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Abstract

Theoretical models of multi-unit, uniform-price auctions assume that the price is given by the highest losing bid. In practice, however, the price is usually given by the lowest winning bid. We derive the equilibrium bidding function of the lowest-winning-bid auction when there are k objects for sale and n bidders, and prove that it converges to the bidding function of the highest-losing-bid auction if and only if the number of losers $n-k$ gets large. When the number of losers grows large, the bidding functions converge at a linear rate and the prices in the two auctions converge in probability to the expected value of an object to the marginal winner.

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1 Introduction

Uniform-price auctions have been extensively used for the sale of homogeneous goods in several countries (e.g., in the sale of Treasury bills and electrical power). In these auctions, the price is usually given by the lowest winning bid. Theoretical models of multi-unit, uniform-price auctions, on the other hand, assume that the price is given by the highest losing bid

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(e.g., Milgrom, 1981, Weber, 1983, Pesendorfer and Swinkels, 1997, Milgrom and Weber, 2000, Kremer, 2002, Jackson and Kremer, 2004, 2006, and Mezzetti, Pekeč and Tsetlin, 2006). A natural question then is: do lowest-winning-bid auctions behave very differently from highest-losing-bid auctions? Do the two auction formats yield similar behavior and prices as the number of bidders increases? An affirmative answer to both questions would provide justification for the theorists's focus on the analytically simpler highest-losing-bid auctions.

When bidders have unit-demand, highest-losing-bid multi-unit auctions behave very much like the second-price auction with a single item for sale. In particular, each bidder bids his expected value for an object conditional on being tied with the price setter. This simplicity is the main reason for their use by theorists.

First, we derive the equilibrium bidding function of the lowest-winning-bid auction in the general affiliated value model with unit demand introduced by Milgrom and Weber (1982); as far as we know, we are the first to study such auctions. Then we show that the bidding functions of the lowest-winning-bid and the highest-losing-bid auction converge as the number of bidders grows large, provided that the number of losers also grows large. More precisely, letting n be the number of bidders and k the number of objects sold, we show that the two bidding functions converge if and only if $n-k$ goes to infinity. As $n-k$ grows, the bidding functions converge at a linear rate. We also show that the prices in the two auctions converge in probability when $n-k$ goes to infinity. They converge to the expected value of an object to the marginal winner; hence, the two auctions become perfectly competitive markets as $n-k$ grows. We conclude that, when the number of losers is large, the highest-losing-bid auction is a good approximation of the uniform auctions used in practice, in which the price is the lowest-winning bid.

The paper is organized as follows. The next section introduces the model. Section 3 derives the bidding function in the k-th price (i.e., the lowest-winning bid) auction for k objects. Section 4 studies the convergence properties of the k-th and $(k+1)$-st price auctions. Section 5 concludes.

2 The Model

We consider a sequence of auctions $\{A_r\}_{r=1}^{\infty}$, where the r-th auction has n_r bidders and k_r objects, with $1 \leq k_r < n_r < n_{r+1}$. Each bidder only demands one object. Bidder i, $i = 1, 2, \ldots, n_r$, observes the realization x_i of a signal X_i. There are m other relevant signals, $W_1, ..., W_m$, which are not observed by the bidders. Denote with $s = (x_1, ..., x_{n_r}, w_1, ..., w_m)$ the vector of signal realizations. Let $s \vee s'$ be the component-wise maximum and $s \wedge s'$ be the component-wise minimum of s and s'. Signals are real random variables drawn from a distribution with a joint pdf $f_r^*(s)$, which satisfies the strict affiliation property (Milgrom and Weber, 1982):

$$f_r^*(s \vee s')f_r^*(s \wedge s') > f_r^*(s)f_r^*(s') \qquad \text{for all } s \neq s'. \tag{1}$$

The support of f_r^* is $[\underline{x}, \overline{x}]^{n_r+m}$, with $-\infty < \underline{x} < \overline{x} < +\infty$. We write $f_r(x)$ for the marginal density of $x = (x_1, ..., x_{n_r})$, and we make the standard assumption that the random variables $X_1, X_2, \ldots$ are symmetric. More precisely, the infinite sequence $\mathbf{X} = (X_1, X_2, ...)$ is exchangeable; that is, for all finite n the joint distribution of $(X_{\pi_1}, \cdots, X_{\pi_n})$ is the same as that of $(X_1, \cdots, X_n)$ for all permutations π.

