University of Leicester

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On the utility representation of asymmetric single-peaked preferences*

Francisco Martínez-Mora, University of Leicester, UK
M. Socorro Puy, Universidad de Málaga, ES

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Asymmetric single-peaked preferences and their utility representation*

Francisco Martínez-Mora†
University of Leicester and FEDEA

M. Socorro Puy‡
Universidad de Málaga

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Abstract

The symmetry of single-peaked preferences is a widely used assumption. When the space of alternatives has a meaningful metric, such restriction is only justified for its analytical convenience. This paper analyzes how to deal with asymmetric preferences in a tractable way. First, we introduce two types of asymmetric preferences (*shortfall* and *excess avoidance*), provide sufficient conditions for preferences to be of one type or the other and a coefficient that measures the degree of asymmetry. Second, we define the family of *generalized distance-metric utility functions* that represents any asymmetric preferences maintaining analytical tractability and we compare it to previous proposals.

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†E-mail: fmm14@leicester.ac.uk. Website: http://www.le.ac.uk/ec/fmm14/
‡E-mail: mps@uma.es. Website: http://sites.google.com/site/socorropuy/

1 Introduction

Models of political economics commonly assume that preferences of voters and policy-makers are single-peaked with respect to a linear order of the policy alternatives. In other words, they assume that agents have a single bliss point and that deviations from it in any direction generate increasing utility losses. Single-peaked preferences have usually been restricted to be symmetric about the bliss point. That is to say, utility losses from symmetric deviations above and below the peak have been assumed identical. In fact, endowing agents with quadratic or euclidean preferences over policy alternatives is the standard assumption in models of political competition, monetary policy making or in the literature on fiscal response to foreign-aid.[1] That is not problematic when the space of policy alternatives does not have a specific and meaningful metric. In other cases, for example when the policy alternatives correspond to levels of spending, taxation or interest rates, analytical convenience is the only justification for the symmetry assumption.

There are indeed important reasons that call such simplification into question in the context of economic policy analysis. On the one hand, it has been shown that symmetric single-peaked preferences cannot be deduced from a standard utility-maximization problem with a well-behaved primitive utility function and a linear constraint (Milyo, 2000). On the other hand, an increasing body of theoretical and empirical literature points towards the relevance of relaxing the symmetry restriction in a number of research areas. In the context of monetary policy, Blinder (1997) notes that: "*Academic macroeconomists tend to use quadratic loss functions for reason of mathematical convenience, without thinking much about their substantive implications. The assumption is not innocuous, [...] practical central bankers and academics would benefit from more serious thinking about the functional form of the loss function*".

This paper offers a formal analysis of asymmetric single-peaked preferences. We believe that our results will be useful in the analysis of economic policy-making and, in general, in any setting where the policy space has

[1] Among the political economy models that consider symmetric preferences: Enelow and Hinich (1982), Ansolabehere and Snyder (2000), Groseclose (2001), Aragonés and Palfrey (2002) who analyze the effect of voters preferences on candidates' personal characteristics; Palfrey (1983), Osborne and Slivinsky (1996) who study the the strategic entry of candidates; Calvert (1985), Bernhardt, Duggan and Squintani (2007) who account for candidates' uncertainty about the location of the median voter.

a specific metric and single-peakedness is an adequate modelling assumption. Within the family of single-peaked preferences, we introduce two basic types of asymmetric preferences: those that show *shortfall avoidance* and those displaying *excess avoidance*.[2] We say that preferences exhibit shortfall avoidance when they favour alternatives above the peak over their symmetric counterparts; in the opposite case, we say that they show excess avoidance.

Our contribution is double-fold. On the one hand, we provide sufficient conditions for the underlying preferences to show *shortfall avoidance* or *excess avoidance,* and identify others that allow the comparison of degrees of asymmetry. Interestingly, we find that the coefficient of prudence proposed by Kimball (defined by the third derivative divided by the second derivative of the utility specification) provides a measure of the asymmetry of preferences. On the other hand, we define a rich family of utility functions, *the generalized distance-metric utility functions,* that can not only represent the two basic types of preference-bias, accommodating any degree of asymmetry, but also more complex asymmetries.[3] Most importantly, our specification inherits the analytical tractability of the distance-functions, including smoothness, for which we believe it could be fruitfully exploited in future theoretical and empirical research. We then compare our proposal to other utility functions that have been used in the literature to represent asymmetric preferences. In particular, we consider the linex function proposed by Varian (1974), and further studied by Zellner (1986), and the piecewise asymmetric function proposed by Waud (1976). We show that, while these proposals are convenient in that the direction and degree of asymmetry depend on a single parameter, they are limited with respect to the asymmetries they can represent.

