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EXPLAINING THE ANOMALIES OF THE EXPONENTIAL DISCOUNTED UTILITY MODEL

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Abstract

In a major contribution, Loewenstein and Prelec (1992) (LP) set the foundations for the behavioral approach to decision making over time. We show that the LP theory is incompatible with two very useful classes of value functions: the HARA class and the constant loss aversion class. Resultingly, the LP theory has been used infrequently in applications, which have largely used the β, δ form of hyperbolic preferences. We propose a more general but equally tractable class of utility functions, the *simple increasing elasticity* (SIE) class, which is compatible with constant loss aversion in a reformulated version of LP. Allowing for reference dependence and different discount rates for gains and losses the SIE class is able to explain impatience, gain-loss asymmetry, magnitude effect, and the delay-speedup asymmetry even under exponential discounting. If combined instead with the (reformulated) LP theory, the SIE class in addition can also explain the common difference effect.

Keywords: Anomalies of the DU model, Intertemporal choice, Generalized hyperbolic discounting, loss aversion, HARA utility functions, SIE value functions.

JEL Classification Codes: C60(General: Mathematical methods and programming); D91(Intertemporal consumer choice).

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1. Introduction

It is well known that the *exponential discounted utility* model of intertemporal choice (henceforth, EDU) is contradicted by a relatively large body of empirical and experimental evidence; see for instance Thaler (1981). Furthermore, it appears that these anomalies are not simply mistakes; see for instance, Frederick, Loewenstein and O'Donoghue (2002). If we wish to develop models that better explain economic behavior, then we have no choice but to take account of these anomalies. Furthermore, certain types of behavior, and several institutional features, can be explained by decision makers attempting to deal with time-inconsistency problems that arise from non-exponential discounting[1].

Loewenstein and Prelec (1992) (henceforth, LP) give a formal statement of the known anomalies of the EDU model. Of the anomalies mentioned in LP, the subsequent literature has focussed largely on the evidence for and implications of declining discount rates (the *common difference effect*); EDU in contrast, assumes constant discount rates. The importance of LP's contribution is that it remains, the leading contender in providing an explanation of the other anomalies of the EDU model, in particular, *gain-loss asymmetry*, the *magnitude effect* and *delay-speedup asymmetry* (these are defined below).

A small, promising, set of recent models also attempt to explain all or a subset of the anomalies, but for a variety of reasons they require further development. Manzini and Mariotti (2006) can explain the magnitude effect in their model of vague time preferences. However, their model relies on a 'vagueness function' as well as 'secondary criteria'; both impose a degree of arbitrariness. By using non-constant marginal utility, Noor (2007) is also able to account for several anomalies of the EDU model. The empirical relevance of Noor's model is not yet clear, for the following reasons. First, in several experiments that illustrate the anomalies of EDU, the time delay is short enough to justify a constant marginal utility. Second, not all experimental results involve students (one motivation used by Noor for his paper) whose marginal utility is most likely to change. Third, it requires a fair degree of sophistication on the part of students (especially in light of bounded rationality models such as the vague time preferences model just considered above) to make precise calculations based on changes in marginal utility. Read and Scholten (2006) use an attributes based model to explain the magnitude effect and the gain-loss asymmetry. Such a model focuses on a 'trade-off' between the time dimension and the outcome dimension. More empirical evidence is needed to narrow down the relevance of these disparate models in explaining the anomalies of the EDU model.

An important contribution of LP is that they give the first statement and axiomatic

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[1]Time inconsistency problems can lead individuals to make suboptimal decisions about, for instance, savings, pensions, retirement etc. The existence of mandatory pension plans, retirement age, compulsory insurance of several sorts etc. are possible institutional responses to these time inconsistency problems; see, for instance, Frederick, Loewenstein and O'Donoghue (2002).

derivation of the *generalized hyperbolic discounting* formula which remains the most general formulation of the *hyperbolic* discounting function. However, the simpler *quasi-hyperbolic* formulation, often referred to as the β, δ form, due to Phelps and Pollack (1968), and later popularized by Laibson (1997), is mainly used in applied theoretical work.

Why is the LP form used so rarely in applied work? Our first contribution is to show that the LP formulation is incompatible with value functions from the *hyperbolic absolute risk aversion class* (HARA) and also with the class of value functions that exhibit *constant loss aversion.* So, for instance, the commonly used utility functions in applied work, the CARA form, the CRRA form, the quadratic form and the logarithmic form are all inconsistent with the LP formulation. Since both classes are tractable and popular in applications, their incompatibility with LP (1992) is potentially a serious handicap. LP themselves do not give a value function that satisfies all their restrictions.

Our second contribution is to restate the LP theory so that it admits discount functions that are not necessarily the same for losses and gains. We show that this reformulation is compatible with value functions that exhibit constant loss aversion.

Our third contribution is to provide a scheme for generating value functions that are compatible with the (reformulated) LP theory. The simplest members of this class are just as tractable as those of the HARA class. We call this class the *simple increasing elasticity* (SIE) class; it is formed by a product of a HARA function and a constant relative risk aversion function (CRRA).

Our fourth contribution is to show that allowing for different discount rates for gains and losses and for reference dependence under exponential discounting, the SIE class is able to explain several well known anomalies of the EDU model, such as impatience, gain-loss asymmetry, magnitude effect, and the delay-speedup asymmetry. Furthermore, if combined instead with generalized hyperbolic discounting, the SIE class, in addition, can also explain the common difference effect.

The scheme of the paper is as follows. In section 2, we discuss the main anomalies of the EDU model. The LP theory and its restatement by al-Nowaihi and Dhami (2006) is explained in section 3.[2] In section 4, we show that the utility functions in the HARA class (which includes as special cases, the CARA, CRRA, logarithmic, and quadratic utility functions) as well as those exhibiting constant loss aversion are not compatible with the LP formulation. In section 5, we derive a utility function that is compatible with the reformulated LP theory and also constant loss aversion, as well as other results. All proofs are contained in the appendix.

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[2]al-Nowaihi and Dhami (2006) point out to three errors in LP. The most important error they point out is that the generalized hyperbolic discount function does not follow from linear delay but rather that it requires quadratic delay. These points are clarified in section 3.

2. Four anomalies of the EDU model

Loewenstein and Prelec (1992) describe the following four anomalies:

1. *Gain-loss asymmetry.* Subjects in a study by Loewenstein (1988b) were, on average, indifferent between receiving $10 immediately and receiving $21 one year later. They were also indifferent between loosing $10 immediately and losing $15 dollars one year later. Letting v be the value function (or felicity, or utility at some instant in time) and $\phi_\pm$ be the discount function for gains/losses, we get: $v(10) = v(21)\phi_+(1)$ and $v(-10) = v(-15)\phi_-(1)$. Since $v(-21) < v(-15)$, we get: $v(-10) > v(-21)\phi_-(1)$.

2. *Magnitude effect.* Thaler (1981) reported that subjects were on average indifferent between receiving $15 immediately and $60 one year later. They were also indifferent between receiving $3000 immediately and receiving $4000 one year later. We thus have $v(15) = v(60)\phi_+(1)$ and $v(3000) = v(4000)\phi_+(1)$. Note that $3000 = 200 \times 15$ and that $200 \times 60 = 12000 > 4000$. Hence $v(200 \times 15) < v(200 \times 60)\phi_+(1)$.