We also make the following uniform boundedness assumption. There exists $\eta_0 > 0$ such that, for all r, x_i, x_i', and $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n_r})$:[2]

$$\eta_0 < \frac{f_r(x_i, x_{-i})}{f_r(x_i', x_{-i})} < \frac{1}{\eta_0}. \tag{2}$$

The value $V_r^i = u_r(X_i, \{X_j\}_{j \neq i}, \{W_h\}_{h=1}^m)$ of an object to bidder i is a function of all signals. The function $u_r(\cdot)$ is non-negative, differentiable, increasing in each variable, and symmetric in the other bidders' signals X_j, $j \neq i$. We also assume that $u_r(\cdot)$ is uniformly bounded and has uniformly bounded partial derivatives; that is, there exist real numbers a

[2] Pesendorfer and Swinkels (1997) make a similar assumption in the context of pure common values and conditionally independent signals.

and b such that, for all r and all $j = 1, \cdots, n_r$:

$$a < u_r(\underline{x}, \cdots, \underline{x}) < u_r(\overline{x}, \cdots, \overline{x}) < b, \tag{3}$$

$$a < \frac{\partial u_r(\cdot)}{\partial x_j} < b. \tag{4}$$

In studying the symmetric equilibrium bidding function in a given auction, it is useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$, and to consider the order statistics associated with the signals of all other bidders. We denote with Y_r^j the j-th highest signal of bidders $2, 3, ..., n_r$ (i.e., all bidders except bidder 1), with $f_r^j(y_j|X_1 = x)$ the marginal density of Y_r^j conditional on $X_1 = x$, and with $F_r^j(y_j|X_1 = x)$ the corresponding cumulative distribution function.

Define

$$v_r^j(x, y) = E\left[V_r^1|X_1 = x, Y_r^j = y\right]. \tag{5}$$

Affiliation implies that $v_r^j(x, y)$ is increasing in both x and y, and hence differentiable almost everywhere (Milgrom and Weber, 1982, Theorem 5).

In a $(k_r + 1)$-st price (or highest-losing-bid) auction, the k_r bidders with the highest bids win at a price equal to the $(k_r + 1)$-st highest bid. Milgrom (1981) showed that the bidding function in such an auction is $v_r^{k_r}(x, x)$. Bidder 1 bids the expected value of an object conditional on his own signal, $X_1 = x$, and on his signal being just high enough to guarantee winning (i.e., being equal to the k_r-th highest signal of all other bidders).

In a k_r-th price (or lowest-winning-bid) auction, the k_r bidders with the highest bids win an object at a price equal to the k_r-th highest bid. In studying equilibrium of such an auction, it is useful to consider another bidder besides bidder 1, say bidder 2 with signal $X_2 = y$. Denote the signals of bidders $3, \ldots, n_r$, ordered descendingly, by $Z_r^1, \ldots, Z_r^{n_r-2}$. Let $f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})$ be the density of X_2 conditional on $X_1 = x$, and $Z_r^{k_r-1} > y > Z_r^{k_r}$; let $F_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})$ be the corresponding cumulative distribution function.[3] Define

$$Q_r(y,x) = (n_r - k_r)\frac{f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}{F_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}. \tag{6}$$

The function $Q_r(y,x)$ will appear in the formula for the bidding function of a k_r-th price auction.

3 The Bidding Function in the Lowest-Winning-Bid Auction

We begin by showing that $Q_r(y,x)$ is increasing in x.

Lemma 1 *$Q_r(y,x)$, defined by (6), is increasing in x.*

PROOF. The proof is essentially the same as the proof of Lemma 1 in Milgrom and Weber (1982). By strict affiliation, for any $y' < y$ and $x' < x$,

$$\frac{f_r^{X_2}(y|X_1 = x', Z_r^{k_r-1} > y > Z_r^{k_r})}{f_r^{X_2}(y'|X_1 = x', Z_r^{k_r-1} > y > Z_r^{k_r})} < \frac{f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}{f_r^{X_2}(y'|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}.$$

Cross multiplying and integrating with respect to y' over the range $\underline{x} \le y' < y$ yields the result. ∎

We now present an auxiliary lemma, providing an alternative formulation of the function $Q_r(y,x)$, that will be used in the proof of Theorem 1.