There are a number of research areas where asymmetric preferences have been found to play critical roles. For example, following Blinder's suggestion, Ruge-Murcia (2003) develops a theoretical model of monetary policy-making which endows the central bank with an asymmetric loss function and finds that some important results (such as the linear relation between unemployment and the average inflation deviation from the target) do not extend to the asymmetric case. Moreover, he provides empirical evidence supporting the existence of a non-symmetric objective function. Along similar lines, Dolado et al. (2004) show that asymmetric preferences are theoretically im-

[2] This terminology is adapted from the one introduced by Cukierman and Muscatelli (2008). These authors refer to inflation avoidance and recession avoidance preferences.

[3] For instance, our proposal can capture cases where preferences display excess avoidance in some parts of the policy space and shortfall avoidance in others.

portant, as they imply the existence of a non-linear monetary policy rule, and provide empirical evidence of such non-linearities in the US. Exploiting the same implication of asymmetric preferences, Surico (2007) offers evidence supporting that the preferences of the Fed were asymmetric with respect to target values before 1979, when output contractions triggered stronger responses than output expansions. Surico stated that " [...] *potential evidence of asymmetries in the central bank objective may be interpreted as evidence of asymmetries in the representative agent's utility.*"

Likewise, within the literature on fiscal response to foreign-aid, Heller (1975) and Feeny (2006) highlight the relevance of policymakers' asymmetric single-peaked preferences over deviations with respect to target spending or tax revenues. We conclude this brief and non-exhaustive review of the literature by recalling the work of Waud (1976) who, in a study of economic policy-making under uncertainty, revealed that the optimal response of the policy-maker to changes in uncertainty crucially hinges on the symmetry or asymmetry of preferences.[4]

The remaining of the paper is organized as follows. The next section presents the environment and definitions. Section 3 studies conditions on the utility representation that guarantee the emergence of asymmetric single-peaked preferences and compares preferences in terms of their degrees of asymmetry. Section 4 proposes the generalized distance-metric utility representation. Section 5 analyzes other utility specifications. Section 6 concludes.

2 The environment and definitions

An agent has preferences defined over alternatives in $\mathbb{R}$.[5] The space of alternatives we consider has a specific and meaningful metric; thus, it is not possible to alter the spatial location of the alternatives.[6] The preference R of the agent on the set of alternatives is a complete preorder. The set of

[4] Prospect theory also accounts for asymmetric preferences, though in a different sense. Following empirical and experimental evidence, this literature assumes that agents are more sensitive to losses than to gains with respect to a reference point (Kahneman and Tversky, 1991; Benartzi and Thaer 1995). In contrast to our analysis, the reference point is not a maximizer.

[5] All our results extend to cases where the set of alternatives is a subset of $\mathbb{R}$. See Appendix.

[6] See Eguia (2010) for an analysis that endogeneizes the spatial representation of the set of alternatives.

complete preorders on the set of alternatives is $\mathcal{R}$.

The strict and indifference preference relations induced by R are denoted by P and I respectively. Given $R \in \mathcal{R}$, the peak of R, when it exists, is an alternative $e \in \mathbb{R}$ strictly preferred to any other alternative. Let e^* denote the peak of R.

Definition 1 *A preference $R \in \mathcal{R}$ satisfies single-peakedness (SP) if there exists a peak of R, and for all $d_0, d_1 > 0$ such that $d_0 < d_1$, $e^* - d_0 \; P \; e^* - d_1$ and $e^* + d_0 \; P \; e^* + d_1$.*

The SP property of preferences requires that, at each side of the peak, alternatives located closer to the peak are preferred to those located further away from it. The set of complete preorders satisfying SP on the set of alternatives is denoted by $\mathcal{R}^{SP}$. We refer to a generic element of $\mathcal{R}^{SP}$ as a SP preference.

Every alternative below the peak can be associated to another alternative above the peak according to the indifference preference relation. In what follows, we interpret $d > 0$ as a deviation off the peak. We next define a function that assigns to every deviation below the peak, a deviation above the peak according to the indifference preference relation.

Definition 2 *The preference-bias function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ associated to $R \in \mathcal{R}^{SP}$ assigns to every deviation d, the corresponding deviation $\delta(d)$ for which $e^* - d \; I \; e^* + \delta(d)$.*

Observe that every preference-bias function associated to a SP preference is a strictly increasing function. We propose three basic types of SP preferences. These are the symmetric SP preferences, the shortfall avoidance preferences and the excess avoidance preferences.

