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AUCTIONS IN WHICH LOSERS SET THE PRICE

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Auctions in which Losers Set the Price

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Abstract

We study auctions of a single asset among symmetric bidders with affiliated values. We show that the second-price auction minimizes revenue among all efficient auction mechanisms in which only the winner pays, and the price only depends on the losers' bids. In particular, we show that the k-th price auction generates higher revenue than the second-price auction, for all $k > 2$. If rationing is allowed, with shares of the asset rationed among the t highest bidders, then the $(t+1)$-st price auction yields the lowest revenue among all auctions with rationing in which only the winners pay and the unit price only depends on the losers' bids. Finally, we compute bidding functions and revenue of the k-th price auction, with and without rationing, for an illustrative example much used in the experimental literature to study first-price, second-price and English auctions.

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1 Introduction

We study auctions of a single asset among symmetric bidders with affiliated values, as in Milgrom and Weber (1982). We focus on the class of auction mechanisms, which we call p-auctions, that satisfy the following three properties: 1. The bidder with the highest signal

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wins. 2. Only the winner pays. 3. The price only depends on the losers' signals. Auction mechanisms in this class are defined by a price function $p(\cdot)$ which maps the signals of the losers into the price paid by the winners. The second-price auction is an example of such a mechanism. Other examples include the k-th price auction, with $k > 2$, in which the highest bidder wins and pays a price equal to the k-th highest bid.

Property 1 says that the auction is efficient. The properties of efficiency and that losers do not pay hold in all standard auctions. The third property, that the price paid by the winner is determined by the losing bids, is a robustness property; it holds in any ex-post incentive compatible mechanism (see Bergemann and Morris, 2005, for a recent discussion of robustness in mechanism design and ex-post incentive compatibility). In an auction that satisfies our third property, a bidder does not need to worry about manipulating the price, because the price does not depend on his bid; his bid only determines whether he wins or loses. This property captures an important feature of an ex-post incentive compatible auction, without going as far as requiring no regret after all possible signal-profile realizations.[1]

We show that the second-price auction minimizes revenue among all p-auction mechanisms. In particular, for all $k > 2$, the k-th price auction generates higher revenue than the second-price auction.

The key building block underlying this result is an *arbitrage principle*. In a p-auction, the price does not depend on the winner's bid. It follows that a small change in his bid only has an impact on a bidder's payoff when he is tied with the winner. Conditioning on this event, the marginal benefit of increasing the bid is the expected value of the object, while the marginal cost is the expected price. The arbitrage principle requires marginal benefit and marginal cost to be equal. As a consequence, a bidder's bid in a second-price auction is equal to his expected price in a p-auction, conditional on his bid being tied with the winner. It then follows from affiliation that the bid in a second-price auction is an underestimate of the expected price in a p-auction.

We also consider p-auctions with rationing. Auctions with rationing have been used to

[1] Ex-post incentive compatible mechanisms have the no-regret property that no buyer would want to revise his decision after observing the rivals' behavior (signals).

model IPO's by Parlour and Rajan (2005). As they point out, in a typical IPO there is excess demand at the offer price, and shares are rationed to investors. Rationing schemes are used more widely than just in IPO's, for example to sell tickets to sport and entertainment events. There are two, formally equivalent, ways of introducing rationing. First, the seller could assign a share of the asset to the t highest bidders (with $t > 1$); this requires that partial ownership of the asset be possible (as, for example, in IPO's). Second, the seller could use a lottery to assign the asset to one of the t highest bidders; this is the only meaningful form of rationing when the asset is an indivisible good (e.g., a ticket to a sporting event).

Parlour and Rajan (2005) studied a sealed-bid, uniform price auction, in which the winners are the t highest bidders (with $t > 1$) and the price is the $(t + 1)$-st highest bid. Each of the t winners receives a share whose value, like in uniform rationing, does not depend on the bids. They showed that rationing may raise the issuer's revenue. In particular, their auction may raise higher revenue than a second-price auction with no rationing. (See also Bulow and Klemperer, 2002, for a discussion of the potential benefits of rationing in common value auctions.) We assume that the share that each winner gets (or the probability of winning the lottery among the t winners of the auction) does not depend on the bids, and show that all p-auctions with rationing yield higher revenue than the auction studied by Parlour and Rajan (2005). Thus, for example, revenue can be raised by leaving the number of winners and rationing rule unchanged, but stipulating that the price is some bid lower than the highest losing bid.[2]

Kagel and Levin (1993) were the first to study a special case of the 3-rd price auction with independent private values. They found such an auction useful from an experimental point of view, because its predictions differ in important ways from those of first- and second-price auctions. Wolfstetter (2001) used revenue equivalence to derive the bidding function in the k-th price auction, with $k > 2$, for the general model with independent private values.

Besides shedding theoretical light on the affiliated values model, our results could prove

[2]Uniform rationing of a single object to the t highest of N bidders is strategically equivalent to selling t objects to N bidders with unit demand. We will only present results for the case of rationing of an asset; their counterparts for multi-unit auctions are obtained by simple reinterpretation.

quite useful in the experimental testing of (Bayesian) Nash equilibrium theory. We elaborate on this point in the concluding section.

The paper is organized as follows. The next section introduces the model. Section 3 derives the arbitrage principle and shows that the second-price auction minimizes revenue among the class of mechanisms we study. Section 4 extends the results to the case of rationing. In Section 5 we use an illustrative example to compare k-th price auctions with the auction with rationing of Parlour and Rajan (2005) and the English auction. Section 6 concludes.

