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DISCOUNTING BY INTERVALS: AN INCONSISTENT THEORY OF INTERTEMPORAL CHOICE?

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Discounting by intervals: An inconsistent theory of intertemporal choice?

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Abstract

We show that the theory developed in Scholten and Read (2006) "Discounting by Intervals: A Generalized Model of Intertemporal Choice", Management Science, 52, 1424-1436, is an inconsistent theory. We suggest a way the inconsistency can be removed.

Keywords: Intertemporal choice, non-additive time discounting..

JEL Classification Codes: C60(General: Mathematical methods and programming); D91(Intertemporal consumer choice).

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Bertrand Russell: An inconsistent theory is useless, because from it we can prove anything.

Heckler: Starting from $4 = 5$, can you prove you are the Bishop of Rome?

Russell: Subtract 3 from each side to get $1 = 2$. The Bishop of Rome and I are two people, two equals one, therefore I am the Bishop of Rome.

1. Introduction

In a clear and beautifully written paper, Scholten and Read (2006) give an account of recent experimental results, including their own experiments, that shed further light on intertemporal choice. In particular, their experimental results support the hypothesis that time discounting is non-additive. They also propose a new discounting function, the *discounting by intervals* function, that encompasses some of the earlier discounting functions, for example, exponential discounting and generalized hyperbolic discounting; and fits the empirical evidence better. However, we show that the theoretical model of Scholten and Read (2006) is inconsistent. We suggest a way in which the inconsistency can be removed.

2. Discounting by intervals

Let $D(t_S, t_L)$ be the discount function from time t_L back to time t_S, $0 \leq t_S \leq t_L$. In particular, $D(0, t_S)$ and $D(0, t_L)$ discount, respectively, from times t_S and t_L back to time 0. Scholten and Read (2006) propose the following *discounting by intervals* function:

$$D(t_S, t_L) = \left[1 + \alpha \left(t_L^\tau - t_S^\tau\right)^\vartheta\right]^{-\frac{\beta}{\alpha}}, \; 0 \leq t_S \leq t_L, \; \alpha > 0, \; \beta > 0, \; \tau > 0, \; \vartheta > 0. \tag{2.1}$$

As special, or limiting, cases of (2.1) we get exponential discounting and a number of other discounting functions including Harvey (1986), Mazur (1987), Rachlin (1989), Lowenstein and Prelec (1992) and Read (2001).

Let $v(x)$ be the utility of x when x is received. Let $V(x, t)$ be the utility, discounted back to time 0, of x received at time t. Assume that v is strictly increasing and, for simplicity, assume $v(0) = 0$. Let

$$x_L > 0, \tag{2.2}$$

$$0 \leq t_S \leq t_L. \tag{2.3}$$

Hence,

$$V(x_L, t_L) = D(0, t_L)\, v(x_L), \tag{2.4}$$

The second equation in the pair of equations (4) in Scholten and Read (2006, p1427) states that

$$V(x_L, t_L) = D(0, t_S)\, D(t_S, t_L)\, v(x_L), \tag{2.5}$$

i.e., the utility of x_L discounted from t_L back to t_S then back to 0 is the same as the utility of x_L discounted from t_L back to 0.

From (2.4) and (2.5) we get

$$D(0, t_L)\, v(x_L) = D(0, t_S)\, D(t_S, t_L)\, v(x_L). \tag{2.6}$$

Since, $x_L > 0$, $v(0) = 0$ and v is strictly increasing, we get $v(x_L) > 0$. In particular, $v(x_L) \neq 0$. Hence, (2.6) gives

$$D(0, t_L) = D(0, t_S)\, D(t_S, t_L), \tag{2.7}$$

Substituting from (2.1) into (2.7) gives

$$\left[1 + \alpha t_L^{\tau\vartheta}\right]^{-\frac{\beta}{\alpha}} = \left[1 + \alpha t_S^{\tau\vartheta}\right]^{-\frac{\beta}{\alpha}} \left[1 + \alpha \left(t_L^{\tau} - t_S^{\tau}\right)^{\vartheta}\right]^{-\frac{\beta}{\alpha}}, \tag{2.8}$$

then successive algebraic steps give

$$1 + \alpha t_L^{\tau\vartheta} = \left(1 + \alpha t_S^{\tau\vartheta}\right) \left[1 + \alpha \left(t_L^{\tau} - t_S^{\tau}\right)^{\vartheta}\right],$$

$$1 + \alpha t_L^{\tau\vartheta} = 1 + \alpha t_S^{\tau\vartheta} + \alpha \left(1 + \alpha t_S^{\tau\vartheta}\right) \left(t_L^{\tau} - t_S^{\tau}\right)^{\vartheta},$$

$$\alpha t_L^{\tau\vartheta} = \alpha t_S^{\tau\vartheta} + \alpha \left(1 + \alpha t_S^{\tau\vartheta}\right) \left(t_L^{\tau} - t_S^{\tau}\right)^{\vartheta},$$

$$t_L^{\tau\vartheta} = t_S^{\tau\vartheta} + \left(1 + \alpha t_S^{\tau\vartheta}\right) \left(t_L^{\tau} - t_S^{\tau}\right)^{\vartheta}. \tag{2.9}$$

The right hand side of (2.9) is a non-constant function of t_S, which can be any real number in the interval $[0, t_L]$. However, t_S does not occur in the left hand side of (2.9). Hence, (2.9) cannot be satisfied (except in the special cases $t_S = 0$ or $t_S = t_L$)[1].