Definition 3 *We say that the preference $R \in \mathcal{R}^{SP}$ is symmetric when $\delta(d) = d$ for all d. We say that the preference $R \in \mathcal{R}^{SP}$ shows shortfall avoidance when $\delta(d) > d$ for all d. We say that the preference $R \in \mathcal{R}^{SP}$ shows excess avoidance when $\delta(d) < d$ for all d.*

Symmetric SP preferences induce indifference between alternatives located symmetrically around the peak. When a SP preference shows shortfall avoidance, the comparison between two alternatives symmetrically located at each side of the peak is such that the alternative located above the peak is higher in the preference ordering than the alternative located below the peak. Of course, it is the opposite when a SP preference shows excess avoidance. The proposed types of SP preferences do not fully classify the set $\mathcal{R}^{SP}$. Our aim, however, is to capture two natural and meaningful types of asymmetries in the set of SP preferences.[7]

Because the preference-bias function is independent of the location of the peak, one can compare degrees of asymmetry between pairs of preferences even when their respective peaks do not coincide. The following definition establishes the binary relation *more avoidance than* on $\mathcal{R}^{SP}$.

Definition 4 *Let $R_1, R_2 \in \mathcal{R}^{SP}$ with δ_1, δ_2 and e_1^*, e_2^* denoting their respective preference-bias functions and peaks. We say that R_2 shows more shortfall avoidance than R_1 when $\delta_1(d) \leq \delta_2(d)$ for all d (or equivalently, R_1 shows more excess avoidance than R_2).*

Thus, according to this definition, degrees of asymmetry are comparable across pairs of preference relations when their associated preference-bias functions are such that one is always above the other, i.e., $\delta_1(d) \leq \delta_2(d)$ for all d. Observe, therefore, that the proposed binary relations are partial preorders (transitive but not complete) on the sets of SP preferences which show shortfall avoidance or excess avoidance.[8]

3 Conditions on the utility representation

Every preference $R \in \mathcal{R}^{SP}$ can be represented by a strictly quasi-concave utility function $V : \mathbb{R} \to \mathbb{R}$ which has a maximizer. The maximizer (or peak of R) satisfies:

$$e^* = \arg\max V(e) \tag{1}$$

In what follows, we consider differentiable utility representations so that the peak of R satisfies $V'(e^*) = 0$.

[7] In particular, these definitions do not cover SP preferences which show shortfall avoidance in some ranges of the domain and excess avoidance in others. It would of course be possible to generalize them to account for any SP preference in $\mathcal{R}$.

[8] Again, the definition could be generalized to compare degrees of asymmetry in specific intervals of the domain of d.

3.1 Conditions for asymmetric SP preferences

In terms of the utility representation, when the SP preference shows shortfall avoidance, then $V(e^* - d) < V(e^* + d)$ for all d, and when the SP preference shows excess avoidance, in turn, $V(e^* - d) > V(e^* + d)$ holds for all d. Figure 1 depicts two examples of utility functions representing shortfall avoidance and excess avoidance respectively.

For SP preferences, $V'(e) > 0$ holds for alternatives below the peak, whereas $V'(e) < 0$ is true for alternatives above the peak.

A characterization of symmetric single-peaked preferences in terms of the properties of the utility representation V is straightforward to establish. Preferences are symmetric if and only if $V'(e^* - d) = -V'(e^* + d)$ for all d.[9] That is to say, if and only if marginal utility at every pair of symmetric deviations above and below the peak coincide. Similarly, certain properties of the slope of V guarantee that preferences show shortfall avoidance or excess avoidance.

Proposition 1 *Let V be the utility representation of $R \in \mathcal{R}^{SP}$.*
Properties (1a) or (2a) guarantee that R shows shortfall avoidance:
(1a) $V'(e^* - d) > -V'(e^* + d)$ for all d
(2a) V' strictly convex for all $e \in \mathbb{R}$
where (2a) $\Rightarrow$(1a).
Properties (1b) or (2b) guarantee that R shows excess avoidance:
(1b) $V'(e^* - d) < -V'(e^* + d)$ for all d
(2b) V' strictly concave for all $e \in \mathbb{R}$
where (2b) $\Rightarrow$(1b).

Proof. The following claims prove our statement.[10]

Claim 1: (1a) $\Rightarrow \delta(d) > d$ for all d.
Proof of Claim 1: Utility loss derived from reducing e^* to $e^* - d$ is measured by $\int_{e^*-d}^{e^*} V'(e)de$, whereas utility loss from increasing e^* to $e^* + d$ is measured by $-\int_{e^*}^{e^*+d} V'(e)de$. By (1a), $\int_{e^*-d}^{e^*} V'(e)de > -\int_{e^*}^{e^*+d} V'(e)de$ for all d. Solving for the integral and simplifying $V(e^* - d) < V(e^* + d)$ for all d, i.e., preferences show shortfall avoidance.

[9] This result is derived by differentiating $V(e^* - d) - V(e^* + d) = 0$ with respect to d.

[10] The analogous statements on preferences showing excess avoidance can be proved following a similar reasoning (that we omit in the interest of brevity).