Lemma 2 *$Q_r(y,x)$, defined by (6), can equivalently be defined as follows:*

$$Q_r(y,x) = \frac{f_r^{k_r}(y|x)}{P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|x)},$$

where $P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|x)$ is the probability that, conditional on $X_1 = x$, $Y_r^{k_r}$ is below y and $Y_r^{k_r-1}$ is above y.

[3] If $k_r = 1$, we let $Z_r^{k_r-1} = \overline{x}$; in such a case the k_r-th price auction is the first-price auction.

PROOF. Because of the symmetry of the $(n_r - 1)$ signals $X_2, \ldots, X_{n_r}$, it is

$$f_r^{k_r}(y|X_1 = x) = (n_r - 1)\binom{k_r - 1}{n_r - 2} \int_{\substack{z_j > y;\ j \leq k_r - 1 \\ z_j < y;\ j \geq k_r}} f_r(y, z_1, \cdots, z_{n_r-2}|X_1 = x)dz_1 \cdots dz_{n_r-2}, \quad (7)$$

and

$$\begin{aligned} P_r(Y_r^{k_r} &< y < Y_r^{k_r-1}|X_1 = x) \\ &= \binom{k_r - 1}{n_r - 1} \int_{\substack{y_j > y;\ j \leq k_r - 1 \\ y_j < y;\ j \geq k_r}} f_r(y_1, \cdots, y_{n_r-1}|X_1 = x)dy_1 \cdots dy_{n_r-1} \\ &= \binom{k_r - 1}{n_r - 1} \int_{\underline{x}}^{y} \int_{\substack{z_j > y;\ j \leq k_r - 1 \\ z_j < y;\ j \geq k_r}} f_r(x_2, z_1, \cdots, z_{n_r-2}|X_1 = x)dz_1 \cdots dz_{n_r-2}dx_2. \end{aligned}$$

As a result, it is

$$\begin{aligned} \frac{f_r^{k_r}(y|X_1 = x)}{P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|X_1 = x)} &= \frac{(n_r - 1)\binom{k_r-1}{n_r-2} \int_{\substack{z_j > y;\ j \leq k_r - 1 \\ z_j < y;\ j \geq k_r}} f_r(y, z_1, \cdots, z_{n_r-2}|X_1 = x)dz_1 \cdots dz_{n_r-2}}{\binom{k_r-1}{n_r-1} \int_{\underline{x}}^{y} \int_{\substack{z_j > y;\ j \leq k_r - 1 \\ z_j < y;\ j \geq k_r}} f_r(x_2, z_1, \cdots, z_{n_r-2}|X_1 = x)dz_1 \cdots dz_{n_r-2}dx_2} \\ &= \frac{(n_r - 1)\binom{k_r-1}{n_r-2} f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}{\binom{k_r-1}{n_r-1} F_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})} \\ &= (n_r - k_r)\frac{f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}{F_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})} \\ &= Q_r(y, x), \end{aligned} \quad (8)$$

where the last equality follows from (6). ∎

Now we derive the equilibrium bidding function of the k_r-th price auction.

Theorem 1 *The increasing symmetric equilibrium of the lowest-winning-bid auction for k_r objects with n_r bidders is:*

$$b_r(x) = \int_{\underline{x}}^{x} v_r^{k_r}(z, z)Q_r(z, z)e^{-\int_z^x Q_r(t,t)dt}dz, \quad (9)$$

where $v_r^{k_r}(\cdot)$ is defined by (5) and $Q_r(\cdot)$ is defined by (6). Letting $\tau_r(z) = v_r^{k_r}(z, z)$ and

$L_r(z) = e^{-\int_z^x Q_r(t,t)dt}$, *the equilibrium bidding function can also be written as*

$$b_r(x) = v_r^{k_r}(x,x) - \int_{\underline{x}}^{x} L_r(z) d\tau_r(z). \tag{10}$$

PROOF. Consider bidder 1 observing signal x. Bidding according to the function $b^*(\cdot)$ corresponds to a symmetric Nash equilibrium if and only if the expected profit of the bidder who observes signal x is maximized at $b = b^*(x)$, when all other bidders follow $b^*(\cdot)$.

