

$$\begin{aligned} &\underset{x_i \geq 0, v_i \geq 0}{Max} u^i\left(x_i, v_i, c^i\left(V_{-i} + v_i, V_{-i} + v_i\right)\right) && \text{(a)} \\ &s.t.\ m_i = x_i + v_i && \text{(b)} \end{aligned} \qquad (1)$$

Using a standard trick, I rewrite i's budget constraint in "full income" form as $m_i + V_{-i} = x_i + v_i + V_{-i}$, i.e., $m_i + V_{-i} = x_i + V$. Using this in the consumer's optimisation to substitute out the budget constraint yields the problem

$$\underset{V \geq V_{-i}}{Max}\ u^i\left(m_i + V_{-i} - V, V - V_{-i}, c^i(V, V)\right) \qquad (2)$$

This maximand is similar to that Kotchen (2007) derived for the impure public good model. But, a crucial difference between that model and a club

[2]None of our results would change if we did not normalise to have unit prices.

model is that a club good is assumed to be (costlessly) excludable. An individual can only consume what he pays for. If he does not buy any, so $v_i = V - V_{-i} = 0$, then the third argument of his utility function should be 0 also:

$$u^i \left(x_i, 0, c^i (V, V)\right) = u^i (x_i, 0, 0) \tag{3}$$

This assumption, made explicitly in the clubs literature (e.g., cf. Berglas, Helpman and Pines (1982)), is perfectly intuitive. But, it is easy to see, it is only guaranteed for a continuous function u^i of the form $u^i \left(x_i, v_i, c^i (y, V)\right) \equiv u^{*i} \left(x_i, h^i (v_i) g^i (y, V)\right)$, for continuous non-negative functions $h^i (v_i)$ and $g^i (y, V)$ with $h^i (0) = 0$. These must satisfy $h^{i/} (v_i) > 0, g^i_1 > 0, g^i_2 < 0$, with $u^{*i}_1 > 0$, $u^{*i}_2 > 0$ to conform with the earlier assumptions on the partial derivatives of u^i and c^i.[3] I assume this form for u^i hereon. The consumer's maximand is transformed once more to a continuous, quasi-concave and differentiable function

$$\underset{V \geq V_{-i}}{Max}\ u^{*i} \left(m_i + V_{-i} - V, h^i (V - V_{-i}) g^i (V, V)\right) \tag{4}$$

Following Bergstrom, Blume and Varian (1986), the solution to (4) is the club's total usage the consumer desires, denoted V_i, given by a continuous best reply function f_i satisfying $V_i = f_i (m_i + V_{-i}, V_{-i}) \equiv f^*_i (V_{-i})$. Then $V_i - V_{-i} = f^*_i (V_{-i}) - V_{-i} = v_i \geq 0$.

3 The Existence and Uniqueness of An Equilibrium Among Club Members

I make the following standard normality assumption at interior solutions:

$$1 > f^{*\prime}_i(V_{-i}) = \partial V_i / \partial V_{-i} > 0 \tag{5}$$

So, as total club usage by others increase (and, at fixed m_i, as does full income), an arbitrary club user's demands for both the club and private goods increase. Given this, I can now show equilibrium existence and uniqueness among club members via Cornes and Hartley's "aggregative games" approach, which much simplifies the analysis. So, I follow them to introduce a *replacement function*. For the arbitrary individual, this is denoted $r_i(V)$ and defined implicitly by

$$f^*_i(V - r_i(V)) = V \tag{6}$$

[3] Berglas, Helpman and Pines (1982) need utility to satisfy (3) for their result that everyone should be included in the club in a single-club economy with identical individuals. They then assume a utility function with the club component taking the multiplicative form as in our text in their numerical example. Fraser (2000, 2005) gives a rationalisation for this form. He shows that it is necessary and sufficient for efficiency to be independent from distribution in the tprovision of a club good when the quality function $c^j(y, V)$ is homogeneous of degree 0. Clearly, if utility is not of this form, there is a discontinuity at $v = 0$ if the same utility function applies to consumption inside and outside the club.