Claim 2: (2a) $\Rightarrow$(1a)
Proof of Claim 2**:** By strict convexity of V' we have $V'(e^*) < \frac{V'(e^*-d)+V'(e^*+d)}{2}$ for all d. By SP, $V'(e^*) = 0$, and substituting in the inequality yields $V'(e^* - d) > -V'(e^* + d)$ for all d. $\blacksquare$

By conditions (1a) and (1b), the comparison of the slopes of the utility representation of SP preferences at every symmetric deviation with respect to the peak reveals the direction of the asymmetry. Conditions (2a) and (2b) also reveal the direction of the asymmetry by checking whether the marginal utility function is strictly concave or strictly convex.[11]

3.2 Conditions for comparing degrees of asymmetry

Figure 2 depicts two utility representations of preferences $R_1, R_2 \in \mathcal{R}^{SP}$ where R_2 (represented by V_2) shows more excess avoidance than R_1 (represented by V_1). We put together both utility representations around their respective peaks in order to compare their associated preference-biased function.

Next, we show that the comparison between the slopes of different utility representations indicates the strength of the asymmetry across different preferences. For each pair of preferences $R_1, R_2 \in \mathcal{R}^{SP}$, we denote by e_1^*, e_2^* their respective peaks.

Proposition 2 *Let V_1, V_2 be two utility representations of $R_1, R_2 \in \mathcal{R}^{SP}$, respectively. If $V_2'(e_2^* + \gamma) \leq V_1'(e_1^* + \gamma)$ for all $\gamma \in \mathbb{R}$, then R_2 shows more excess avoidance than R_1(or equivalently, R_1 shows more shortfall avoidance than R_2).*

Proof. Condition $V_2'(e_2^* - d) \leq V_1'(e_1^* - d)$ for all d implies that the utility loss derived from reducing e_2^* to $e_2^* - d$, which is measured by $\int_{e_2^*-d}^{e_2^*} V_2'(e)de = B$, in comparison to the utility loss derived from reducing e_1^* to $e_1^* - d$, which is measured by $\int_{e_1^*-d}^{e_1^*} V_1'(e,)de = A$, is such that $B \leq A$. By definition of the preference-bias function, there is $\delta_1 > 0$ for which $A = \int_{e_1^*}^{e_1^*+\delta_1} V_1'(e)de$, at the

[11]For a concrete application of conditions (2a) and (2b) see Cukierman and Muscatelli (2008) where central banks' preferences show excess avoidance over inflation and shortfall avoidance over output. As shown by these authors, this asymmetry of preferences is not innocuous as it induces a non-linear Taylor rule.

same time, and given that $0 > V_1'(e_1^* + d) \geq V_2'(e_2^* + d)$ for all d,

$$\int_{e_2^*}^{e_2^*+\delta_1} V_2'(e)de \geq A. \tag{2}$$

By definition of the preference-bias function there is $\delta_2 > 0$ for which

$$\int_{e_2^*}^{e_2^*+\delta_2} V_2'(e)de = B. \tag{3}$$

Because $B \leq A$, conditions (2) and (3) imply $\int_{e_2^*}^{e_2^*+\delta_2} V_2'(e)de \leq \int_{e_2^*}^{e_2^*+\delta_1} V_2'(e)de$ from where we derive that $\delta_2(d) \leq \delta_1(d)$ for all d. ■

In other terms, the proposed sufficient condition for R_2 to show more excess avoidance than R_1 implies that $V_2'(e_2^*-d) \leq V_1'(e_1^*-d)$ and $V_2'(e_2^*+d) \leq V_1'(e_1^* + d)$, for all d; i.e., below the peak, equal-distance deviations generate more disutility with V_1 than with V_2, and above the peak, equal-distance deviations generate more disutility with V_2 than with V_1.

The degree of asymmetry can also be compared using the degree of concavity or convexity of the marginal utility function. For this, one of the marginal utility specifications must be obtained as an increasing transformation of the other.

Proposition 3 *Let V_1, V_2 be strictly concave utility representations of $R_1, R_2 \in \mathcal{R}^{SP}$. If V_1', V_2' are strictly concave and $\frac{V_2'''(e)}{V_2''(e)} \geq \frac{V_1'''(e-\Lambda)}{V_1''(e-\Lambda)}$ for all $e \in \mathbb{R}$ where $\Lambda = e_2^* - e_1^*$, then R_2 shows more excess avoidance than R_1.*[12]

Proof. By strict concavity of V_1 and V_2, the functions V_1' and V_2' are strictly decreasing functions and they can be related by a strictly increasing transformation g such that $V_2'(e) = g(V_1'(e - \Lambda))$ where $\Lambda = e_2^* - e_1^*$. This implies that when $e = e_2^*$, $g(0) = 0$.[13] Differentiating the expression,

$$V_2''(e) = g'(V_1'(e - \Lambda))V_1''(e - \Lambda)$$
$$V_2'''(e) = g''(V_1'(e - \Lambda)) \left[V_1''(e - \Lambda)\right]^2 + g'(V_1'(e - \Lambda))V_1'''(e - \Lambda).$$

[12] Similarly, if V_1', V_2' are strictly convex and $-\frac{V_2'''(e)}{V_2''(e)} \geq -\frac{V_1'''(e-\Lambda)}{V_1''(e-\Lambda)}$ for all $e \in \mathbb{R}$, then R_2 shows more shortfall avoidance than R_1.