Define

$$v_r^{k_r-1,k_r}(x, y_{k_r-1}, y_{k_r}) = E[V_r^1 | X_1 = x, Y_r^{k_r-1} = y_{k_r-1}, Y_r^{k_r} = y_{k_r}].$$

The expected profit $\Pi(b;x)$ of bidder 1, who observes signal x and bids b, while all other bidders follow $b^*(\cdot)$, is:[4]

$$\begin{aligned}
\Pi(b;x) &= E[\big(V_r^1 - b^*(Y_r^{k_r-1})\big)\, I_{b^*(Y_r^{k_r-1})<b} | X_1 = x] + E[\big(V_r^1 - b\big)\, I_{b^*(Y_r^{k_r})<b<b^*(Y_r^{k_r-1})} | X_1 = x] \\
&= E[E[\big(V_r^1 - b^*(Y_r^{k_r-1})\big)\, I_{b^*(Y_r^{k_r-1})<b} | X_1, Y_r^{k_r-1}] | X_1 = x] \\
&\quad + E[E[\big(V_r^1 - b\big)\, I_{b^*(Y_r^{k_r})<b<b^*(Y_r^{k_r-1})} | X_1, Y_r^{k_r-1}, Y_r^{k_r}] | X_1 = x] \\
&= E[\big(v_r^{k_r-1}(X_1, Y_r^{k_r-1}) - b^*(Y_r^{k_r-1})\big)\, I_{b^*(Y_r^{k_r-1})<b} | X_1 = x] \\
&\quad + E[\big(v_r^{k_r-1,k_r}(X_1, Y_r^{k_r-1}, Y_r^{k_r}) - b\big)\, I_{b^*(Y_r^{k_r})<b<b^*(Y_r^{k_r-1})} | X_1 = x] \\
&= \int_{\underline{x}}^{b^{*^{-1}}(b)} \big(v_r^{k_r-1}(x, y_{k_r-1}) - b^*(y_{k_r-1})\big)\, f_r^{k_r-1}(y_{k_r-1} | X_1 = x) dy_{k_r-1} \\
&\quad + \int_{\underline{x}}^{b^{*^{-1}}(b)} \int_{b^{*^{-1}}(b)}^{\overline{x}} \big(v_r^{k_r-1,k_r}(x, y_{k_r-1}, y_{k_r}) - b\big)\, f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r} | X_1 = x) dy_{k_r-1} dy_{k_r},
\end{aligned}$$

where $f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r} | X_1 = x)$ is the joint density of Y^{k_r-1} and Y^{k_r} conditional on $X_1 = x$.

Let

$$\begin{aligned}
\Pi_1(b;x) &= \int_{\underline{x}}^{b^{*^{-1}}(b)} \big(v_r^{k_r-1}(x, y_{k_r-1}) - b^*(y_{k_r-1})\big)\, f_r^{k_r-1}(y_{k_r-1} | X_1 = x) dy_{k_r-1}, \\
\Pi_2(b;x) &= \int_{\underline{x}}^{b^{*^{-1}}(b)} \int_{b^{*^{-1}}(b)}^{\overline{x}} \big(v_r^{k_r-1,k_r}(x, y_{k_r-1}, y_{k_r}) - b\big)\, f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r} | X_1 = x) dy_{k_r-1} dy_{k_r},
\end{aligned}$$

[4] $I_{b^*(Y^{k_r-1})<b}$ is an indicator function: it equals one if $b^*(Y^{k_r-1}) < b$ and zero otherwise.

so that

$$\Pi(b;x) = \Pi_1(b;x) + \Pi_2(b;x).$$

The derivative of $\Pi_1(b;x)$ with respect to b is

$$\Pi_{1b}(b;x) = \frac{1}{b^{*\prime}(b^{*^{-1}}(b))} \left(v_r^{k_r-1}(x, b^{*^{-1}}(b)) - b\right) f_r^{k_r-1}(b^{*^{-1}}(b)|X_1 = x),$$