2 The Model

N bidders participate in the auction of a single object. Bidder i, $i = 1, 2, \ldots, N$, observes the realization x_i of a signal X_i. There are M other relevant signals, $Z_1, ..., Z_M$, which are not observed by the bidders. Denote with $s = (x_1, ..., x_N, z_1, ..., z_M)$ the vector of signal realizations. Let $s \vee s'$ be the component-wise maximum and $s \wedge s'$ be the component-wise minimum of s and s'. As in Milgrom and Weber (1982), signals are drawn from a distribution with a joint pdf $f(s)$, which is symmetric in $x_1, ..., x_N$ and satisfies the affiliation property:

$$f(s \vee s')f(s \wedge s') \geq f(s)f(s') \qquad \text{for all } s, s'. \tag{1}$$

If the inequality holds strictly, we say that the signals are strictly affiliated. The support of f is $[\underline{x}, \overline{x}]^{N+M}$, with $-\infty < \underline{x} < \overline{x} < +\infty$. We will abuse notation and write $f(x_1, ..., x_N)$ for the marginal density of $x_1, ..., x_N$. We will also assume, for simplicity, that f is differentiable.

The value V_i of the object to bidder i is a function of all signals: $V_i = u(X_i, \{X_j\}_{j \neq i}, \{Z_h\}_{h=1}^{M})$. The function $u(\cdot)$ is non-negative, bounded, differentiable, increasing in each variable, and symmetric in the other bidders' signal realizations x_j, $j \neq i$. The model with affiliated private values corresponds to valuation function $u(X_i, \{X_j\}_{j \neq i}, \{Z_h\}_{h=1}^{M}) = X_i$; that is, bidder i's valuation depends only on his own signal.

In studying the equilibrium of a given auction, it is useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$, and to consider the order statistics associated with the signals of all other bidders. We denote with Y^n the n-th highest signal of bidders $2, 3, ..., N$ (i.e., all bidders except bidder 1).

Define

$$v_t(x, y) = E\left[V_1 | X_1 = x, Y^t = y\right].$$

Affiliation implies that $v_t(x, y)$ is increasing in both arguments, and hence differentiable almost everywhere (see Milgrom and Weber, 1982, Theorem 5).

3 p-Auctions

By the revelation principle (see Myerson, 1981), given any auction, or mechanism, there is an equivalent direct mechanism where bidders directly report their signals to a designer, and it is an equilibrium for all bidders to report truthfully. A direct mechanism can be thought of as a proxy auction in which each bidder reports a signal to a proxy bidder who then bids on his behalf in the true auction.

We are interested in the class of (direct) auction mechanisms, called p-auctions, which satisfy the following three properties: 1. The bidder with the highest signal value wins. 2. Only the winner pays. 3. The price does not depend on the winner's signal and is a weakly increasing function of the losers' signals. These three properties are a mix of efficiency, simplicity, and robustness requirements. Properties 1 and 2 are satisfied by all standard auctions. Property 1 implies that the auction is efficient. Property 3 captures an important feature of an ex-post incentive compatible auction, without going as far as requiring no regret after all possible signal-profile realizations. In an auction that satisfies it, bidders cannot directly manipulate the price.

Let $r_1, ..., r_N$ be the bidders' reported signal values, in decreasing order ($r_1 \geq r_2 \geq ... \geq r_N$). In a p-auction the winner is the bidder with the highest signal and he pays a price $p(r_2, r_3, ..., r_N)$. If $r_i > r_i'$ for all $i = 2, ..., N$, then $p(r_2, r_3, ..., r_N) > p(r_2', r_3', ..., r_N')$.

The k-th price auction, with $k \geq 2$, in which the highest bidder wins and pays a price equal to the k-th highest bid, corresponds to a p-auction with a price function $p(r_k)$ that only depends on r_k. The first-price auction, clearly, is not equivalent to any p-auction. The English auction, on the other hand, corresponds to a p-auction with a price function $p(r_2, ..., r_N)$ that depends on the reports of all losers.

We now derive a (necessary) equilibrium condition that must be satisfied by a p-auction.

Theorem 1 (*The Arbitrage Principle*) *A p-auction must satisfy the following condition*

$$E\left[V_1|X_1 = x, Y^1 = x\right] = E\left[p\left(Y^1, ..., Y^{N-1}\right)|X_1 = x, Y^1 = x\right], \tag{2}$$

together with the boundary condition

$$p(\underline{x}, ..., \underline{x}) = E\left[V_1|X_1 = \underline{x}, Y^1 = \underline{x}\right]. \tag{3}$$

Proof. Let $v(x, y_1, ..., y_{N-1})$ be the expected value of the object to bidder 1 when his signal is x and the (ordered) signals of the other bidders are $y_1, ..., y_{N-1}$. Let $f_{1:N-1}(y_1, ..., y_{N-1}|X_1 = x)$ denote the marginal density of $Y^1, ..., Y^{N-1}$ conditional on $X_1 = x$, and $f_1(y_1|X_1 = x)$ denote the marginal density of Y^1 conditional on $X_1 = x$. If all bidders different from bidder 1 truthfully bid their signals, then type x of bidder 1's payoff when bidding r is

$$\begin{aligned} &U(x, r) \\ &= \int_{\underline{x}}^{r}\int_{\underline{x}}^{y_1}..\int_{\underline{x}}^{y_{N-2}} \left[v(x, y_1, .., y_{N-1}) - p\left(y_1, .., y_{N-1}\right)\right] f_{1:N-1}\left(y_1, .., y_{N-1}|X_1 = x\right) dy_{N-1}..dy_2 dy_1. \end{aligned}$$