3. Removing the inconsistency

The source of the inconsistency is as follows. The Scholten and Read discounting by intervals function (2.1) is designed to be in line with the empirical evidence that time discounting is non-additive. Of course, Scholten and Read (2006) do not, *in general*, assume additivity of time preferences. However, the second equation of their pair of equations (4), reproduced above as (2.5), is a *particular instance* of the additivity assumption. Unfortunately, this one instance is sufficient to produce the inconsistency.

[1]The special case $t_S = 0$ and $\tau = \vartheta = 1$ gives the generalized hyperbolic discounting function of Loewenstein and Prelec (1992).

As in section 2, $D(t_S, t_L)$ is the discount function from time t_L back to time t_S, $0 \leq t_S \leq t_L$. In particular, $D(0, t_S)$ and $D(0, t_L)$ discount, respectively, from times t_S and t_L back to time 0.

Let

$$V(x_n; t_1, t_2, ..., t_n),\ 0 \leq t_1 \leq t_2 \leq ... \leq t_n, \tag{3.1}$$

be the discounted value at time t_1 of the utility of x_n, received at time t_n, and discounted back to t_1 through the intermediate time points $t_2, ..., t_{n-1}$. For example, for $n = 2$, $x_2 = x_S$, $t_1 = 0$ and $t_2 = t_S$, (3.1) becomes $V(x_S; 0, t_S)$. For $n = 3$, $x_3 = x_L$, $t_1 = 0$, $t_2 = t_S$, and $t_3 = t_L$, (3.1) becomes $V(x_L; 0, t_S, t_L)$, where $0 \leq t_S \leq t_L$. Non-additivity introduces a non-Markov feature, so in the authors' notation "$V(x_S, t_S)$" and "$V(x_L, t_L)$" are ambiguous. By contrast, our suggested notation (3.1) explicitly gives the history of the discounting process.

To remove the inconsistency, we replace (2.4) by

$$V(x_S; 0, t_S) = D(0, t_S)\, v(x_S), \tag{3.2}$$

and replace (2.5) by

$$V(x_L; 0, t_S, t_L) = D(0, t_S)\, D(t_S, t_L)\, v(x_L), \tag{3.3}$$

where $V(x_L; 0, t_S, t_L)$ is, in general, not the same as $V(x_L; 0, t_L)$.

To derive the rest of their model, consider the two allocations:

Allocation SS: x_S is received at time t_S and is discounted back to time 0 in one step, (3.4)

Allocation LL: x_L is received at time t_L, discounted back to time t_S, then to time 0. (3.5)

The allocations (3.4) and (3.5) are chosen by experimental design so that they are equivalent, i.e.,

$$V(x_S; 0, t_S) = V(x_L; 0, t_S, t_L). \tag{3.6}$$

This appears to be in accord with the authors' intension. We quote from their first paragraph under "Discounting by Intervals", p1427:

> "According to our model [Scholten and Read (2006)], SS is discounted over the interval $0 \rightarrow t_S$, while LL is discounted over the consecutive intervals $0 \rightarrow t_S$ and $t_S \rightarrow t_L$. As a result, discounting over the interval $t_S \rightarrow t_L$ has a primitive, rather than derivative status."

The rest of their model follows without any further inconsistencies.

In brief, non-additivity, which is the experimental finding in Read (2001) and in Scholten and Read (2006) introduces a non-Markov feature. The result is that 'discounted utility' is no more a function of the final state but of the entire history. The theoretical model of Scholten and Read (2006), however, misses this critical insight. For that reason, (2.4), (2.5), lead to an additive formulation (which is counter to the experimental findings) and, hence, to a fundamental inconsistency, which can be deduced from (2.9). Our proposed reformulation in section 3 explicitly takes account of the non-Markov feature arising from non-additivity. This is reflected in (3.2) and (3.3). $V(x_L, t_L)$ is defined to be the present value of the utility of x_L, received at time t_L, discounted to time 0. However, crucially, and unlike Scholten and Read (2006), $V(x_L, t_L)$ cannot then be used to represent allocation LL which denotes the present value of the utility of x_L, received at time t_L, discounted back to time t_S, then to time 0. This leads us to resolve the inconsistency.

4. Conclusion

Decision makers may exhibit inconsistent behavior (and there is a large body of experimental evidence supporting this). However, the theories (or models) constructed to explain such behavior have to be consistent (as all theories/models have to be). The 'discounting by interval' *function* of Scholten and Read (2006) may describe well actual behavior of decision makers. But the 'discounting by intervals' *theory* of intertemporal choice that Scholten and Read (2006) construct is inconsistent. We suggest a minimal modification that would render their theory consistent.

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