[13] If $e_2^* = e_1^*$, then $\Lambda = e_2^* - e_1^* = 0$, and it is possible to compare degrees of asymmetry just by comparing $\frac{V_1'''(e)}{V_1''(e)}$ to $\frac{V_2'''(e)}{V_2''(e)}$.

From where

$$\frac{V_2'''(e)}{V_2''(e)} = \frac{V_1'''(e-\Lambda)}{V_1''(e-\Lambda)} + \frac{g''(V_1'(e-\Lambda))V_1''(e-\Lambda)}{g'(V_1'(e-\Lambda))}.$$

By strict concavity of V_1' and V_2', $V_1''' < 0$ and $V_2''' < 0$. Then, $\frac{V_2'''(e)}{V_2''(e)} \geq \frac{V_1'''(e-\Lambda)}{V_1''(e-\Lambda)}$ implies $g''(V_1'(e-\Lambda)) \leq 0$ (g concave).
By definition of δ_2, it follows that $V_2(e_2^* - d) = V_2(e_2^* + \delta_2(d))$, or equivalently, $\int_{e_2^*-d}^{e_2^*+\delta_2(d)} V_2'(e)de = 0$. Substituting function g,

$$0 = \int_{e_2^*-d}^{e_2^*+\delta_2(d)} V_2'(e)de = \int_{e_2^*-d}^{e_2^*+\delta_2(d)} g(V_1'(e-\Lambda))de.$$

By concavity of g, $\int_{e_2^*-d}^{e_2^*+\delta_2(d)} g(V_1'(e-\Lambda))de \leq g(\int_{e_2^*-d}^{e_2^*+\delta_2(d)} V_1'(e-\Lambda)de)$, where $\int_{e_2^*-d}^{e_2^*+\delta_2(d)} V_1'(e-\Lambda)de = V_1(e_1^*+\delta_2(d)) - V_1(e_1^*-d)$. Since g is strictly increasing and $g(0) = 0$, then $V_1(e_1^* + \delta_2(d)) - V_1(e_1^* - d) \geq 0$. Since $V_1' < 0$ for all $e > e_1^*$, and by definition of $\delta_1(d)$, $V_1(e_1^* - d) = V_1(e_1^* + \delta_1(d))$, we deduce that $\delta_1(d) \geq \delta_2(d)$ for all d. ■

This proposition reveals an analogy between our analysis and the theories developed by Arrow (1971), Pratt (1964) and Kimball (1990). In Arrow-Pratt's theory of risk aversion, concavity of the utility function over consumption indicates the presence of risk aversion, while according to Kimball's theory of precautionary savings, concavity of the marginal utility function entails precautionary saving behavior. In each case, the degree of concavity of the utility function or the degree of concavity of the marginal utility function measures risk aversion or precautionary savings respectively.[14] These behavioral traits become thus comparable across pairs of concave functions such that one is a concave transformation of the other. In our context, as long as V is strictly concave, the curvature of the marginal utility function determines the degree of asymmetry. We can therefore apply the coefficient of prudence proposed by Kimball (1990) to measure whether preferences show more shortfall avoidance for the *more convex* marginal utility representation, and more excess avoidance for the *more concave* marginal utility representation.

[14] Arrow-Pratt's coefficient of risk aversion is defined by $-\frac{u''}{u'}$, whereas Kimball's coefficient of prudence is defined by $-\frac{u'''}{u''}$ where u measures utility over private consumption.

Definition 5 *Let V be strictly concave utility representation of $R \in \mathcal{R}^{SP}$. If V' is strictly concave, we refer to the ratio $\frac{V'''(e)}{V''(e)}$ as the coefficient of excess avoidance. If V' is strictly convex, we refer to the ratio $-\frac{V'''(e)}{V''(e)}$ as the coefficient of shortfall avoidance.*

The proposed coefficients of asymmetry can be classified into three types: constant, increasing or decreasing in e. Thus, we say that V satisfies constant excess avoidance when $\frac{V'''(e)}{V''(e)}$ is constant for all $e \in \mathbb{R}$, and we say that V satisfies increasing (decreasing) excess avoidance when it is increasing (decreasing) in e. Likewise, we say that V satisfies constant shortfall avoidance when $-\frac{V'''(e)}{V''(e)}$ is constant for all $e \in \mathbb{R}$; and we say that V satisfies increasing (decreasing) shortfall avoidance when it is increasing (decreasing) in e.