The first-order condition for maximization with respect to r can be written as:

$$\begin{aligned} 0 &= v_1(x, r) f_1(r|X_1 = x) \\ &\quad - \int_{\underline{x}}^{r}..\int_{\underline{x}}^{y_{N-2}} p\left(r, y_2, .., y_{N-1}\right) f_{1:N-1}\left(r, y_2, .., y_{N-1}|X_1 = x\right) dy_{N-1}..dy_2, \end{aligned} \tag{4}$$

or, dividing by $f_1(r|X_1 = x)$,

$$v_1(x, r) = E\left[p\left(Y^1, ..., Y^{N-1}\right)|X_1 = x, Y^1 = r\right].$$

In equilibrium, bidder 1 must bid truthfully; hence it must be $r = x$, and (2) holds. ▮

Let P be the price in a p-auction with price function $p(\cdot)$. If bidder 1 wins the auction, then $P = p\left(Y^1, ..., Y^{N-1}\right)$. It follows that condition (2) can be written as:

$$E\left[V_1 | X_1 = x, Y^1 = x\right] = E\left[P | X_1 = x, Y^1 = x\right].$$

This equation can be interpreted as an *arbitrage principle*. In a p-auction the price does not depend on the winner's bid; a bidder's own bid only influences his probability of winning. Thus, at the margin, a bidder's payoff is only affected by his own bid when he is tied for a win. In such a case, the marginal benefit of winning (i.e., of a small raise in his bid) is $E\left[V_1 | X_1 = x, Y^1 = x\right]$, while the marginal cost is $E\left[P | X_1 = x, Y^1 = x\right]$. Optimality, the arbitrage principle, requires the two to be equal. To put it differently, in a p-auction each bidder bids as if his signal were just high enough to win; that is, as if it were tied with the highest signal of his opponents. Conditioning on this event, a bidder chooses a bid that makes him indifferent between winning and losing; that is, such that the expected price is equal to the expected value of the object.

Lemma 1, proven in the Appendix, shows that if either there are affiliated private values, or an additional assumption is satisfied, then conditions (2) and (3) are also sufficient for a p-auction to be well defined (i.e., for a truthful equilibrium to exist).

Assumption 1 *Let $f_{2:N-1}(y_2, ..., y_{N-1} | X_1 = x, Y^1 = r)$ be the density of $Y^2, ..., Y^{N-1}$ conditional on $X_1 = x$ and $Y^1 = r$. The function $\varsigma(\cdot)$ defined by*

$$\varsigma(x, r) = \frac{v_1(x, r)}{f_{2:N-1}\left(r, r, ..., r | X_1 = x, Y^1 = r\right)} \tag{5}$$

is increasing in x for all values of r.

Observe that Assumption 1 would be satisfied if signals were independent, because the denominator of $\varsigma(x, r)$ would not depend on x, while $v_1(x, r)$ and hence the numerator of $\varsigma(x, r)$ increases with x.

Lemma 1 *Suppose that either there are private values, or Assumption 1 holds. Then conditions (2) and (3) are sufficient for a p-auction to be well defined.*

Recall that in a second-price auction the bidder with the highest bid wins at a price equal to the second highest bid; denote with $\beta_2(\cdot)$ its symmetric equilibrium bidding function. Milgrom and Weber (1982) showed that it is:

$$\beta_2(x) = E\left[V_1 | X_1 = x, Y^1 = x\right] = v_1(x, x). \tag{6}$$

Bidder 1 bids the expected value of the object conditional on his own signal, $X_1 = x$, and on his signal being just high enough to guarantee winning (i.e., being equal to the highest signal of all other bidders).

Thus, we can write the arbitrage condition (2) as

$$E\left[P | X_1 = x, Y^1 = x\right] = \beta_2(x).$$

In any p-auction, the expected price of bidder 1 with signal x, conditional on the highest signal of his opponents also being x, is equal to the bid of bidder 1 with signals x in a second-price auction. By affiliation, $\beta_2(x)$ is an increasing function of x. If values are private, then $\beta_2(x) = x$.

We are now ready to derive one of our main results. We will show that any p-auction generates higher revenue than the second-price auction. Thus, in particular, a k-th price auction generates higher revenue than the second-price auction, for all $k > 2$.

Theorem 2 *The second-price auction generates the lowest expected revenue among all p-auctions.*

Proof: Let R be the revenue in a p-auction with price function $p(\cdot)$, and let R_2 be the revenue in the second-price auction. It follows from (2) and (6) that, conditional on $X_1 \geq x = Y^1$,

$$\begin{aligned} E\left[R_2 | X_1 \geq x = Y^1\right] &= \beta_2(x) \\ &= E\left[p\left(Y^1, ..., Y^{N-1}\right) | X_1 = x, Y^1 = x\right] \\ &\leq E\left[p\left(Y^1, ..., Y^{N-1}\right) | X_1 \geq x, Y^1 = x\right] \\ &= E\left[R | X_1 \geq x = Y^1\right], \end{aligned}$$

where the inequality follows from affiliation. Taking expectations of both sides yields $E[R_2] \leq E[R]$. ∎

Remark 1 *Under strict affiliation $E[R_2] < E[R]$, unless $p(r_2, \ldots, r_N) = p(r_2)$.*

The bidder with the second highest signal, say bidder 1, is the price setter in a second-price auction. It follows directly from the arbitrage principle that bidder 1's bid in a second-price auction is equal to the expected price in a p-auction, conditional on bidder 1's signal being tied with the winner's signal. However, because signals are affiliated and bidder 1 has the second highest signal, the expected price in a p-auction conditional on bidder 1 being tied with the highest bidder is an underestimate of the true expected price. It follows that expected revenue is higher in a p-auction than in a second-price auction.