3. *Common difference effect.* Thaler (1981): a person might prefer one apple today to two apples tomorrow, but at the same time prefer two apples in 51 days to on apple in 50 days. Thus we have $v(1) > v(2)\phi_+(1)$ but $v(1)\phi_+(50) < v(2)\phi_+(1 + 50)$.

4. *Delay-speedup asymmetry.* Loewenstein (1988a) reported that, in general, the amount required to compensate for delaying receiving a real reward by a given interval, from s to $s + t$, was two to four times greater than the amount subjects were willing to sacrifice to speed consumption from $s + t$ to s. Consider the two consumption streams: $((0,0),(c,s),(0,s+t))$ and $((0,0),(0,s),(c,s+t))$, where $c > 0$. If the consumer expects the first stream but gets the second, he will code that as a loss of $-c$ in the second period but a gain of c in the third period. Let the resulting value be $V((0,0),(-c,s),(c,s+t))$.[3] On the other hand, if the consumer expects the second stream but gets the first, he will code that as a gain of c in the second period but a loss of c in the third period. Let the resulting value be $V((0,0),(c,s),(-c,s+t))$. Classically, of course, $V((0,0),(c,s),(-c,s+t)) = -V((0,0),(-c,s),(c,s+t))$ but what is observed is $V((0,0),(c,s),(-c,s+t)) < -V((0,0),(-c,s),(c,s+t))$.

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[3]We will use capital V to denote the intertemporal utility from a consumption profile (lowercase v is used for the felicity, or utility and is sometimes called a value function). Hence, $V((0,0),(-c,s),(c,s+t))$ means the intertemporal utility from receiving $0 today, $-\$c$, s periods from now and $\$c$, $s + t$ periods from now.

3. The Loewenstein-Prelec theory of intertemporal choice

Consider a decision maker who, at time t_0, formulates a plan to choose c_i at time t_i, $i = 1, 2, ..., n$, where $t_0 < t_1 < ... < t_n$. LP assume that the intertemporal utility to the decision maker, at time t_0, is given by:

$$V\left((c_1, t_1), (c_2, t_2), ..., (c_n, t_n)\right) = \Sigma_{i=1}^n v\left(x_i\right) \varphi_\pm\left(t_i\right),$$

$$x_i = c_i - c_i^r, \tag{3.1}$$

were c_i^r is the reference point in period i.[4] If $x_i \geq 0$, then $\varphi_\pm\left(t_i\right) = \varphi_+\left(t_i\right)$. If $x_i < 0$, then $\varphi_\pm\left(t_i\right) = \varphi_-\left(t_i\right)$. LP assume from the outset that $\varphi_-\left(t_i\right) = \varphi_+\left(t_i\right)$. By contrast, in our reformulation of LP, we allow the discount function for gains, $\varphi_+\left(t_i\right)$, to be different from the discount function for losses, $\varphi_-\left(t_i\right)$.

We get the standard EDU model for the special case $c_i^r = 0$ and exponential discounting:

$$\varphi\left(t_i\right) = e^{-\beta t_i}, \beta > 0. \tag{3.2}$$

Aside from its tractability, the main attraction of EDU is that it leads to time-consistent choices (at least, in non-game-theoretic situations). If the plan $(c_1, t_1), (c_2, t_2), ..., (c_n, t_n)$ is optimal at time t_0, then at time t_k the plan $(c_{k+1}, t_{k+1}), (c_{k+2}, t_{k+2}), ..., (c_n, t_n)$ is also optimal. But this may no longer be true for more general specifications of the discount function φ.

LP adopt the utility function (3.1) taking v to be the value function introduced by Kahneman and Tversky (1979). Thus v satisfies:

$$v : (-\infty, \infty) \rightarrow (-\infty, \infty) \text{ is continuous, strictly increasing,}$$

$$v\left(0\right) = 0 \text{ and is twice differentiable except at } 0 \tag{3.3}$$

They define the *elasticity* of v (LP's second equation number 15, p583) by:

$$\epsilon_v\left(x\right) = \frac{x}{v}\frac{dv}{dx}, x \neq 0. \tag{3.4}$$

LP introduce five assumptions, all with good experimental bases (LP, II pp574-578). The first assumption, A0 below, is only implicit in LP and is, of course also satisfied in the EDU model. However, it is essential.

A0 *Impatience.* $\varphi_+ : [0, \infty) \overset{onto}{\rightarrow} (0, 1]$ and $\varphi_- : [0, \infty) \overset{onto}{\rightarrow} (0, 1]$ are strictly decreasing. If $0 < x < y$, then $v\left(x\right) = v\left(y\right) \varphi_+\left(t\right)$ for some $t \in [0, \infty)$. If $y < x < 0$, then $v\left(x\right) = v\left(y\right) \varphi_-\left(t\right)$ for some $t \in [0, \infty)$.[5]

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[4]Our formulation (3.1) differs from LP (LP (9), p579) only in that LP's formulation is in terms of $x_i = c_i - c_i^r$, from the outset.

[5]It is sufficient that φ_+ be strictly decreasing in some interval: $(a, a + \delta)$, $a \geq 0, \delta > 0$ and, similarly, for ϕ_-.

The next four assumptions (A1 to A4, below) correspond to anomalies 1 to 4, above. Thus, what is regarded as anomalous behavior from the point of view of EDU, is at the core of the LP theory.[6]

A1 *Gain-loss asymmetry.* If $0 < x < y$ and $v(x) = v(y)\varphi_+(t)$, then $v(-x) > v(-y)\varphi_-(t)$.

A2 *Magnitude effect.* If $0 < x < y$, $v(x) = v(y)\varphi_+(t)$ and $a > 1$, then $v(ax) < v(ay)\varphi_+(t)$. If $y < x < 0$, $v(x) = v(y)\varphi_-(t)$ and $a > 1$, then $v(ax) > v(ay)\varphi_-(t)$.

A3 *Common difference effect.* If $0 < x < y$, $v(x) = v(y)\varphi_+(t)$ and $s > 0$, then $v(x)\varphi_+(s) < v(y)\varphi_+(s+t)$. If $y < x < 0$, $v(x) = v(y)\varphi_-(t)$ and $s > 0$, then $v(x)\varphi_-(s) > v(y)\varphi_-(s+t)$.

To derive the LP formula for generalized hyperbolic discounting (LP (15), p580), a stronger form of A3 is needed. We adopt:

A3a *Common difference effect with quadratic delay.* If $0 < x < y$, $v(x) = v(y)\varphi_+(t)$ and $s > 0$, then $v(x)\varphi_+(s) = v(y)\varphi_+(s+t+\alpha_+ st)$, $\alpha_+ > 0$. If $y < x < 0$, $v(x) = v(y)\varphi_-(t)$ and $s > 0$, then $v(x)\varphi_-(s) = v(y)\varphi_-(s+t+\alpha_- st)$, $\alpha_- > 0$.

A4 *Delay-speedup asymmetry.* For $c > 0$, $s > 0$ and $t > 0$, $V((0,0),(c,s),(-c,s+t)) < -V((0,0),(-c,s),(c,s+t))$.