and the derivative of $\Pi_2(b;x)$ with respect to b is

$$\begin{aligned}
\Pi_{2b}(b;x) &= \frac{1}{b^{*\prime}(b^{*^{-1}}(b))} \int_{b^{*^{-1}}(b)}^{\overline{x}} \left(v_r^{k_r-1,k_r}(x, y_{k_r-1}, b^{*^{-1}}(b)) - b\right) f_r^{k_r-1,k_r}(y_{k_r-1}, b^{*^{-1}}(b)|X_1 = x) dy_{k_r-1} \\
&\quad - \frac{1}{b^{*\prime}(b^{*^{-1}}(b))} \int_{\underline{x}}^{b^{*^{-1}}(b)} \left(v_r^{k_r-1,k_r}(x, b^{*^{-1}}(b), y_{k_r}) - b\right) f_r^{k_r-1,k_r}(b^{*^{-1}}(b), y_{k_r}|X_1 = x) dy_{k_r} \\
&\quad - \int_{\underline{x}}^{b^{*^{-1}}(b)} \int_{b^{*^{-1}}(b)}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r}|X_1 = x) dy_{k_r-1} dy_{k_r} \\
&= \frac{1}{b^{*\prime}(b^{*^{-1}}(b))} \left(v_r^{k_r}(x, b^{*^{-1}}(b)) - b\right) f_r^{k_r}(b^{*^{-1}}(b)|X_1 = x) \\
&\quad - \frac{1}{b^{*\prime}(b^{*^{-1}}(b))} \left(v_r^{k_r-1}(x, b^{*^{-1}}(b)) - b\right) f_r^{k_r-1}(b^{*^{-1}}(b)|X_1 = x) \\
&\quad - \int_{\underline{x}}^{b^{*^{-1}}(b)} \int_{b^{*^{-1}}(b)}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r}|X_1 = x) dy_{k_r-1} dy_{k_r}.
\end{aligned}$$

Therefore, the derivative of $\Pi_b(b;x)$ with respect to b is

$$\Pi_b(b;x) = \frac{\left(v_r^{k_r}(x, b^{*^{-1}}(b)) - b\right) f_r^{k_r}(b^{*^{-1}}(b)|X_1 = x)}{b^{*\prime}(b^{*^{-1}}(b))} - \int_{\underline{x}}^{b^{*^{-1}}(b)} \int_{b^{*^{-1}}(b)}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r}|X_1 = x) dy_{k_r-1} dy_{k_r}. \tag{11}$$

Note that $\int_{\underline{x}}^{y} \int_{y}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r}|X_1 = x) dy_{k_r-1} dy_{k_r} = P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|X_1 = x)$, the probability that, conditional on $X_1 = x$, $Y_r^{k_r}$ is below y and $Y_r^{k_r-1}$ is above y. Therefore, by setting $\Pi_b(b,x)|_{b=b^*(x)} = 0$, the differential equation for the candidate for an increasing symmetric equilibrium is

$$\frac{1}{b^{*\prime}(x)} \left(v_r^{k_r}(x,x) - b^*(x)\right) f_r^{k_r}(x|X_1 = x) - P_r(Y_r^{k_r} < x < Y_r^{k_r-1}|X_1 = x) = 0,$$

or

$$b^{*\prime}(x) = \left(v_r^{k_r}(x,x) - b^*(x)\right) \frac{f_r^{k_r}(x|X_1 = x)}{P_r(Y_r^{k_r} < x < Y_r^{k_r - 1}|x)}. \tag{12}$$

By Lemma 2, we can then write (12) as

$$b^{*\prime}(x) = \left[v_r^{k_r}(x,x) - b^*(x)\right] Q_r(x,x). \tag{13}$$

The solution of this differential equation, with the boundary condition $b^*(\underline{x}) = v_r^{k_r}(\underline{x},\underline{x})$, yields (9). Integration by parts yields (10).

It only remains to show that deviations from (9) are not profitable. From (11), by setting ${b^*}^{-1}(b) = y$, we get

$$\begin{aligned}
\Pi_b(b;x) &= \frac{1}{b^{*\prime}(y)} \left(v_r^{k_r}(x,y) - b^*(y)\right) f_r^{k_r}(y|X_1 = x) - P_r(Y_r^{k_r} < y < Y_r^{k_r - 1}|X_1 = x) \\
&= \frac{f_r^{k_r}(y|X_1 = x)}{b^{*\prime}(y)} \left(v_r^{k_r}(x,y) - b^*(y) - \frac{b^{*\prime}(y)}{Q_r(y,x)}\right).
\end{aligned}$$

By Lemma 1, $Q_r(y,x)$ is increasing in x, so $\Pi_b(b;x)$ is positive for $x > y$ and negative for $x < y$, which implies that setting $b = b^*(x) = b_r(x)$ maximizes the expected profit of bidder 1.