In the special case of affiliated private values, the English and the second-price auction are equivalent and yield the same revenue. It follows that in such a case any p-auction not identical to the second-price auction (for example, the k-th price auction) yields higher revenue than the English auction.

4 p-Auctions with Rationing

In this section we will extend our results to the case of rationing. There are two ways of modelling auctions with rationing. If, as in IPO's, partial ownership is possible, then the seller could assign a share of the asset to the t highest bidders (with $t > 1$). Alternatively, the seller could use a lottery to assign the asset to one of the t highest bidders. The two versions of the model are formally equivalent. In the remainder of the paper we will use the first version, but all results apply to the second version as well.

Auctions with rationing of shares were introduced by Harstad and Bordley (1996) and have been used to model bookbuilding and rationing in IPO's by Parlour and Rajan (2005). As they point out, in a typical IPO there is excess demand at the offer price and shares are rationed to investors. They model IPO's with rationing as a sealed-bid, uniform-price auction in which the winners are the t highest bidders (with $t > 1$) and the unit price is the $(t+1)$-st highest bid. Each of the t winners receives a share of the asset whose value does not depend on the bids (uniform sharing, where each winner receives a share $1/t$, is a special case), and pays his share of the unit price. As Harstad and Bordley (1996) and Parlour and Rajan (2005) showed, the bidding function in such an auction is

$$\beta_{t+1}(x) = E\left[V_1 | X_1 = x, Y^t = x\right]. \tag{7}$$

Note that this is the same as the bidding function in a uniform auction for t objects with bidders having unit demand and the price being the $(t+1)$-st bid (e.g., see Milgrom 1981). More generally, after rescaling payoff functions by $1/t$, uniform rationing of a single object to the t highest of N bidders is strategically equivalent to selling t objects to N bidders with unit demand. We will present results for the case of rationing of a single object; obtaining their counterparts for multi-unit auctions is a simple matter of reinterpretation.

We study p-auctions with rationing of the t winners, called p^t-auctions, which satisfy the following three properties: 1. The bidders with the t highest signals win and the share that each winner gets does not depend on the bids. 2. Only the winners pay; they pay their share of the unit price. 3. The uniform unit price p does not depend on the winners' signals and is an increasing function of the losers' signals.

We can now derive the arbitrage principle for p^t-auctions, characterized by the price function $p^t(r_{t+1}, \ldots, r_N)$.

Theorem 3 (*The Arbitrage Principle*) *A p^t-auction must satisfy the following condition*

$$E\left[V_1|X_1=x,Y^t=x\right]=E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1=x,Y^t=x\right]. \qquad (8)$$

together with the boundary condition

$$p^t(\underline{x},...,\underline{x})=E\left[V_1|X_1=\underline{x},Y^t=\underline{x}\right]. \qquad (9)$$

Proof. The proof is analogous to the proof of Theorem 1. Let $f_{t:N-1}\left(y_t,...,y_{N-1}|X_1=x\right)$ denote the marginal density of $Y^t,...,Y^{N-1}$ conditional on $X_1=x$, and $f_t(y_t|X_1=x)$ denote the marginal density of Y^t conditional on $X_1=x$. If all bidders different from bidder 1 truthfully bid their signals, then type x of bidder 1's payoff when bidding r is proportional to[3]

$$U(x,r)=\int_{\underline{x}}^{r}E\left[\left(v_t(x,Y^t)-p^t\left(Y^t,...,Y^{N-1}\right)\right)|X_1=x,Y^t=y_t\right]f_t(y_t|X_1=x)dy_t.$$

The first-order condition for maximization with respect to r can be written as:

$$v_t(x,r)=E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1=x,Y^t=r\right].$$

In equilibrium, bidder 1 must bid $r=x$; hence (8) holds. ∎

With only minor changes in the notation, we can repeat the arguments in the proof of Lemma 1 to show that under private values conditions (8) and (9) are sufficient for a p^t-auction to be well defined, while when values are not private the following counterpart of Assumption 1 is sufficient, together with (8) and (9).

Assumption 2 *Let* $f_{t+1:N-1}(y_{t+1},...,y_{N-1}|X_1=x,Y^t=r)$ *be the density of* $Y^{t+1},...,Y^{N-1}$

[3] The constant of proportionality equals the expected share of the asset that bidder 1 would get, were he to win.

conditional on $X_1 = x$ and $Y^t = r$. The function $\varsigma(\cdot)$ defined by

$$\varsigma(x,r) = \frac{v_t(x,r)}{f_{t+1:N-1}\left(r,r,...,r|X_1 = x, Y^t = r\right)}$$

is increasing in x for all values of r.

We are now ready to show that the auction with rationing studied by Harstad and Bordley (1996) and Parlour and Rajan (2005) minimizes revenue among all p^t-auctions. Thus, for example, the issuer of an IPO would raise revenue by leaving the number of winners and the rationing rule unchanged, but stipulating that the price is some bid lower than the highest losing bid.

Theorem 4 *The p^t-auction with rationing in which the unit price is the $(t+1)$-st bid, generates the lowest expected revenue among all p^t-auctions.*

Proof: The proof follows along the same lines as the proof of Theorem 2. Let R^t be the revenue in a p^t-auction with price function $p^t(\cdot)$, and let R^t_{t+1} be the revenue in the Harstad-Bordley and Parlour-Rajan auction. It follows from (8) and (7) that, conditional on $X_1 \geq x = Y^t$,

$$\begin{aligned} E\left[R^t_{t+1}|X_1 \geq x = Y^t\right] &= \beta^t_{t+1}(x) \\ &= E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1 = x, Y^t = x\right] \\ &\leq E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1 \geq x, Y^t = x\right] \\ &= E\left[R^t|X_1 \geq x = Y^t\right], \end{aligned}$$

where the inequality follows from affiliation. Taking expectations of both sides yields $E\left[R^t_{t+1}\right] \leq E\left[R^t\right]$. Under strict affiliation the inequality is strict unless $p^t(r_{t+1},\ldots,r_N) = p^t(r_{t+1})$. ∎

Theorem 4 does not contradict the main message of Parlour and Rajan (2005). They showed that with common values rationing may raise the issuer's revenue. In particular, their auction may raise higher revenue than the second-price auction. Theorem 4 shows that there are many auctions with rationing that yield even higher revenue than the auction they proposed.