Note that A3a $\Rightarrow$ A3 and that $\alpha_+ \to 0$ and $\alpha_- \to 0$ gives exponential discounting. As mentioned above, LP assume that $\varphi_+ = \varphi_-$. While this is consistent with their theory, it does not follow from it. Also, LP state the assumptions explicitly only for the case $0 < x < y$. Three theorems follow:

Proposition 1 : *Suppose $\varphi_- = \varphi_+$. Then A0 and A1 imply that losses are discounted less heavily than gains in the following sense: $0 < x < y \Rightarrow \frac{v(x)}{v(y)} > \frac{v(-x)}{v(-y)}$. If $\varphi_- = \varphi_+$, then A0 and A1 also imply that the value function is more elastic for losses than for gains: $x > 0 \Rightarrow \epsilon_v(-x) > \epsilon_v(x)$.*

Proposition 2 : *A0 and A2 imply that the value function is subproportional: $(0 < x < y$ or $y < x < 0) \Rightarrow \frac{v(x)}{v(y)} > \frac{v(ax)}{v(ay)}$, for $a > 1$, and is more elastic for outcomes of larger absolute magnitude: $(0 < x < y$ or $y < x < 0) \Rightarrow \epsilon_v(x) < \epsilon_v(y)$.[7]*

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[6]More recent work attempts to derive the LP assumptions from more basic psychological principles. See, for example, Dasgupta and Maskin (2002, 2005) and Fudenberg and Levine (2006).

[7]This proposition is stated incorrectly in al-Nowaihi and Dhami (2006).

Proposition 3 : *A0 and A3a imply that the discount function is a generalized hyperbola:*
$\varphi_+(t) = (1 + \alpha_+ t)^{-\frac{\beta_+}{\alpha_+}}$, $\varphi_-(t) = (1 + \alpha_- t)^{-\frac{\beta_-}{\alpha_-}}$, $\beta_+ > 0$, $\beta_- > 0$, $t \geq 0$ (α_+ *and* α_- *are as in A3a).*

Corollary 1 : *A0 and A3a imply that* $-\frac{\dot{\varphi}_+}{\varphi_+} = \frac{\beta_+}{1 + \alpha_+ t}$ *and* $-\frac{\dot{\varphi}_-}{\varphi_-} = \frac{\beta_-}{1 + \alpha_- t}$. *Hence, the discount rate is positive and declining.*

We now add two standard assumption from prospect theory. The first is that the value function is strictly concave for gains and strictly convex for losses (Kahneman and Tversky, 1979):

A5 *Declining sensitivity.* For $x > 0$, $v''(x) < 0$ (strict concavity for gains). For $x < 0$, $v''(x) > 0$ (strict convexity for losses).

Combining A5 with Proposition 2 we get:

Corollary 2 : $0 < \epsilon_v < 1$.

The second assumption that we add from prospect theory is constant loss aversion. While this is not core to prospect theory, it is very useful and has good empirical support (Tversky and Kahneman, 1992):

A6 *Constant loss aversion.* $v(-x) = -\lambda v(x)$, $\lambda > 1$, for $x > 0$,

With the aid of these two extra assumptions, we get the following two theorems:

Proposition 4 : $t > 0 \Rightarrow \varphi_+(t) < \varphi_-(t)$, $0 < x < y \Rightarrow \frac{v(x)}{v(y)} = \frac{v(-x)}{v(-y)}$ *and* $x > 0 \Rightarrow \epsilon_v(-x) = \epsilon_v(x)$.

Proposition 5 : *Assumption A4 follows from the other assumptions.*

As mentioned above, LP assume that $\varphi_+ = \varphi_-$. While this is consistent with their theory, it does not follow from it. Assuming that $\varphi_+ = \varphi_-$ is obviously attractive. However, it implies (Proposition 1) that *gain-loss asymmetry* (A1) can only be satisfied if $\epsilon_v(-x) > \epsilon_v(x)$, for $x > 0$. In the light of Proposition 4, this would exclude value functions exhibiting constant loss aversion. While *constant* loss aversion is not core to prospect theory, this auxiliary assumption considerably simplifies application of the theory and is consistent with the evidence (Tversky and Kahneman, 1992).

The generalized hyperbola given in Proposition 3 does not follow from the assumptions made in LP.[8] Our A3a is a corrected version of the one that appears in LP (LP (11), p579) and enables us to derive the required generalized hyperbola. As far as we know, Corollary 2 and Proposition 4 are new results.

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[8]See al-Nowaihi and Dhami (2006).

4. Incompatibility of HARA value functions with the Loewenstein-Prelec theory

A popular family of utility functions is the family of *hyperbolic absolute risk aversion* functions (HARA). Proposition 6, below, shows that each member of this family exhibits constant or declining elasticity, contradicting Loewenstein and Prelec's Proposition 2. Hence, none of this family is compatible with the Loewenstein-Prelec theory. First, we give the definitions and main properties of this family of functions, followed by the main result of this section: Proposition 6.

Notation: We use the notation, ρ_A and ρ_R respectively, for the coefficients of absolute risk aversion and relative risk aversion. So for a utility function $v(x)$, $\rho_A = -\frac{v''(x)}{v'(x)}$ and $\rho_R = -\frac{xv''(x)}{v'(x)}$.

4.1. Constant relative risk aversion functions (CRRA)

$$
\begin{aligned}
v(x) &= \frac{x^{1-\gamma}}{1-\gamma}, \; 0 < \gamma < 1, \\
v'(x) &= x^{-\gamma} > 0; \; v''(x) = -\gamma x^{-\gamma-1} < 0, \\
\rho_R &= -\frac{xv''(x)}{v'(x)} = \gamma, \\
\epsilon_v(x) &= \frac{xv'(x)}{v(x)} = 1 - \gamma.
\end{aligned}
\tag{4.1}
$$

The general restriction is that $\gamma \neq 1$. However, we need the stronger restriction, $0 < \gamma < 1$, in order to satisfy Corollary 2. It is clear, from the last line of (4.1), that members of the CRRA class of functions violate Proposition 2 and, hence, are not compatible with the Loewenstein-Prelec theory.

4.2. Hyperbolic absolute risk aversion functions (HARA)

$$
\begin{aligned}
v(x) &= \frac{\gamma}{1-\gamma}\left[\left(\mu + \frac{\theta x}{\gamma}\right)^{1-\gamma} - \mu^{1-\gamma}\right], \; \theta > 0, \; \mu > 0, \; 0 < \gamma < 1, \; x \geq 0, \\
v'(x) &= \theta\left(\mu + \frac{\theta x}{\gamma}\right)^{-\gamma} > 0; \; v''(x) = -\theta^2\left(\mu + \frac{\theta x}{\gamma}\right)^{-\gamma-1} < 0, \\
\epsilon_v(x) &= \frac{xv'(x)}{v(x)} = \frac{(1-\gamma)\frac{\theta x}{\gamma}\left(\mu + \frac{\theta x}{\gamma}\right)^{-\gamma}}{\left(\mu + \frac{\theta x}{\gamma}\right)^{1-\gamma} - \mu^{1-\gamma}}, \\
\rho_A &= -\frac{v''(x)}{v'(x)} = \frac{\gamma\theta}{\gamma\mu + \theta x}.
\end{aligned}
\tag{4.2}
$$

7

The general restrictions are $\theta > 0$, $\left(\mu + \frac{\theta x}{\gamma}\right)^{1-\gamma} > 0$, $\gamma \neq 1$. Since we allow $x \in [0, \infty)$, the restriction $\left(\mu + \frac{\theta x}{\gamma}\right)^{1-\gamma} > 0$ implies that $\mu > 0$ and $\gamma > 0$. We then also need $\gamma < 1$ in order to satisfy Corollary 2.