4 Asymmetric SP utility representation

In this section, we show that a generalization of any distance-metric utility function allows for the utility representation of any asymmetric SP preference relation.

A *distance-metric utility function* is defined by $V(e) = -f(e - e^*)$ where f is a continuous and strictly increasing distance function between the peak e^* and the alternative e. Particular examples of f are the quadratic function, in which $f(e - e^*) = [e - e^*]^2$, or the distance function induced by a norm, in which $f(e - e^*) = \|e - e^*\|$, or any function $f(e - e^*) = |e - e^*|^\lambda$ where $\lambda > 0$.

Given a preference $R \in \mathcal{R}^{SP}$, the preference-bias function δ associated to R assigns to each deviation below the peak, the deviation above the peak for which the agent is indifferent. Let $\delta^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ be the inverse of the preference-bias function.[15] The *generalized distance-metric utility function* is defined by

$$V(e) = \begin{cases} -f(e - e^*) & \text{when } e \leq e^* \\ -f(\delta^{-1}(e - e^*)) & \text{when } e > e^* \end{cases}.$$

Below the peak, the proposed utility function coincides with the distance-metric utility function. In order to capture the preference-bias, the level

[15] Because the preference satisfies the SP condition, the preference-bias function δ is biyective and it has an inverse.

of utility derived from any alternative above the peak is equal to the corresponding distance-metric utility value at its indifferent alternative below the peak. If the SP preference relation is symmetric, i.e., $\delta^{-1}(d) = d$, the generalized distance-metric utility function collapses to the distance-metric utility function. In the Appendix we specify the generalized distance-metric utility function for the case in which the set of alternatives is bounded.

Arguably, an advantage of the generalized distance-metric utility function is that it is derived from the set of indifferent alternatives of the primitive preference relation. Another is that it maintains the tractability of the distance functions (including smoothness if the distance function is smooth). Moreover, it can accommodate every SP preference, including those which satisfy our definition of shortfall and excess avoidance.

Theorem 1: *Every preference $R \in \mathcal{R}^{SP}$ can be represented by the generalized distance-metric utility function. Furthermore, this utility specification can be used to compare pairs of preferences such that one of them shows more shortfall avoidance (or less excess avoidance) than the other.*

Proof. First, we show that every preference relation $R \in R^{SP}$ is represented by the generalized distance-metric utility function. The preference ordering across alternatives located at the same side of the peak is captured by the distance-metric utility function below the peak, and above the peak, by a function that is strictly decreasing in distance (given that δ^{-1} is a strictly increasing function in all its domain). The preference ordering of pairs of alternatives located at opposite sides of the peak can be deduced by identifying the pairs of alternatives yielding equal utility. Thus, $V(e^* + \delta(d)) = -f\left(\delta^{-1}(e^* + \delta(d) - e^*)\right)$ and simplifying $V(e^* + \delta(d)) = -f(d)$. Since $V(e^* - d) = -f(d)$, we deduce that $e^* - d \; I \; e^* + \delta(d)$ for all d. Second, we show that the generalized distance-metric utility function can be used to compare degrees of asymmetries across pairs of preferences. Suppose that $\delta_1(d) \leq \delta_2(d)$ for all d. Then, because δ is always a strictly increasing function, $\delta_1(d) \leq \delta_2(d)$ implies that $\delta_1^{-1}(d) \geq \delta_2^{-1}(d)$. Plugging this inequality into the generalized distance-metric utility function we obtain that $V_2'(e^* + d) \leq V_1'(e^* + d)$ for all d. In addition, $V_2'(e^* - d) = V_1'(e^* - d)$ for all d.[16] We deduce, using Proposition 2, that R_2 shows more shortfall avoidance than R_1. ■

For instance, if we take a linear preference bias-function $\delta(d) = kd$ with

[16] Observe that, off the peak, all the proposed distance functions are differentiable.

$k > 0$, the corresponding generalized distance-metric utility function is

$$V(e) = \begin{cases} -f(e - e^*) & \text{when } e \leq e^* \\ -f(\frac{e-e^*}{k}) & \text{when } e > e^* \end{cases}.$$

Under this utility specification, $k > 1$ represents a particular class of SP preferences that show shortfall avoidance, while $k < 1$ corresponds to another class of SP preferences that exhibit excess avoidance.