Note also that if values are private, then the Parlour and Rajan auction with rationing always yields less revenue than the second-price auction. In such a case $\beta_{t+1}(x) = x$ and hence revenue in the second-price auction is the expected value of the second order statistic out of the N bidders' signals, while in the Parlour and Rajan auction revenue is the expected value of the $(t+1)$-st order statistic, with $t > 1$.

5 An Illustrative Example

In this section, we discuss the best known analytically solvable example of auctions with affiliated values. We will derive equilibrium bidding functions and revenue results for the k-th price ($k \geq 2$) and the English auctions with rationing.

In a k-th price auction with rationing, N bidders submit sealed bids, the t highest bidders win ($N \geq k > t \geq 1$), and each winner is allocated a given share and pays his share of the k-th highest bid. In an English auction with rationing, bidding stops when there are only t bidders left. Each of them is allocated a share of the asset and pays a share of the unit price, the bid of the last bidder to drop out of the auction. In both auctions, no rationing corresponds to $t = 1$.

Example 1 *There is a single object and N bidders. Conditional on $V = v$, each bidder's signal is drawn independently from a uniform distribution on $[v - \frac{1}{2}, v + \frac{1}{2}]$, where the random variable (or signal) V corresponds to the object's common value component. Bidder i's payoff consists of a private-value and a common-value component, with weights λ and $(1-\lambda)$ respectively, $0 \leq \lambda \leq 1$. It is $u(X_i, \{X_j\}_{j \neq i}, V) = \lambda X_i + (1-\lambda)V$. The random variable V has a diffuse prior; that is, it is uniformly distributed on $[-M, M]$ with $M \to \infty$.*

This example has been extensively used in the experimental literature to study first-price, second-price, and English auctions in the two polar cases of pure private ($\lambda = 1$) and pure common values ($\lambda = 0$); see Kagel, Harstad and Levin (1987), Kagel and Levin (2002), and Parlour et al. (2007). Klemperer (2004, pp. 55-57) presents the equilibria and revenue comparisons of first-price, second-price and English auctions for the pure common-value case in which $\lambda = 0$. Parlour and Rajan (2005) study a few variations of this example with $\lambda = 0$, including some in which the signal distribution is not uniform and the random variable V has finite support, rather than being diffuse over the real line. These variations have the advantage of making the model more realistic (e.g., V is bounded above and below), but come at the cost of having to resort to numerical methods in order to calculate bidding functions near the boundary of the signal support and expected revenue. In the interior of the signal support, on the other hand, the bidding functions correspond to the analytically solvable version of the example we study. While our intent in this section is mostly illustrative, our results could prove useful in future experimental research.[4]

Proposition 5 *In Example 1, the bidding function in a k-th price auction with rationing is given by*

$$\beta_k^t(x) = x + \frac{k-1}{N} - \frac{1}{2} + \lambda\left[\frac{1}{2} - \frac{t}{N}\right]. \tag{10}$$

The expected revenue in a k-th price auction with rationing, conditional on $V = v$, is

$$E[R_k^t|V = v] = v + \lambda\left[\frac{1}{2} - \frac{t}{N}\right] - \frac{N+1-k}{N(N+1)}. \tag{11}$$

The proof is in the appendix. The bidding function and revenue in a k-th price auction with rationing satisfy the following properties. (1) The bid and revenue are increasing functions of k. (2) The bid decreases (and revenue need not increase) with the number of bidders N. (3) The bid and revenue increase with the weight λ attached to the private-value component if and only if $t < N/2$. (4) For fixed k and $\lambda > 0$, the bid and revenue decrease with the rationing parameter t.

Wolfstetter (2001) showed that the first two properties of the bidding function hold in the

[4] We should stress that only for the case of λ "sufficiently close" to 1, we have been able to establish existence of equilibrium (i.e., that the second order conditions hold). For other values of λ the bidding functions we present in Proposition 5 are the only increasing symmetric equilibrium candidates.

(general) model with independent private values. As is well known, with independent signals revenue equivalence holds, and hence for fixed t revenue does not depend on k.

Property (4) shows that, at least in this example, rationing is not beneficial in a k-th price auction: any auction with rationing in which the price is the k-th highest bid yields less revenue than the very same auction without rationing ($t = 1$).

Property (4) does not contradict Parlour and Rajan (2005), who claimed that rationing raises bids. They assumed $k = t + 1$, and if one makes such an assumption, then indeed the bid increases with rationing (i.e., with t), provided $\lambda < 1$; that is, provided values are not purely private.