Remark 1 : *Note that, traditionally, the HARA class is defined by* $v(x) = \frac{\gamma}{1-\gamma}\left(\mu + \frac{\theta x}{\gamma}\right)^{1-\gamma}$, *and that* $\epsilon_v(x) = (1 - \gamma)\left(1 + \frac{\gamma\mu}{\theta x}\right)^{-1}$, *which is increasing in* x, *as required by Proposition 2. While an additive constant, of course, makes no difference in expected utility theory; its absence here would violate the assumption* $v(0) = 0$. *However, including the constant* $-\frac{\gamma}{1-\gamma}\mu^{1-\gamma}$, *to make* $v(0) = 0$, *results in* $\epsilon_v(x)$ *decreasing with* x, *as shown by Proposition 6, and, hence, violating Proposition 2.*

The following three classes of functions are also regarded members of the HARA family.

4.3. Constant absolute risk aversion functions (CARA)

$$
\begin{aligned}
v(x) &= 1 - e^{-\theta x},\ \theta > 0,\ x \geq 0,\\
v'(x) &= \theta e^{-\theta x} > 0;\ v''(x) = -\theta^2 e^{-\theta x} < 0,\\
\rho_A &= -\frac{v''(x)}{v'(x)} = \theta,\\
\epsilon_v(x) &= \frac{xv'(x)}{v(x)} = \frac{\theta x}{e^{\theta x} - 1} = \frac{1}{\sum_{n=1}^{\infty} \frac{(\theta x)^{n-1}}{n!}}. \tag{4.3}
\end{aligned}
$$

From the last line of (4.3), we see that $\epsilon_v(x)$ is decreasing with x. Hence, the CARA class is not compatible with the Loewenstein-Prelec theory.

4.4. Logarithmic functions

$$
\begin{aligned}
v(x) &= \ln(1 + \theta x),\ \theta > 0,\ x \geq 0,\\
v'(x) &= \theta(1 + \theta x)^{-1} > 0;\ v''(x) = -\theta^2(1 + \theta x)^{-2} < 0,\\
\epsilon_v(x) &= \frac{xv'(x)}{v(x)} = \frac{\theta x}{(1 + \theta x)\ln(1 + \theta x)}. \tag{4.4}
\end{aligned}
$$

Proposition 6 establishes that $\epsilon_v(x)$ is decreasing with x. Hence this class is not compatible with the Loewenstein-Prelec theory.

4.5. Quadratic functions

$$
\begin{aligned}
v(x) &= \frac{1}{2}\mu^2 - \frac{1}{2}(\mu - \theta x)^2,\ \theta > 0,\ 0 \leq x < \frac{\mu}{\theta},\\
v'(x) &= \theta(\mu - \theta x)^2 > 0;\ v''(x) = -2\theta^2(\mu - \theta x) < 0,\\
\epsilon_v(x) &= \frac{xv'(x)}{v(x)} = \frac{2\theta x(\mu - \theta x)^2}{\mu^2 - (\mu - \theta x)^2}. \tag{4.5}
\end{aligned}
$$

Proposition 6 establishes that $\epsilon_v(x)$ is decreasing with x. Hence this class is also not compatible with the Loewenstein-Prelec theory.

Proposition 6 : *For members of the CRRA class of value functions (4.1), $\epsilon_v(x)$ is constant. For members of the HARA (4.2), CARA (4.3), logarithmic (4.4) and quadratic (4.5) classes of functions, $\epsilon_v(x)$ is declining. Hence none of the general family of hyperbolic absolute risk aversion functions is compatible with the Loewenstein-Prelec theory.*

5. A value function compatible with the Loewenstein-Prelec theory

The following method can be used to generate candidates for value functions compatible with the Loewenstein-Prelec theory. Choose a function, $f(x)$, satisfying:

$$0 < f(x) < 1,\ f'(x) > 0, \tag{5.1}$$

then solve the following differential equation for $v(x)$:

$$\frac{x}{v}\frac{dv}{dx} = f(x). \tag{5.2}$$

This method only yields candidate value functions, which then have to be verified. For example, choose

$$
\begin{aligned}
f(x) &= \frac{ax}{b+x} + c,\ x \geq 0, \\
a &> 0,\ b > 0,\ c > 0,\ a + c \leq 1.
\end{aligned} \tag{5.3}
$$

Substituting from (5.3) into (5.2), separating variables, then integrating, gives:

$$
\begin{aligned}
\frac{x}{v}\frac{dv}{dx} &= \frac{ax}{b+x} + c, \\
\int \frac{dv}{v} &= a\int \frac{dx}{b+x} + c\int \frac{dx}{x}, \\
\ln v &= a\ln(b+x) + c\ln x + \ln K, \\
v(x) &= K(b+x)^a x^c.
\end{aligned} \tag{5.4}
$$

Choosing $a = 1 - \gamma$, $b = \frac{\gamma\mu}{\theta}$, $c = \sigma$ and $K = \frac{\gamma}{1-\gamma}\left(\frac{\theta}{\gamma}\right)^{1-\gamma}$ produces a value function:

$$v(x) = \frac{\gamma x^\sigma}{1-\gamma}\left(\mu + \frac{\theta}{\gamma}x\right)^{1-\gamma},\ x \geq 0, \tag{5.5}$$

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The restrictions $a > 0$, $b > 0$, $c > 0$, $a + c \leq 1$ give: $0 < \sigma \leq \gamma < 1$ and $\mu/\theta > 0$. To ensure that $v' > 0$, take $\theta > 0$. Hence $\mu > 0$. For $x < 0$, define $v(x)$ by $v(x) = -\lambda v(-x)$, where $\lambda > 1$. Putting all these together gives the candidate value function

$$v(x) = \frac{\gamma x^\sigma}{1-\gamma}\left(\mu + \frac{\theta}{\gamma}x\right)^{1-\gamma}, \ x \geq 0,$$

$$v(x) = -\lambda v(-x)$$

$$= -\lambda\frac{\gamma(-x)^\sigma}{1-\gamma}\left(\mu - \frac{\theta}{\gamma}x\right)^{1-\gamma}, \ x < 0,$$

$$\mu > 0, \ \theta > 0, \ \lambda > 1, \ 0 < \sigma \leq \gamma < 1, \tag{5.6}$$

where λ is the (constant) coefficient of loss aversion.[9]

Proposition 7 : From (5.6) it follows that:

(a) $v : (-\infty, \infty) \to (-\infty, \infty)$, $v(0) = 0$, v is continuous, v is C^∞ except at $x = 0$.