Of course, the preference-bias function does not need to be linear; we can think of diverse functional forms yielding different asymmetric SP preferences. Thus, it is possible to represent more complex asymmetries than the basic types previously defined. For instance, we can represent preferences that display excess avoidance for some values of d, and shortfall avoidance for others.[17]

Our proposal accommodates every continuous and strictly increasing distance metric function. For instance, a rich family of utility specifications are given by:

$$V(e) = \begin{cases} -\left|e - e^*\right|^\lambda & \text{when } e \leq e^* \\ -\left|\delta^{-1}(e - e^*)\right|^\lambda & \text{when } e > e^* \end{cases}$$

where different values of $\lambda > 0$ yield different utility functions. It is straightforward to check that these functions are smooth, i.e. that they are continuously differentiable in all the domain, including the peak.

5 Other asymmetric SP utility representations

In this section we analyze other utility representations of asymmetric SP preferences in the literature and compare them to our proposal. We study the linex loss function proposed by Varian (1974) and further studied by Zellner (1986), and a simple piecewise asymmetric function proposed by Waud (1976).

[17] A preference-bias function that creates such kind of asymmetry is $\delta(d) = [d(1-a)]^\beta$, where $a \in [0, 1)$, $\beta > 0$ and $\beta \neq 1$; β determines whether preferences exhibit excess avoidance near the peak and shortfall avoidance from some distance onwards, or viceversa. In particular, if $\beta < 1$, then $\delta(d) \geq d$, $\forall d \leq d^*$ and $\delta(d) < d$, $\forall d > d^*$ so that preferences show shortfall avoidance below d^* and excess avoidance above d^*. The opposite case occurs when $\beta > 1$, which implies that $\delta(d) \leq d$ $\forall d \leq d^*$, and $\delta(d) > d$ $\forall d > d^*$. The threshold distance is given by $d^* = (1-a)^{\frac{\beta}{1-\beta}}$.

5.1 The linex loss function

The linex (for linear-exponential) loss function has been used to represent policy makers' preferences over inflation and output gap (e.g. Ruge-Murcia, 2003; Dolado et al, 2004; Surico, 2007) as well as in Bayesian econometrics (Zellner, 1986) and optimal forecasting (Christoffersen and Diebold, 1997). According to our notation, this utility specification is defined by:

$$V(e) = -\exp(\alpha\,[e^* - e]) + \alpha\,[e^* - e] + 1$$

where $\alpha \in \mathbb{R}$. Observe that this specification yields $V(e^*) = 0$.

Solving for the first, second and third derivatives we obtain:

$$\begin{aligned} V' &= \alpha \exp(\alpha\,[e^* - e]) - \alpha \\ V'' &= -\alpha^2 \exp(\alpha\,[e^* - e]) \\ V''' &= \alpha^3 \exp(\alpha\,[e^* - e]) \end{aligned}$$

The second derivative guarantees concavity of the utility specification. By Proposition 1, the sign of the third derivative implies that the linex utility function represents shortfall avoidance when $\alpha > 0$ and excess avoidance when $\alpha < 0$. Additionally, using Proposition 3, we can obtain the coefficient of asymmetry, which is equal to $-\alpha$ when there is excess avoidance and to α when there is shortfall avoidance. Note that the coefficient of asymmetry of the linex function is constant.

Therefore, according to our results, the linex function is able to represent preferences with the two types of asymmetry (shortfall and excess avoidance). Moreover, given that its coefficient of asymmetry is constant, it is easy to represent and compare different degrees of asymmetry just by changing or comparing the value of α. This utility representation, however, cannot represent every preference $R \in \mathcal{R}^{SP}$. For instance, it cannot represent preferences which show excess avoidance for some values of d, and shortfall avoidance for others. Likewise, it cannot represent preferences with increasing or decreasing avoidance.

5.2 The piecewise asymmetric function of Waud

Waud (1976) proposes another utility representation of asymmetric preferences. Using our notation, this utility specification is defined by:

$$V(e) = \begin{cases} -\lambda f(e^* - e) & \text{when } e \leq e^* \\ -f(e - e^*) & \text{when } e > e^* \end{cases} \tag{4}$$

where f is a strictly increasing and convex function, $f(0) = 0$ and $\lambda > 0$. Observe that $\lambda < 1$ represents preferences showing shortfall avoidance, while $\lambda > 1$ corresponds to preferences showing excess avoidance. Additionally, by Proposition 2, when $\lambda > 1$, the higher the coefficient λ, the more excess avoidance the preferences show (likewise, when $\lambda < 1$, the smaller the coefficient λ, the more shortfall avoidance the preferences display). Thus, the function proposed by Waud can represent preferences of the two basic asymmetric types (shortfall and excess avoidance). Furthermore, it allows for comparisons of degrees of asymmetry just by checking the value of λ. Nevertheless, this utility representation cannot represent every preference $R \in \mathcal{R}^{SP}$.