Cornes and Hartley and Kotchen motivate $r_i(V)$ and show that $r_i\prime(V) = 1 - \left[f_i^*\prime(V - r_i(V))\right]^{-1} \in (-1, 0)$ by normality.

With a public good, pure or impure, it is natural to assume that each individual will contribute something to providing it if no-one else does, given the good is assumed essential. With a non-essential club good, this is not so. Let $\underline{v}_i = f_i(m_i, 0) = f_i^*(0) = f_i^*(\underline{v}_i - r_i(\underline{v}_i))$ be the amount of the club facility individual i will supply and use if no-one else supplies and uses any. This could be 0 but we assume consumers with $\underline{v}_i > 0$ exist. Let $\underline{V} = Max\{\underline{v}_i\}$ and define the aggregate replacement function by $R(V) = \sum_i r_i(V)$. Clearly, as the sum of continuous decreasing functions, $R(V)$ is also continuous and decreasing. Also, $\sum_i \underline{v}_i \geq R(V) \geq \underline{V}$ and $R\prime(V) \leq 0$. A Nash equilibrium is a total usage of the club, V^*, such that consumers' best response to that in aggregate, given by the sum of their replacement functions at V^*, satisfies $R(V^*) = V^*$.

Proposition 1 *Suppose the normality assumption holds. Then a unique Nash equilibrium in usage of the club exists.*

Proof. This follows from $R(V)$ being continuous and strictly decreasing on its domain to its minimum value ($\underline{V}$). So there is an unique fixed point. ∎

4 Will Normality Prevail?

Equilibrium existence and uniqueness is predicated on the normality assumption. Will normality hold? To see, I examine the arbitrary consumer's behaviour more closely. First, who will join the club? At arbitrary club usage by others of V_{-i}, the ith person joins if their utility is increased by doing so - i.e., if:[4]

$$-u_1^{i*}(m_i, 0) + u_2^{i*}(m_i, 0)\, h^i\prime(0)\, g^i(V_{-i}, V_{-i}) > 0 \tag{7}$$

Conversely, that person will not join if the reverse holds - i.e., if

$$-u_1^{i*}(m_i, 0) + u_2^{i*}\left(m_i^i, 0\right) h^i\prime(0)\, g^i(V_{-i}, V_{-i}) \leq 0 \tag{8}$$

It follows that, in equilibrium, arbitrary person i is a non-joiner if $-u_1^{i*}(m_i, 0) + u_2^{i*}(m_i, 0)\, h^i\prime(0)\, g^i(V^*, V^*) \leq 0$ holds. They will have joined if

$$\begin{gathered} -u_1^{i*}\left(m_i + V_{-i} - V^*, h^i(V^* - V_{-i})\, g^i(V^*, V^*)\right) \\ +u_2^{i*}\left(m_i + V_{-i} - V^*, h^i(V^* - V_{-i})\, g^i(V^*, V^*)\right) \text{x} \\ \left\{h^i\prime(V^* - V_{-i})\, g^i(V^*, V^*) + h^i(V^* - V_{-i})\left(g_1^i(V^*, V^*) + g_2^i(V^*, V^*)\right)\right\} = 0 \end{gathered} \tag{9}$$

and $V^* - V_{-i} = v_i^* > 0$.[5] How increasing full income affects demand for the club good is found by differentiating through (9) with respect to V_{-i} and (omitting function arguments, indices and asterisks in u^{i*} for brevity) rearranging to

[4] The next two inequalities characterising the joiners and non-joiners are from the first derivative of utility at $v_i = 0$, $-u_1^i(m_i, 0) + u_2^i(m_i, 0)\left\{h^i\prime(0)\, g^i(V_{-i}, V_{-i}) + h^i(0)\left(g_1^i(V_{-i}, V_{-i}) + g_2^i(V_{-i}, V_{-i})\right)\right\}$ and using $h^i(0) = 0$.