(b) $v'(x) = \frac{\sigma\gamma x^{\sigma-1}}{1-\gamma}\left(\mu + \frac{\theta}{\gamma}x\right)^{1-\gamma} + \theta x^\sigma \left(\mu + \frac{\theta}{\gamma}x\right)^{-\gamma} > 0$, $x > 0$,

(c) $v''(x) = \left(\mu + \frac{\theta}{\gamma}x\right)^{-\gamma-1}\left[-x^{\sigma-2}\left[\sigma\mu + (\sigma-\gamma)\frac{\theta x}{\gamma}\right]^2 - \frac{x^{\sigma-2}\sigma(\gamma-\sigma)}{1-\gamma}\left(\mu+\frac{\theta}{\gamma}x\right)^2\right] < 0$, $x > 0$,

(d) $v'(x) = \lambda v'(-x) > 0$, $x < 0$,

(e) $v''(x) = -\lambda v''(-x) > 0$, $x < 0$,

(f) $\epsilon_v(x) = \frac{x}{v}\frac{dv}{dx} = \sigma + \frac{1-\gamma}{1+\frac{\gamma\mu}{\theta x}} > 0$, $\epsilon'_v(x) > 0$, $x > 0$,

(g) $\epsilon_v(x) = \frac{x}{v}\frac{dv}{dx} = \sigma + \frac{1-\gamma}{1-\frac{\gamma\mu}{\theta x}} > 0$, $\epsilon'_v(x) < 0$, $x < 0$,

(h) $\rho_A = -\frac{v''(x)}{v'(x)} = \frac{(1-\gamma)\left[\sigma\mu+(\sigma-\gamma)\frac{\theta x}{\gamma}\right]^2+\sigma(\gamma-\sigma)\left(\mu+\frac{\theta}{\gamma}x\right)^2}{\sigma\gamma x\left(\mu+\frac{\theta}{\gamma}x\right)^2+\theta(1-\gamma)x^2\left(\mu+\frac{\theta}{\gamma}x\right)} > 0$, $x > 0$,

(i) $\rho_R = -\frac{xv''(x)}{v'(x)} = \frac{(1-\gamma)\left[\sigma\mu+(\sigma-\gamma)\frac{\theta x}{\gamma}\right]^2+\sigma(\gamma-\sigma)\left(\mu+\frac{\theta}{\gamma}x\right)^2}{\sigma\gamma\left(\mu+\frac{\theta}{\gamma}x\right)^2+\theta(1-\gamma)x\left(\mu+\frac{\theta}{\gamma}x\right)} > 0$, $x > 0$,

(j) $\rho_A = -\frac{v''(x)}{v'(x)} = \frac{v''(-x)}{v'(-x)} < 0$, $x < 0$,

(k) $\rho_R = -\frac{xv''(x)}{v'(x)} = \frac{(-x)v''(-x)}{v'(-x)} < 0$, $x < 0$,

Corollary 3 : From (f) and (g) of Proposition 6, we get that $\epsilon_v(x) \to \sigma$ as $x \downarrow 0$ and as $x \uparrow 0$. Hence, $\epsilon_v(x)$ is defined for all $x \in (-\infty, \infty)$, $\epsilon_v(x) = \sigma + \frac{1-\gamma}{1+\frac{\gamma\mu}{\theta|x|}} > 0$, $\epsilon_v(x)$ is increasing in $|x|$ and $\epsilon_v(x) \to \sigma + 1 - \gamma \leq 1$, as $|x| \to \infty$.

Remark 2 : In the light of Corollary 3, we may call the value function (5.6) a *simple increasing elasticity (SIE)* value function.

---

[9]It may be interesting to note that (5.6) is a product of a CRRA function, x^σ, and a HARA function, $v(x) = \frac{\gamma x^\sigma}{1-\gamma}\left(\mu + \frac{\theta}{\gamma}x\right)^{1-\gamma}$.

As our discount function, we adopt one of the following:

$$\phi_+(t) = e^{-\beta_+ t}, \; \phi_-(t) = e^{-\beta_- t}, \; 0 < \beta_- < \beta_+, \text{ or} \tag{5.7}$$

$$\phi_+(t) = (1 + \alpha_+ t)^{-\frac{\beta_+}{\alpha_+}}, \; \phi_-(t) = (1 + \alpha_- t)^{-\frac{\beta_-}{\alpha_-}}, \text{ where,} \tag{5.8}$$

$$0 < \alpha_- \le \alpha_+, \; 0 < \frac{\beta_-}{\alpha_-} \le \frac{\beta_+}{\alpha_+} \text{ and, at least, one of the inequalities,}$$

$$\alpha_- \le \alpha_+, \; \frac{\beta_-}{\alpha_-} \le \frac{\beta_+}{\alpha_+}, \text{ is strict.}$$

Corollary 4 *It follows from (5.7) and (5.8) that $0 < \phi_+(t) < \phi_-(t)$ for $t > 0$.*

Proposition 8 *: (a) Under exponential discounting (5.7) (but with different discount rates for gains and losses), the SIE value function (5.6) satisfies assumptions A0, A1, A2, A4, A5 and A6, i.e., all the assumptions except A3, the Common difference effect.*
(b) Under hyperbolic discounting (5.8), the SIE value function (5.6) satisfies assumptions A0 to A6, i.e., all the assumptions.

6. Conclusions

In a seminal contribution, Lowenstein and Prelec (1992) (LP) provide the foundations for the study of choice of over time, taking into account the observed stylized facts on anomalies of the EDU model. A small literature, using promising alternative frameworks, has also attempted to explain the observed anomalies. But further work is needed to develop these alternative frameworks. For the moment, though, LP remains the most accepted framework to resolve the EDU anomalies. Furthermore, it provides an axiomatic derivation of the generalized hyperbolic discounting formula that forms the basis of much recent research in temporal choice.

We show that LP (1992) is incompatible with value functions from the hyperbolic absolute risk aversion class (HARA) and also with the class of value functions that exhibit constant loss aversion. Since both classes are tractable and popular in applications, their incompatibility with LP (1992) is potentially a serious handicap. We restate the LP theory so that it admits discount functions that are not necessarily the same for losses and gains.

We show that this reformulation is compatible with value functions that exhibit constant loss aversion. We provided a scheme for generating value functions compatible with the (reformulated) LP theory. The simplest members of this class are just as tractable as those of the HARA class. They are formed by a product of a HARA function and a constant relative risk aversion function (CRRA). We call this class the class of *simple increasing elasticity* (SIE) value functions, because they are the simplest class compatible with the (reformulated) LP theory that we could find. Their main feature is that they

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exhibit increasing elasticity. If we are willing to modify the exponential discouting model to allow for differential discount rates for gains and losses, and reference dependence, then the SIE class is able to resolve several well known anomalies, such as impatience, gain-loss asymmetry, magnitude effect, and the delay-speedup asymmetry. Furthermore, if combined instead with generalized hyperbolic discounting, the SIE class in addition can also explain the common difference effect.