By the definition of the preference-bias function, $V(e^* - d) = V(e^* + \delta(d))$ for all d, which implies $\lambda = \frac{f(\delta(d))}{f(d)}$ for all d. Given a convex function f, that condition imposes a restriction on the preference-bias function and thereby on the preferences it can represent. To illustrate such restriction, let f be the quadratic function; then, the preference-bias function must satisfy $\delta(d) = \lambda^{\frac{1}{2}} d$; i.e., δ has to be linear in d. This example illustrates that this utility specification cannot represent every preference $R \in \mathcal{R}^{SP}$. Additionally, there are preference-bias functions for which no function f exists such that Expression (4) represents these preferences (consider again the case of preferences which show excess avoidance for some values of d, and shortfall avoidance for others).

6 Conclusion

Symmetric single-peaked preferences are usually represented by distance-metric utility functions. Our analysis revealed that such utility specifications can be easily transformed to accommodate single-peaked preferences with any direction and degree of asymmetry, including the two basic types (shortfall and excess avoidance) as well as more complex asymmetries. Our utility specification, which we named the *generalized distance-metric utility function*, maintains the analytical tractability and smoothness of the distance functions. Another advantage is that it is directly derived from the set of indifferent alternatives of the primitive preference relation. Our proposal can be used to capture the preferences of policy-makers, investors, politicians, the media or, in general, in any context where the policy space has a meaningful metric and where single-peakedness is a relevant modelling assumption. This is for instance the case in the models studied by Blinder (1997), Ruge-Murcia

(2003), Feeny (2006) or Dolado et al. (2004) among others. We believe our proposal could also be fruitfully exploited in Bayesian econometrics and in optimal forecasting, as it would allow to incorporate richer asymmetries in the loss function.

On the other hand, we identified sufficient conditions on standard utility representations that reveal the direction (if any) of the asymmetry of preferences and another that allows the comparison of degrees of asymmetry across different preference relations. We found that an analogous of Kimball's coefficient of prudence can be used to measure the degree of asymmetry of preferences and, hence, to compare degrees of shortfall or excess avoidance. Finally, we analyzed two proposals of asymmetric utility representations – the linex function and the piecewise asymmetric function of Waud (1976). We explained that the sign and degree of asymmetry of these utility specifications depend on a single parameter, for which they are able to represent preferences exhibiting different degrees of shortfall or excess avoidance. For the same reason and in contrast to our proposal, however, these utility (or loss) functions are limited in the range of single-peaked preferences they can represent.

Appendix

If the set of alternatives were bounded, as for instance it is the case when $e \in [0, \bar{e}]$, we could also define the distance-metric utility function. For that, suppose that we take $\bar{e}$ sufficiently large as to guarantee that there exists an alternative $\tilde{e} \leq \bar{e}$ such that $0 \; I \; \tilde{e}$, i.e., such that the agent is indifferent between 0 and $\tilde{e}$. This implies that $e^* \in (0, \tilde{e})$. The preference-bias function is then defined so that $\delta : [0, e^*] \to [0, \tilde{e} - e^*]$.[18] The inverse of the preference-bias function δ^{-1} is then bounded above by $\delta^{-1}(\tilde{e} - e^*) = e^*$, which corresponds to the indifference relation $0 \; I \; \tilde{e}$. We can extend the domain of δ^{-1} to every $d \in (\tilde{e} - e^*, \bar{e} - e^*]$ in a strictly increasing way. Thus, the *generalized distance-metric utility function* is defined by

$$V(e) = \begin{cases} -f(e - e^*) & \text{when } e \in [0, e^*] \\ -f(\delta^{-1}(e - e^*)) & \text{when } e \in (e^*, \overline{e}] \end{cases}.$$

[18] The assumption that $0 \; I \; \tilde{e}$, $\tilde{e} \in (e^*, \bar{e}]$ is made only to simplify the exposition. Alternatively, if there exists an alternative $\widehat{e} \in [0, e^*)$ such that $\overline{e} \; I \; \widehat{e}$ then the preference-bias function is defined on $\delta : [0, e^* - \widehat{e}] \to [0, \overline{e} - e^*]$ and the analysis remains unchanged.

Observe that the utility derived from alternatives $e \in (\tilde{e}, \bar{e}]$ is below $V(\tilde{e})$ and that the utility function is strictly decreasing in $e \in (\tilde{e}, \bar{e}]$. If we take a linear preference bias-function $\delta(d) = kd$ with $k > 0$, the corresponding generalized distance-metric utility function would be

$$V(e) = \begin{cases} -f(e - e^*) & \text{when } e \in [0, e^*] \\ -f(\frac{e-e^*}{k}) & \text{when } e \in (e^*, \bar{e}] \end{cases} .$$

Note that this function extends, in a natural way, the domain of δ^{-1} to alternatives in the interval $e \in (\tilde{e}, \bar{e}]$.

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