[5] Conditions on incomes, u, h, and g can be found to ensure there are joiners and non-joiners. However, for brevity, such issues are not pursued here.

$$\partial V^*/\partial V_{-i} = N/D \tag{10}$$

where

$$\begin{array}{c} N \equiv u_{11} - u_{12}gh\prime - (u_{21} - u_{22}gh\prime)\left(gh\prime + h\left(g_1 + g_2\right)\right) + u_2\left(gh\prime\prime + h\prime\left(g_1 + g_2\right)\right) \\ D \equiv u_{11} - u_{12}gh\prime - u_{12}h\left(g_1 + g_2\right) + \\ + \left\{gh\prime + h\left(g_1 + g_2\right)\right\}\left\{-u_{12} + u_{22}\left(gh\prime + \left(g_1 + g_2\right)\right)\right\} \\ +u_2\left(gh\prime\prime + 2h\prime\left(g_1 + g_2\right) + h\left(g_{11} + 2g_{12} + g_{22}\right)\right) < 0 \end{array} \tag{11}$$

Here $D < 0$ from the second-order condition for the maximisation. I assume $\partial V^*/\partial V_{-i} \geq 0$, so $N \leq 0$: an increase in full income does not decrease the total usage. This does not preclude $\partial v_i^*/\partial V_{-i} = \partial V^*/\partial V_{-i} - 1 < 0$.

4.1 Homogeneous quality functions g^i

It is difficult to put bounds on the daunting expression for $\partial V^*/\partial V_{-i}$ in general. However, insight is available in special cases. Suppose $g_i(V,V)$ is homogeneous in its arguments for all i. As is well known, homogeneity is a convenient intuitive device allowing insight into the consequences of returns to scale. Here it is "qualitative returns": the impact on the club's perceived quality of equiproportonate changes in investment in its facility and in its usage.

If $g\left(V,V\right)$ is homogeneous of arbitrary degree δ ("h.o.d.δ") in (V,V), Euler's Theorem on homogeneous functions states

$$\begin{array}{lr} g_1\left(V,V\right)V + g_2\left(V,V\right)V = \delta g\left(V,V\right) & \text{(a)} \\ g_{i1}\left(V,V\right)V + g_{i2}\left(V,V\right)V = \left(\delta - 1\right)g_i\left(V,V\right),\ i = 1,2 & \text{(b)} \end{array} \tag{12}$$

So,

$$\begin{array}{c} g_{11}\left(V,V\right) + 2g_{12}\left(V,V\right) + g_{22}\left(V,V\right) \\ = \left(\delta - 1\right)\frac{\left[g_1(V,V) + g_2(V,V)\right]}{V} = \left(\delta - 1\right)\delta\frac{g(V,V)}{V^2} \end{array} \tag{13}$$

Using (12)-(13), N and D can be rewritten as

$$\begin{array}{c} N = u_{11} - u_{12}gh\prime - (u_{21} - u_{22}gh\prime)\left(gh\prime + h\delta g/V\right) + gu_2\left(h\prime\prime + h\prime g/V\right) \\ D = u_{11} - u_{12}g\left(h\prime + h\frac{\delta}{V}\right) + g\left(h\prime + h\frac{\delta}{V}\right)\left\{gu_{22}\left(h\prime + h\frac{\delta}{V}\right) - u_{21}\right\} \\ +gu_2\left(h\prime\prime + 2h\prime\frac{\delta}{V} + h\left(\delta - 1\right)\frac{\delta}{V^2}\right) \end{array} \tag{14}$$

Comparing N and D, D contains the following extra terms:

$$\begin{array}{c} E \equiv -u_{12}\delta\frac{gh}{V} + g\left(h\prime + \frac{\delta h}{V}\right)\delta\frac{gh}{V}u_{22} + \delta g\frac{u_2}{V}\left(h\prime + \frac{\delta h}{V}\right) - \delta g\frac{u_2 h}{V^2} \\ = \underbrace{-u_{12}\delta\frac{gh}{V}}_{(a)} + \underbrace{\delta\frac{g}{V}\left(h\prime + \frac{\delta h}{V}\right)}_{(b)}\underbrace{\left(ghu_{22} + u_2\right)}_{(c)}\underbrace{-\delta g\frac{u_2 h}{V^2}}_{(d)} \end{array} \tag{15}$$

Writing (9) compactly using (12), $-u_1 + gu_2\left(h\prime + \frac{\delta h}{V}\right) = 0$. So, $g\left(h\prime + \frac{\delta h}{V}\right) > 0$. I now make the following common comparative static assumptions:

(A.1) Utility functions are supermodular: $u_{12} \geq 0$;(A.2) $-ghu_{22}/u_2 \geq 1$.