7. Appendix: Proofs

We define two functions, $\widetilde{v}_+$ and $\widetilde{v}_-$, associated with the value function (3.3),

$$
\begin{aligned}
\widetilde{v}_+ (x) &= \ln v\left(e^x\right), \ -\infty < x < \infty, \\
\widetilde{v}_- (x) &= \ln\left(-v\left(-e^x\right)\right), \ -\infty < x < \infty.
\end{aligned}
\tag{7.1}
$$

Proof of Proposition 1: Suppose $0 < x < y$. By A0, $v(x) = v(y)\varphi_+(t)$ for some $t \in [0, \infty)$. Hence, $\frac{v(x)}{v(y)} = \varphi_+(t)$. By A1, $v(-x) > v(-y)\varphi_-(t)$. Since $-y < 0$, it follows that $v(-y) < 0$ and, hence, $\frac{v(-x)}{v(-y)} < \varphi_-(t)$. $\varphi_+(t) = \varphi_-(t)$ then gives $\frac{v(-x)}{v(-y)} < \frac{v(x)}{v(y)}$. Taking logs gives $\ln v(y) - \ln v(x) < \ln(-v(-y)) - \ln(-v(-x))$ and, hence, $\frac{\ln v(y) - \ln v(x)}{\ln y - \ln x} < \frac{\ln(-v(-y)) - \ln(-v(-x))}{\ln y - \ln x}$. Letting $\widetilde{x} = \ln x$ and $\widetilde{y} = \ln y$, we get $\frac{\widetilde{v}_+(\widetilde{y}) - \widetilde{v}_+(\widetilde{x})}{\widetilde{y} - \widetilde{x}} < \frac{\widetilde{v}_-(\widetilde{y}) - \widetilde{v}_-(\widetilde{x})}{\widetilde{y} - \widetilde{x}}$. Take limits as $\widetilde{y} \to \widetilde{x}$, to get $\frac{d\widetilde{v}_+}{d\widetilde{x}} \le \frac{d\widetilde{v}_-}{d\widetilde{x}}$, from which it follows that $\epsilon_v(x) \le \epsilon_v(-x)$. $\square$

Proof of Proposition 2: Let

$$
0 < x < y \ (\text{or } y < x < 0).
\tag{7.2}
$$

By A0, there is a time, t, such that the consumer is indifferent between receiving the increment x now and receiving the increment y, t-periods from now. Then, letting v be the value function and φ_+ the discount function, we get

$$
v(x) = v(y)\varphi_+(t) \ \left(v(x) = v(y)\varphi_-(t)\right).
\tag{7.3}
$$

Let

$$
a > 1,
\tag{7.4}
$$

then the *magnitude effect*, A2, predicts that

$$
v(ax) < v(ay)\varphi_+(t) \ \left(v(ax) > v(ay)\varphi_-(t)\right),
\tag{7.5}
$$

(7.3) gives

$$
\frac{v(x)}{v(y)} = \varphi_+(t) \ \left(\frac{v(x)}{v(y)} = \varphi_-(t)\right).
\tag{7.6}
$$

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Since y, a are positive (resp. y is negative), it follows that ay, and hence, $v(ay)$ are also positive (resp. negative). Hence, (7.5) gives

$$\frac{v(ax)}{v(ay)} < \varphi_+(t) \quad \left(\frac{v(ax)}{v(ay)} > \varphi_-(t)\right), \tag{7.7}$$

(7.6) and (7.7) give[10]

$$\frac{v(x)}{v(y)} > \frac{v(ax)}{v(ay)}, \ 0 < x < y \ (\text{or } y < x < 0), \ a > 1. \tag{7.8}$$

It follows from (7.8) that the value function, v, is *subproportional*[11]. It then follows (see Appendix B) that the value function is more elastic for outcomes that are larger in absolute magnitude:

$$(0 < x < y \text{ or } y < x < 0) \Rightarrow \epsilon_v(x) < \epsilon_v(y).$$

Since the value function is *subproportional* we have, for $0 < x < y$, $a > 1$ (or $-y < -x < 0$):

$$\frac{v(ay)}{v(ax)} > \frac{v(y)}{v(x)}. \tag{7.9}$$

Start with the case $0 < x < y$. Since $a > 0, x > 0$ and $y > 0$, and since $v(0) = 0$ and v is strictly increasing, it follows that $v(x), v(y), v(ax)$ and $v(ay)$ are all positive. Hence, we can take their logs. Let $\tilde{x} = \ln x$, $\tilde{y} = \ln y$, $\tilde{a} = \ln a$ (hence, $x = e^{\tilde{x}}$, $y = e^{\tilde{y}}$, $a = e^{\tilde{a}}$). Then, since $x < y$ and $a > 1$, it follows that $\tilde{x} < \tilde{y}$, $\tilde{a} > 0$. Let $\tilde{v}(\tilde{x}) = \ln v(e^{\tilde{x}})$, then, by taking logs, and recalling that the logarithmic function is strictly increasing and changes multiplication to addition, subproportionality. (7.9) gives:

$$\ln v(ay) - \ln v(ax) > \ln v(y) - \ln v(x),$$

$$\ln v\left(e^{\tilde{a}}e^{\tilde{y}}\right) - \ln v\left(e^{\tilde{a}}e^{\tilde{x}}\right) > \ln v\left(e^{\tilde{y}}\right) - \ln v\left(e^{\tilde{x}}\right),$$

$$\ln v\left(e^{\tilde{a}+\tilde{y}}\right) - \ln v\left(e^{\tilde{a}+\tilde{x}}\right) > \ln v\left(e^{\tilde{y}}\right) - \ln v\left(e^{\tilde{x}}\right).$$

Since $\tilde{v}(x) = \ln v(e^x)$, we get

$$\tilde{v}(\tilde{y} + \tilde{a}) - \tilde{v}(\tilde{x} + \tilde{a}) > [\tilde{v}(\tilde{y}) - \tilde{v}(\tilde{x})],$$

$$\tilde{v}(\tilde{y} + \tilde{a}) - \tilde{v}(\tilde{x} + \tilde{a}) - [\tilde{v}(\tilde{y}) - \tilde{v}(\tilde{x})] > 0. \tag{7.10}$$

Take $\delta x > 0$, $\tilde{a} = \delta x$, $\tilde{y} = \tilde{x} + \delta x$, then (7.10) gives:

$$\tilde{v}(\tilde{x} + 2\delta x) - \tilde{v}(\tilde{x} + \delta x) - [\tilde{v}(\tilde{x} + \delta x) - \tilde{v}(\tilde{x})] > 0,$$

---

[10]In the course of their proof, LP derive, incorrectly, the formula (LP (18) p583): $\frac{v(x)}{v(y)} < \frac{v(ax)}{v(ay)}, 0 < x < y; a > 1$ (the first $<$ should be $>$)

[11]See Kahneman and Tversky (1979, p282) for the definition of subproportionality. Note that our, and LP's, $a > 1$ corresponds to their $0 < r < 1$.

$$\frac{\widetilde{v}\left(\widetilde{x}+2\delta x\right)-\widetilde{v}\left(\widetilde{x}+\delta x\right)-\left[\widetilde{v}\left(\widetilde{x}+\delta x\right)-\widetilde{v}\left(\widetilde{x}\right)\right]}{\left(\delta x\right)^{2}}>0. \tag{7.11}$$

Now

$$\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}}=\lim_{\delta x\to 0}\frac{\widetilde{v}\left(\widetilde{x}+\delta x\right)-v\left(\widetilde{x}\right)}{\delta x},$$

$$\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}+\delta x}=\lim_{\delta x\to 0}\frac{\widetilde{v}\left(\widetilde{x}+2\delta x\right)-\widetilde{v}\left(\widetilde{x}+\delta x\right)}{\delta x},$$

$$\left[\frac{d^{2}\widetilde{v}}{dx^{2}}\right]_{x=\widetilde{x}}=\lim_{\delta x\to 0}\frac{\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}+\delta x}-\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}}}{\delta x}.$$