∎

In the lowest-winning-bid auction, a bidder bids his expected value of the object, conditional on being tied with the marginal (or highest bidding) loser, minus a shading factor. This is similar to the equilibrium bidding function of a first-price auction when there is a single object for sale.

4 Convergence

We now study the convergence of the bidding function $b_r(x)$ of the lowest-winning-bid auction to the bidding function $v_r^{k_r}(x,x)$ of the highest-losing-bid auction, as r grows large.

Theorem 2 *The bidding function of the lowest-winning-bid auction, $b_r(x)$ given by (10), converges to the bidding function of the highest-losing-bid auction, $v_r^{k_r}(x,x)$ given by (5), if and only if the number of losing bidders n_r-k_r goes to infinity. When n_r-k_r goes to infinity, $b_r(x)$ converges to $v_r^{k_r}(x,x)$ at a linear rate.*

Proof. By (10) and (6):

$$\begin{aligned}
b_r(x)-v_r^{k_r}(x,x) &= -\int_{\underline{x}}^{x} L_r(z)d\tau_r(z)\\
&= -\int_{\underline{x}}^{x} e^{-\int_z^x Q_r(t,t)dt}d\tau_r(z)\\
&= -\int_{\underline{x}}^{x} e^{-(n_r-k_r)\int_z^x \frac{f_r^{X_2}(t|X_1=t,Z_r^{k_r-1}>t>Z_r^{k_r})}{F_r^{X_2}(t|X_1=t,Z_r^{k_r-1}>t>Z_r^{k_r})}dt}d\tau_r(z).
\end{aligned}$$

Using the mean value theorem for integrals, we have that there exists t' such that:

$$\begin{aligned}
\frac{f_r^{X_2}(t|X_1=t,Z_r^{k_r-1}>t>Z_r^{k_r})}{F_r^{X_2}(t|X_1=t,Z_r^{k_r-1}>t>Z_r^{k_r})} &= \frac{\int_{\substack{z_j>t;\ j\le k_r-1\\ z_j<t;\ j\ge k_r}} f_r(t,t,z_1,\cdots,z_{n_r-2})dz_1\cdots dz_{n_r-2}}{\int_{\underline{x}}^{t}\int_{\substack{z_j>t;\ j\le k_r-1\\ z_j<t;\ j\ge k_r}} f_r(t,x_2,z_1,\cdots,z_{n_r-2})dz_1\cdots dz_{n_r-2}dx_2}\\
&= \frac{\int_{\substack{z_j>t;\ j\le k_r-1\\ z_j<t;\ j\ge k_r}} f_r(t,t,z_1,\cdots,z_{n_r-2})dz_1\cdots dz_{n_r-2}}{(t-\underline{x})\int_{\substack{z_j>t;\ j\le k_r-1\\ z_j<t;\ j\ge k_r}} f_r(t,t',z_1,\cdots,z_{n_r-2})dz_1\cdots dz_{n_r-2}}.
\end{aligned}$$

By the boundedness assumption (2) on the density function f_r, it is

$$\eta_0 f_r(t,t',z_1,\cdots,z_{n_r-2}) < f_r(t,t,z_1,\cdots,z_{n_r-2}) < f_r(t,t',z_1,\cdots,z_{n_r-2})\frac{1}{\eta_0},$$

and hence

$$-\int_{\underline{x}}^{x} e^{-(n_r-k_r)\eta_0\int_z^x \frac{1}{(t-\underline{x})}dt}d\tau_r(z) < b_r(x)-v_r^{k_r}(x,x) < -\int_{\underline{x}}^{x} e^{-(n_r-k_r)\frac{1}{\eta_0}\int_z^x \frac{1}{(t-\underline{x})}dt}d\tau_r(z). \quad (14)$$