Proposition 6 *In Example 1, suppose bidder* 1 *with signal* x *is left with* t *opponents in an English auction with rationing, and hence all signals* $Y^{t+1}, ..., Y^{N-1}$ *have been revealed during the bidding. Then, bidder 1 bids:*

$$\beta_E^t(x) = x + (1-\lambda)\left[(y_{N-1} + 1 - x)\frac{t}{t+1} - \frac{1}{2}\right]. \tag{12a}$$

The expected revenue in an English auction with rationing, conditional on $V = v$, *is*

$$E[R_E^t|V = v] = v + \frac{1}{2} - \lambda\frac{t+1}{N+1} - (1-\lambda)\left(\frac{1}{(N+1)(t+1)} + \frac{1}{2}\right). \tag{13}$$

The proof is in the appendix. The bidding function and revenue in an English auction with rationing satisfy the following properties. (1) Revenue increases with the number of bidders N. (2) The bid and revenue may increase or decrease with the weight λ attached to the private-value component. (3) For $\lambda < 1$, the bid increases with the rationing parameter t. Revenue increases with the rationing parameter t if and only if $t < \sqrt{\frac{1-\lambda}{\lambda}} - 1$. In particular, with common values ($\lambda = 0$) an increase in the rationing parameter increases both bids and revenue, while with private values an increase in rationing reduces revenue. Note here the contrast with the k-th price auction, where rationing is never beneficial.

According to the standard interpretation of the "linkage principle" (see Milgrom and

Weber, 1982, Milgrom, 1987, Krishna and Morgan, 1997, Krishna, 2002, and Klemperer, 2004), if the price the winner pays in an efficient auction with affiliated signals and common values is more statistically linked to the other bidders' signals, then expected revenue is higher. Since in a k-th price auction the price only depends on "one other bidder's information," this would seem to imply that the expected revenue is higher in an ascending than in any k-th price auction. It is thus interesting to observe that in the case of common values (i.e., $\lambda = 0$) and without rationing (i.e., $t = 1$) the revenue in an English auction is higher than in a k-th price auction if and only if $k < \frac{N+2}{2}$.

This result and the result that with private values the k-th price auction always generates higher revenue than the English auction are related to Lopomo (2000). He showed, using a two-bidder example, that there are auctions yielding greater revenue than the English auction, in which losers do not pay. However, the mechanism in Lopomo's example does not satisfy the property that the price only depends on the losers' bids; it is substantially more complex than p-auctions (especially k-th price auctions), and it is not easy to generalize beyond the two-bidder case.

6 Conclusions

We have studied efficient auctions among symmetric bidders with affiliated values, in which the price paid by the winners depends only on the losing bids, and losers do not pay. Examples of such auctions include the second-price auction and the k-th price auction, with $k > 2$. Auctions in this class satisfy the robustness property that a bidder does not need to worry about manipulating the price he must pay if he wins.

We have shown that the following arbitrage principle holds. At the margin, a change in a bidder's bid only matters when the bidder's signal is tied with the highest signal of the opponents. Conditioning on this event, the expected value of the object (i.e., the marginal benefit of a bid increase) must be equal to the expected price (the marginal cost of a bid increase). A consequence of the arbitrage principle (and signal affiliation) is that the second-price auction minimizes revenue among all auctions in the class we study (the set of p-auctions). We have also considered auctions with rationing, and extended the arbitrage principle to such auctions. When an asset is rationed to t bidders, setting the unit price to be the $(t+1)$-st bid minimizes revenue among the class of p-auctions with rationing.

In studying bidders' behavior in the first-price, second-price, and English auctions with affiliated values, experimentalists have typically used a pure private-value and a pure common-value version of a simple example of the general model. We have provided closed form solutions of the bid function and revenue of the k-th price auction for a generalization of this example, in which values have a private and a common value component and rationing is allowed. We have derived several additional predictions that could prove useful in experimental studies (e.g., in a k-th price auction with rationing of the asset to the top t bidders, the bid and revenue increase with k and decrease with the rationing parameter t, while in an English auction with rationing the bid always increases with t unless values are purely private, and revenue increases with the rationing parameter t if there are common values and decreases with t if there are private values).

Experimental testing of (Bayesian) Nash equilibrium theory is a potentially important application of our results. It is well known that, in common-value auctions, experimental subjects (especially inexperienced ones) do not behave fully in accordance with the predictions of equilibrium theory. Instead, they fall prey of the *winner's curse*; that is, they do not entirely take into account that winning conveys the bad news that all other bidders have lower value estimates (e.g., see Kagel and Levin, 2002). Kagel, Harstad and Levin (1987) showed that similar overbidding also takes place in the second-price auction with affiliated private values. On the other hand, Kagel and Levin (1993) showed that in independent private-value auctions equilibrium theory provides a reasonably accurate prediction of bidders' behavior. Inexperienced subjects, it seems, have difficulties solving the more difficult statistical inference problem associated with affiliated values.

Parlour et al. (2007) tested equilibrium theory in the example of a common value auction with rationing discussed in Section 5, in which one object is assigned with equal probability to the t highest bidders, at a price equal to the $(t+1)$-st bid. Experimental subjects overbid, thus suffering from the winner's curse, but they adjusted their bids in the direction predicted by the theory as the degree of rationing (t) changed. In the equilibrium of a k-th price auction with or without rationing (with $k > t$) a bidder must bid above his value estimate conditional on being tied with the winner (unlike a second-price auction, or the auction studied by Parlour et al., 2007). It seems then reasonable to conjecture that in such auctions there might be less overbidding relative to the equilibrium prediction; an underestimate of the strategic need to bid above value may counteract the winning curse. Testing experimentally this conjecture and the other theoretical results concerning k-th price auctions (with and without rationing) could lead to interesting new insights about the predictive power of Bayesian Nash equilibrium theory.