Hence,

$$\left[\frac{d^{2}\widetilde{v}}{dx^{2}}\right]_{x=\widetilde{x}}=\lim_{\delta x\to 0}\frac{\lim_{\delta x\to 0}\frac{\widetilde{v}(\widetilde{x}+2\delta x)-\widetilde{v}(\widetilde{x}+\delta x)}{\delta x}-\lim_{\delta x\to 0}\frac{\widetilde{v}(\widetilde{x}+\delta x)-\widetilde{v}(\widetilde{x})}{\delta x}}{\delta x},$$

$$\left[\frac{d^{2}\widetilde{v}}{dx^{2}}\right]_{x=\widetilde{x}}=\lim_{\delta x\to 0}\frac{\frac{\widetilde{v}(\widetilde{x}+2\delta x)-\widetilde{v}(\widetilde{x}+\delta x)}{\delta x}-\frac{\widetilde{v}(\widetilde{x}+\delta x)-\widetilde{v}(\widetilde{x})}{\delta x}}{\delta x},$$

$$\left[\frac{d^{2}\widetilde{v}}{dx^{2}}\right]_{x=\widetilde{x}}=\lim_{\delta x\to 0}\frac{\widetilde{v}\left(\widetilde{x}+2\delta x\right)-\widetilde{v}\left(\widetilde{x}+\delta x\right)-\left[\widetilde{v}\left(\widetilde{x}+\delta x\right)-\widetilde{v}\left(\widetilde{x}\right)\right]}{\left(\delta x\right)^{2}}. \tag{7.12}$$

Since the limit of a converging sequence of positive numbers in non-negative, we get, from (7.11) and (7.12):

$$\left[\frac{d^{2}\widetilde{v}}{dx^{2}}\right]_{x=\widetilde{x}}\geq 0,$$

$$\left[\frac{d}{dx}\left(\frac{d}{dx}\widetilde{v}\left(x\right)\right)\right]_{x=\widetilde{x}}\geq 0.$$

$$\frac{d}{dx}\left(\epsilon\left(x\right)\right)\geq 0.$$

If $\epsilon\left(x\right)$ were constant on some non-empty open interval, then the value function would take the form $v\left(x\right)=cx^{\gamma}$ on that interval, and subproportionality. (7.9) would be violated. Hence $\epsilon'\left(x\right)>0$ almost everywhere. Thus $\epsilon\left(x\right)$ increases with x.

Now consider the case $y<x<0$. Then (7.9) still holds. But now we define $\widetilde{x}=\ln\left(-x\right)$, $\widetilde{y}=\ln\left(-y\right)$ and $\widetilde{v}\left(\widetilde{x}\right)=-\ln\left(-v\left(-e^{\widetilde{x}}\right)\right)$. As before, (7.10) holds and $v'\left(x\right)>0$ almost everywhere. Thus $\epsilon\left(x\right)$ increases with x.

It then follows that the value function is more elastic for outcomes that are larger in absolute magnitude. $\square$

Proof of Corollary 2: That $\epsilon_{v}>0$, follows from (3.3) and (3.4). Also from (3.4) we get:

$$v''\left(x\right)=\frac{v\left(x\right)}{x}\left[\epsilon_{v}'-\frac{\epsilon_{v}\left(1-\epsilon_{v}\right)}{x}\right]. \tag{7.13}$$

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If $x > 0$ then $v(x) > 0$, $v''(x) < 0$, $\epsilon'_v(x) \geq 0$. From (7.13) it follows that, necessarily, $\epsilon_v < 1$.

If $x < 0$ then $v(x) < 0$, $v''(x) > 0$, $\epsilon'_v(x) \leq 0$. From (7.13), it follows that, again, $\epsilon_v < 1$.
$\square$

Proof of Proposition 3: Let

$$0 < x < y. \tag{7.14}$$

By A0, there is a time, t, such that the consumer is indifferent between receiving the increment x now and receiving the increment y, t-periods from now. Then, letting v be the value function and φ the discount function, we get

$$v(x) = v(y)\,\varphi_+(t). \tag{7.15}$$

Multiply (7.15) by $\varphi_+(s)$, where $s > 0$, to get

$$v(x)\,\varphi_+(s) = v(y)\,\varphi_+(s)\,\varphi_+(t), \tag{7.16}$$

A3a, (7.14)and (7.15) give

$$v(x)\,\varphi_+(s) = v(y)\,\varphi_+(s+t+\alpha_+ st),\ \alpha_+ > 0, \tag{7.17}$$

(7.16) and (7.17) give

$$\varphi_+(s+t+\alpha_+ st) = \varphi_+(s)\,\varphi_+(t) \tag{7.18}$$

Let

$$X = 1 + \alpha_+ s, Y = 1 + \alpha_+ t. \tag{7.19}$$

Hence

$$s = \frac{X-1}{\alpha_+}, \quad t = \frac{Y-1}{\alpha_+}, \quad s+t+\alpha_+ st = \frac{XY-1}{\alpha_+}. \tag{7.20}$$

Define the function $G : [1, \infty) \to (0, \infty)$ by

$$G(X) = \varphi_+\left(\frac{X-1}{\alpha_+}\right). \tag{7.21}$$

Hence,

$$G(Y) = \varphi_+\left(\frac{Y-1}{\alpha_+}\right), G(XY) = \varphi_+\left(\frac{XY-1}{\alpha_+}\right). \tag{7.22}$$

From (7.18),(7.20),(7.21),(7.22)

$$G(XY) = \varphi_+\left(\frac{XY-1}{\alpha_+}\right) = \varphi_+(s+t+\alpha_+ st) = \varphi_+(s)\,\varphi_+(t)$$

$$= \varphi_+\left(\frac{X-1}{\alpha_+}\right)\varphi_+\left(\frac{Y-1}{\alpha_+}\right) = G(X)\,G(Y).$$

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Define the function $h : [0, \infty) \rightarrow (0, \infty)$ by

$$h(x) = G(e^x), x \geq 0 \tag{7.23}$$

Hence, and in the light of A0, h satisfies[12]:

$$h : [0, \infty) \rightarrow (0, \infty) \text{ is strictly decreasing and } h(x + y) = h(x)h(y). \tag{7.24}$$

As is well known, see for example Corollary 1.4.11 in Eichhorn (1978), the solution of (7.24) is the exponential function

$$h(x) = e^{c_+ + x}, x \geq 0, c_+ < 0, \tag{7.25}$$

(7.19), (7.21), (7.23), (7.25) give

$$\varphi_+(t) = (1 + \alpha_+ t)^{c_+}. \tag{7.26}$$

Let

$$\beta_+ = -\alpha_+ c_+, \tag{7.27}$$

(7.26), (7.27) give

$$\varphi_+(t) = (1 + \alpha_+ t)^{-\frac{\beta_+}{\alpha_+}}, \alpha_+, \beta_+ > 0, t \geq 0, \tag{7.28}$$

where $\beta > 0$ because $\alpha > 0$ and $c < 0$. $\square$

Proof of Proposition 4: Suppose $0 < x < y$. By A0, $v(x) = v(y)\varphi_+(t)$ for some $t \in [0, \infty)$. By A1, $v(-x) > v(-y)\varphi_-(t)$. Hence, from A6, $-\lambda v(x) > -\lambda v(y)\varphi_-(t)$. Hence, $v(y)\varphi_-(t) > v(x) = v(y)\varphi_+(t)$. It follows that $\varphi_-(t) > \varphi_+(t)$. We also get $\frac{v(-x)}{v(-y)} = \frac{-\lambda v(x)}{-\lambda v(y)} = \frac{v(x)}{v(y)}$. Finally, $\epsilon_v(-x) = \frac{-x}{v(-x)}v'(-x) = \frac{-x}{-\lambda v(x)}\left(-\frac{-\lambda dv}{dx}\right) = \frac{x}{v(x)}\frac{dv}{dx} = \epsilon_v(x)$.
$\square$