Given (A.1)-(A.2), in (15): (i) $(c) \leq 0$; (ii) either $(b)(c) = 0$, or $(b)(c) \{<, =, >\} 0$ as $\delta \{>, =, <\} 0$; (iii) either $(a) = 0$ or $(a) \{<, =, >\} 0$ as $\delta \{>, =, <\} 0$; (iv) $(d) \{<, =, >\} 0$ as $\delta \{>, =, <\} 0$. Hence: (v) $E \{<, =, >\} 0$ as $\delta \{>, =, <\} 0$. Using (v) and (14), $|N| \{<, =, >\} |D|$ as $\delta \{>, =, <\} 0$. Using (15), (14) and (10), we have the following result:

Proposition 2 *Given (A.1)-(A.2), if utility functions are h.o.d.δ,* $(0 <) \partial V^* / \partial V_{-i} \{<, =, >\} 1$ *as* $\delta \{>, =, <\} 0$.

Corollary 3 *Given (A.1)-(A.2), $\delta > 0$ for all consumers is sufficient, but unnecessary, for a unique Nash equilibrium in club membership to exist.*

Proof. This follows from Propositions 1 and 2. ■

An interesting case is the "knife-edge" one where $\delta = 0$. That $\partial V^* / \partial V_{-i} = 1$ then is seen directly from inspecting (9). Using (12)(a), this reduces to

$$\begin{array}{c} u_2^i \left(m_i^i + V_{-i} - V^*, h^i \left(V^* - V_{-i} \right) g^i \left(V^*, V^* \right) \right) h^{i\prime} \left(V^* - V_{-i} \right) g^i \left(V^*, V^* \right) \\ -u_1^i \left(m_i + V_{-i} - V^*, h^i \left(V^* - V_{-i} \right) g^i \left(V^*, V^* \right) \right) = 0 \end{array} \tag{16}$$

If V_{-i} increases and there is an exactly offsetting increase in V^*, no magnitude in (16) changes. $V^* - V_{-i}$ is unchanged, as is $g^i \left(V^*, V^* \right)$ (because of zero homogeneity). So (16) continues to be satisfied. Intuition for the other cases is also simple. Take $\delta > 0$. Then, if V_{-i} increases, g^i increases at unchanged v_i and subutility from the club, $g^i h^i$, also increases. So, i rebalances consumption by decreasing v_i and increasing x_i. Though $\partial v_i / \partial V_{-i} < 0$, so $\partial h^i \left(v_i \right) / \partial V_{-i} < 0$, $g^i h^i$ can then still increase if $1 > \partial V^* / \partial V_{-i} = 1 + \partial v_i / \partial V_{-i} > 0$, thus both components of utility increase in the process.

5 Conclusion

I have shown the equivalence of a heterogeneous member, single-club model and the "joint production" impure public good model. I have used this equivalence and Cornes and Hartley's "aggregative games" approach to show that the club good's normality is sufficient for Nash equilibrium existence and uniqueness in the club model. When consumers' club quality functions are homogeneous, I show that normality holds if these functions are homogeneous of any positive degree. To the best of my knowledge, Konishi (2010) is the only other general analysis of Nash equilibrium existence and uniqueness in a heterogeneous club model. There are important differences between our models. Unlike me, he assumes quasilinear utility, with the club good being essential and satisfying an Inada condition. He also assumes a fixed membership. In my model, as the club good is not essential, membership is determined endogenously to be just those who choose to buy the club good in equilibrium. However, he considers non-linear (two-part) pricing; I only consider linear pricing. Clearly, having established conditions for existence and uniqueness in a heterogeneous club model with equivalences to the canonical impure public good model, there is a foundation for exploring other properties of this model in further research.

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