Appendix

Proof of Lemma 1. We only need to show that if all other bidders participating in the p-auction bid truthfully, then it is optimal for bidder 1 also to bid truthfully. Consider first the case of private values. Suppose type x of bidder 1 bids r. It follows from (4) that the rate of change of bidder 1's payoff with respect to r can be written as

$$\frac{\partial U(x;r)}{\partial r} = v_1(x,r) f_1(r|X_1 = x) - \tag{14}$$
$$f_1(r|X_1 = x) \int_{\underline{x}}^{r} .. \int_{\underline{x}}^{y_{N-2}} p(r, y_2, .., y_{N-1}) f_{2:N-1}\left(y_2, .., y_{N-1}|X_1 = x, Y^1 = r\right) dy_{N-1}..dy_2$$

$$= f_1(r|X_1 = x)\left[v_1(x,r) - E\left[p\left(r, Y^2, ..., Y^{N-1}\right)|X_1 = x, Y^1 = r\right]\right],$$

where the second equality follows after noting that

$$f_{2:N-1}(y_2, ..., y_{N-1}|X_1 = x, Y^1 = r) = \frac{f_{1:N-1}(r, y_2..., y_{N-1}|X_1 = x)}{f_1(r|X_1 = x)} > 0.$$

By affiliation, $E\left[p\left(x, Y^2, ..., Y^{N-1}\right)|X_1 = x, Y^1 = r\right]$ is increasing in r. Furthermore, with private values it is $v_1(x,r) = v_1(x,x)$. Thus, taking into account (2), $\frac{\partial U(x;r)}{\partial r}$ has the same sign as $x - r$. This completes the proof for the case of private values.

Consider now the general model when Assumption 1 holds. Suppose that all other bidders bid truthfully, and type x of bidder 1 bids r. If $r > x$, then it follows from (14) that

$$\frac{\partial U(x;r)}{\partial r}\frac{1}{f_1(r|X_1 = x)}$$
$$= v_1(x,r) -$$
$$\int_{\underline{x}}^{r} .. \int_{\underline{x}}^{y_{N-2}} p(r, y_2, .., y_{N-1}) f_{2:N-1}\left(y_2, ..|X_1 = Y^1 = r\right) \frac{f_{2:N-1}\left(y_2, ..|X_1 = x, Y^1 = r\right)}{f_{2:N-1}\left(y_2, ..|X_1 = r, Y^1 = r\right)} dy_{N-1}..dy_2$$

$$
\begin{aligned}
&\leq v_1(x,r) - \\
&\left(\int_{\underline{x}}^{r} .. \int_{\underline{x}}^{y_{N-2}} p(r, y_2, .., y_{N-1}) f_{2:N-1}\left(y_2, ..|X_1 = Y^1 = r\right) dy_{N-1}..dy_2\right) \frac{f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right)}{f_{2:N-1}\left(r, .., r|X_1 = Y^1 = r\right)} \\
&= v_1(x,r) - v_1(r,r)\frac{f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right)}{f_{2:N-1}\left(r, .., r|X_1 = Y^1 = r\right)} \\
&= \left[\frac{v_1(x,r)}{f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right)} - \frac{v_1(r,r)}{f_{2:N-1}\left(r, .., r|X_1 = r, Y^1 = r\right)}\right] f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right) \\
&\leq 0,
\end{aligned}
$$

where the first inequality holds because, by affiliation, for $r > x$ and all $y_j < r$, $j = 2, ..., N-1$, it is

$$
\frac{f_{2:N-1}\left(y_2, .., y_{N-1}|X_1 = x, Y^1 = r\right)}{f_{2:N-1}\left(y_2, .., y_{N-1}|X_1 = r, Y^1 = r\right)} \geq \frac{f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right)}{f_{2:N-1}\left(r, .., r|X_1 = r, Y^1 = r\right)}, \tag{15}
$$

the second equality follows from (2), and the last inequality holds by Assumption 1. This shows that when $r > x$ it is profitable for bidder 1 to reduce his bid below r.

Now suppose that $r < x$. Then

$$
\begin{aligned}
&\frac{\partial U(x;r)}{\partial r}\frac{1}{f_1(r|X_1 = x)} \\
&\geq v_1(x,r) - \\
&\left(\int_{\underline{x}}^{r} .. \int_{\underline{x}}^{y_{N-2}} p(r, y_2, .., y_{N-1}) f_{2:N-1}\left(y_2, ..|X_1 = Y^1 = r\right) dy_{N-1}..dy_2\right) \frac{f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right)}{f_{2:N-1}\left(r, .., r|X_1 = r, Y^1 = r\right)} \\
&= v_1(x,r) - v_1(r,r)\frac{f_{2:k-1}\left(r, .., r|X_1 = x, Y^1 = r\right)}{f_{2:N-1}\left(r, .., r|X_1 = r, Y^1 = r\right)} \\
&= \left[\frac{v_1(x,r)}{f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right)} - \frac{v_1(r,r)}{f_{2:N-1}\left(r, .., r|X_1 = r, Y^1 = r\right)}\right] f_{2:N-1}\left(r, .., r|X_1 = x, Y^1 = r\right) \\
&\geq 0,
\end{aligned}
$$

where the first inequality holds because, by affiliation, the inequality in (15) is reversed when $r < x$, and the last inequality holds by Assumption 1. It follows that when $r < x$ it is profitable for bidder 1 to increase his bid above r. This completes the proof. ∎

Proof of Proposition 5. One can show (e.g., see Klemperer, 2004) that:

$$E\left[V|X_1 = x, Y^t = x\right] = x - \frac{1}{2} + \frac{t}{N}.$$

Furthermore, since $E\left[Y^{k-1}|V\right]$ is equal to the $(k-1)$-st highest value out of $N-1$ draws from a uniform on $\left[V - \frac{1}{2}, V + \frac{1}{2}\right]$, it is

$$E\left[Y^{k-1}|V\right] = V + \frac{1}{2} - \frac{k-1}{N},$$

and hence it follows that

$$\begin{aligned} E\left[Y^{k-1}|X_1 = x, Y^t = x\right] &= E\left[E\left[Y^{k-1}|V\right]|X_1 = x, Y^t = x\right] \\ &= E\left[V + \frac{1}{2} - \frac{k-1}{N}|X_1 = x, Y^t = x\right] \\ &= x - \frac{k-(1+t)}{N} \end{aligned}$$