Proof of Proposition 5: Consider the two consumption streams: $((0, 0), (c, s), (0, s + t))$ and $((0, 0), (0, s), (c, s + t))$, where $c > 0$. If the consumer receives $((0, 0), (0, s), (c, s + t))$ when he was expecting $((0, 0), (c, s), (0, s + t))$ then, according to prospect theory, he codes the postponement of c as a loss in period 2 but a gain in period 3. According to (3.1), $V((0, 0), (0, s), (c, s + t)) = v(0)\varphi_+(0) + v(-c)\varphi_-(s) + v(c)\varphi_+(s + t) = -\lambda v(c)\varphi_-(s) + v(c)\varphi_+(s + t)$, where the last inequality comes from $v(0) = 0$ and $v(-c) = -\lambda v(c)$. On the other hand, if the consumer receives $((0, 0), (c, s), (0, s + t))$ when he was expecting $((0, 0), (0, s), (c, s + t))$, he codes the bringing forward of consumption as a gain in period 2 but a loss in period 3. Hence, $V((0, 0), (c, s), (0, s + t)) = v(0)\varphi_+(0) + v(c)\varphi_+(s) + v(-c)\varphi_-(s + t) = v(c)\varphi_+(s) - \lambda v(c)\varphi_-(s + t)$. We thus have: $V((0, 0), (c, s), (0, s + t)) + V((0, 0), (0, s), (c, s + t)) = v(c)\varphi_+(s) - \lambda v(c)\varphi_-(s + t) -$

---

[12]It is sufficient that h be strictly decreasing in some interval: $(a, a + \delta), a \geq 0, \delta > 0$.

$\lambda v\left(c\right)\varphi_-\left(s\right)+v\left(c\right)\varphi_+\left(s+t\right)=-\left[\lambda\left(\varphi_-\left(s+t\right)+\varphi_-\left(s\right)\right)-\left(\varphi_+\left(s\right)+\varphi_+\left(s+t\right)\right)\right]v\left(c\right)<$ 0, since $v\left(c\right)>0$, $\varphi_-\left(s+t\right)+\varphi_-\left(s\right)\geq\varphi_+\left(s+t\right)+\varphi_+\left(s\right)$ (Proposition 4) and $\lambda>1$ (A6). Hence, $V\left(\left(0,0\right),\left(c,s\right),\left(0,s+t\right)\right)<-V\left(\left(0,0\right),\left(0,s\right),\left(c,s+t\right)\right)$. $\square$

Proof of Proposition 6: The result is obvious for the CRRA and CARA classes, from (4.1) and (4.3), respectively. We shall concentrate on giving the proof for the HARA class (4.2). For the remaining two classes: the logarithmic (4.4) and the quadratic (4.5), the proof is similar but easier and, so, will be omitted. Let $f\left(y\right)=\ln y-\gamma\ln\left(\mu+y\right)-\ln\left[\left(\mu+y\right)^{1-\gamma}-\mu^{1-\gamma}\right]$, $y=\frac{\theta x}{\gamma}$, $y\geq0$. Then, from (4.2), $\epsilon_v\left(x\right)=\left(1-\gamma\right)e^{f\left(\frac{\theta x}{\gamma}\right)}$. Hence, ϵ_v is decreasing if, and only if, $f\left(y\right)$ is decreasing. Let $g\left(y\right)=\mu^{2-\gamma}+\left(1-\gamma\right)\mu^{1-\gamma}y-\mu\left(\mu+y\right)^{1-\gamma}$, then it is straightforward to show that $f'\left(y\right)<0$ if, and only if, $g\left(y\right)>0$. Simple calculations show that $g\left(0\right)=0$, $g'\left(0\right)=0$ and $g''\left(y\right)=\gamma\mu\left(1-\gamma\right)\left(\mu+y\right)^{-\gamma-1}>0$. Hence, $g\left(y\right)>0$ for $y>0$. Hence f and, thus, also ϵ_v, is decreasing. $\square$

Proof of Proposition 7: Follow from (5.6) by direct calculation. $\square$

Proof of Proposition 8: Can be verified by direct calculation using (5.6), (5.7) and (5.8). $\square$

References

[1] al-Nowaihi, A., Dhami, S., 2006. A note on the Loewenstein-Prelec theory of intertemporal choice. Mathematical Social Sciences 52, 99-108.

[2] Dasgupta, P., Maskin, E., 2002. Uncertainty, waiting costs and hyperbolic discounting. Mimeo. University of Cambridge.

[3] Dasgupta, P., Maskin, E., 2005. Uncertainty and hyperbolic discounting. Mimeo. University of Cambridge.

[4] Eichhorn, W., 1978. Functional Equations in Economics, Addison-Wesley.

[5] Frederick, S., Loewenstein, G., O'Donoghue, T., 2002. Time discounting and time preference: a critical review. Journal of Economic Literature 40, 351-401.

[6] Fudenberg, D., Levine, K., 2006. A dual self model of impulse control. American Economic Review 96, 1449-1476.

[7] Kahneman, D., Tversky, A., 1979. Prospect theory : an analysis of decision under risk. Econometrica 47, 263-291.

[8] Kahneman D., Tversky A., 2000. Choices, Values and Frames, Cambridge University Press, Cambridge.

[9] Laibson, D., 1997. Golden eggs and hyperbolic discounting. Quarterly Journal of Economics 112, 443-477.

[10] Loewenstein G., 1988a. Frames of mind in intertemporal choice. Management Science 34, 200-214.

[11] Loewenstein G., 1988b. The weighting of waiting: response mode effects in intertemporal choice. Working paper, Center for Decision Research, University of Chicago.

[12] Loewenstein G., Prelec D., 1992. Anomalies in intertemporal choice : evidence and an interpretation. The Quarterly Journal of Economics 107, 573-597.

[13] Manzini, P. and M. Mariotti, 2006. A Vague Theory of Choice over Time. Advances in Theoretical Economics. Berkeley Electronic Press vol. 6(1), pages 1265-1265.

[14] Noor, J., 2007. Hyperbolic discounting and the standard model. Mimeo. Boston University.

[15] Phelps, E.S., Pollak, R., 1968. On second-best national saving and game-equilibrium growth. Review of Economic Studies 35, 185-199.

[16] Read, D. and Scholten, M., 2006. Beyond Discounting: The Tradeoff Model of Intertemporal. Choice. Working Paper LSEOR 06.88.

[17] Thaler, R.H., 1981. Some empirical evidence on dynamic consistency. Economic Letters 8, 201-207.

[18] Tversky, A., and Kahneman, D., 1992. Advances in prospect theory: cumulative representation of uncertainty. Journal of Risk and Uncertainty 5, 297-323.