Looking for a linear equilibrium $\beta_k^t(x) = a + bx$ of the k-th price auction with rationing, we can write equation (8) as

$$\lambda x + (1-\lambda) E\left[V|X_1 = x, Y^t = x\right] = a + bE\left[Y^{k-1}|X_1 = x, Y^t = x\right],$$

or,

$$\lambda x + (1-\lambda)\left[x - \frac{1}{2} + \frac{t}{N}\right] = a + b\left[x - \frac{k-(1+t)}{N}\right].$$

Hence it is $b = 1$ and $a = \frac{k-1}{N} - \frac{1}{2} + \lambda\left[\frac{1}{2} - \frac{t}{N}\right]$. This gives the bidding function.

Letting Y_N^k be the k-th highest value out of N draws from a uniform on $\left[V - \frac{1}{2}, V + \frac{1}{2}\right]$, the expected revenue in a k-th price auction with rationing, conditional on $V = v$, is

$$\begin{aligned} E[R_k^t|V=v] &= E[\beta_k^t\left(Y_N^k\right)|V=v] \\ &= \left[v+\frac{1}{2}-\frac{k}{N+1}\right]+\frac{k-1}{N}-\frac{1}{2}+\lambda\left[\frac{1}{2}-\frac{t}{N}\right] \\ &= v+\lambda\left[\frac{1}{2}-\frac{t}{N}\right]-\frac{N+1-k}{N(N+1)}. \end{aligned}$$

This completes the proof. ▮

Proof of Proposition 6. Let $f(x|v)$ be the density of x conditional on v; it is equal to 1 for $v \in \left[x-\frac{1}{2}, x+\frac{1}{2}\right]$ and zero otherwise. Its associated distribution in the interior of the support is $F(x|v) = x - v + \frac{1}{2}$. Suppose bidder 1 with signal x is left with t opponents, and hence all signals $Y^{t+1}, ..., Y^{N-1}$ have been revealed during the bidding. Then bidder 1 knows that $v \in \left[x-\frac{1}{2}, y_{N-1}+\frac{1}{2}\right]$; he bids

$$\begin{aligned} \beta_E(x, y_t, ..., y_{N-1}) &= \lambda x+(1-\lambda)E[V|X_1=x, Y^t=x, Y^{t+1}=y_{t+1}, ..., Y^{N-1}=y_{N-1}] \\ &= \frac{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}}[\lambda x+(1-\lambda)v]f^2(x|v)[1-F(x|v)]^{t-1}f(y_{t+1}|v)...f(y_{N-1}|v)dv}{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}}f^2(x|v)[1-F(x|v)]^{t-1}f(y_{t+1}|v)...f(y_{N-1}|v)dv} \\ &= \lambda x+(1-\lambda)\frac{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}}v\left(\frac{1}{2}-x+v\right)^{t-1}dv}{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}}\left(\frac{1}{2}-x+v\right)^{t-1}dv} \\ &= \lambda x+(1-\lambda)\frac{\int_0^{y_{N-1}-x+1}\left(z+x-\frac{1}{2}\right)z^{t-1}dz}{\int_0^{y_{N-1}-x+1}z^{t-1}dz} \\ &= \lambda x+(1-\lambda)\left(x-\frac{1}{2}+\frac{\int_0^{y_{N-1}-x+1}z^t dz}{\int_0^{y_{N-1}-x+1}z^{t-1}dz}\right) \\ &= \lambda x+(1-\lambda)\left(x-1/2+\frac{t}{t+1}\frac{(y_{N-1}-x+1)^{t+1}}{(y_{N-1}-x+1)^t}\right) \\ &= \lambda x+(1-\lambda)\left[x-1/2+(y_{N-1}-x+1)\frac{t}{t+1}\right]. \end{aligned}$$

This gives the bidding function.

Let Y_N^m be the m-th highest value out of N draws from a uniform on $\left[V-\frac{1}{2}, V+\frac{1}{2}\right]$, and recall that $E[Y_N^m|V=v] = v + \frac{1}{2} - \frac{m}{N+1}$. Then, the revenue in the English auction conditional on $V = v$ is

$$E[R_E^t|V=v] = E\left[Y_N^{t+1}\left(1 - \frac{(1-\lambda)t}{t+1}\right) + \frac{(1-\lambda)t}{t+1}Y_N^N + (1-\lambda)\left(\frac{t}{t+1} - \frac{1}{2}\right)|V=v\right]$$

$$\begin{aligned}
&= \left(v + \frac{1}{2} - \frac{t+1}{N+1}\right)\left(1 - \frac{(1-\lambda)t}{t+1}\right) + \frac{(1-\lambda)t}{t+1}\left(v + \frac{1}{2} - \frac{N}{N+1}\right) + (1-\lambda)\left(\frac{t}{t+1} - \frac{1}{2}\right) \\
&= v + \frac{1}{2} - \frac{1+\lambda t}{N+1} - \frac{N}{N+1}\frac{(1-\lambda)t}{t+1} + (1-\lambda)\left(\frac{t}{t+1} - \frac{1}{2}\right) \\
&= v + \frac{1}{2} - \lambda\frac{t+1}{N+1} - (1-\lambda)\left(\frac{1}{(N+1)(t+1)} + \frac{1}{2}\right).
\end{aligned}$$

This completes the proof. ∎

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