Observe that $-\int_z^x \frac{1}{(t-\underline{x})}dt = \ln\frac{z-\underline{x}}{x-\underline{x}}$; it then follows from (14) that

$$-\int_{\underline{x}}^{x}\left(\frac{z-\underline{x}}{x-\underline{x}}\right)^{(n_r-k_r)\eta_0}d\tau_r(z) < b_r(x)-v_r^{k_r}(x,x) < -\int_{\underline{x}}^{x}\left(\frac{z-\underline{x}}{x-\underline{x}}\right)^{(n_r-k_r)\frac{1}{\eta_0}}d\tau_r(z). \quad (15)$$

By the uniform boundedness assumptions (3) and (4) on the payoff function u_r, the left and right hand side of (15) converge linearly to zero as $n_r - k_r$ grows. This shows that $b_r(x)$ converges to $v_r^{k_r}(x, x)$ if and only if $n_r - k_r$ goes to infinity, and that convergence is at a linear rate. ∎

The intuition behind Theorem 2 is the following. In a k_r-th price auction for k_r objects, a bidder bids the expected value of the object, conditional on his bid being tied with the bid of the marginal loser, minus a shading factor. As the number of losers in the auction increases, the shading factor decreases linearly, reflecting increased competition for the last object. In the limit, the bid in the k_r-th price auction coincides with the bid in the $(k_r + 1)$-st price auction: the expected value of the object conditional on being tied with the marginal loser. In a $(k_r + 1)$-st price auction, the marginal loser is the price setter.

We now show that the prices in the two auctions converge in probability, and they converge to the expected value of an object to the marginal winner. This is because the k_r-th and the $(k_r + 1)$-st order statistic converge in probability as $n_r - k_r$ grows large.

Let X_r^j be the j-th highest signal among all bidders in auction A_r. Consider the marginal winner in auction r, the bidder with k_r-th highest signal; his expected value for an object conditional on his signal being x is $E\left[v_r^{k_r}(x, X_r^{k_r+1}) | X_r^{k_r} = x\right]$.

Theorem 3 *The prices of the lowest-winning-bid auction and the highest-losing-bid auction converge in probability when $n_r - k_r$ goes to infinity; they converge to the expected value of an object to the marginal winner.*

Proof. It suffices to show that the $(k_r + 1)$-st order statistic (i.e., X^{k_r+1}) converges to the k_r-th order statistic in probability when $n_r - k_r$ grows large. To see this note first that the price in a k_r-th price auction is $b_r(X^{k_r})$, while the price in a $(k_r + 1)$-st price auction is $v_r^{k_r}(X_r^{k_r+1}, X_r^{k_r+1})$. By Theorem 2, the prices converge when the order statistics converge. Second, the expected value of an object to the marginal winner with signal x, $E\left[v_r^{k_r}(x, X_r^{k_r+1}) | X_r^{k_r} = x\right]$, converges to $v_r^{k_r}(x, x)$ if the order statistics converge.

Consider the infinite sequence of exchangeable random variables $\mathbf{X} = (X_1, X_2, \cdots)$. By de Finetti's exchangeability theorem (e.g., see Kingman, 1978, for a simple exposition) there exists a real random variable ζ with distribution $H(\zeta)$ and a conditional distribution function $G(\cdot|\zeta)$ such that, for all n, the joint distribution of the random variables $X_1, X_2, \cdots, X_n$ is:

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) = \int_{-\infty}^{+\infty} G(x_1|\zeta)G(x_2|\zeta)\cdots G(x_n|\zeta)dH(\zeta). \tag{16}$$

By (16), the probability that the (k_r+1)-st order statistic is less than $x-\varepsilon$, conditional on the k_r-th order statistic being equal to x, is given by

$$\begin{aligned} P_r(X_r^{k_r+1} &\leq x-\varepsilon | X_r^{k_r} = x) \\ &= \frac{\int_{-\infty}^{+\infty} \left(\frac{G(x-\varepsilon|\zeta)}{G(x|\zeta)}\right)^{n_r-k_r} (1-G(x|\zeta))^{k_r-1} g(x|\zeta) G(x|\zeta)^{n_r-k_r} dH(\zeta)}{\int_{-\infty}^{+\infty} (1-G(x|\zeta))^{k_r-1} g(x|\zeta) G(x|\zeta)^{n_r-k_r} dH(\zeta)}, \end{aligned} \tag{17}$$

where $g(\cdot|\zeta)$ is the density of $G(\cdot|\zeta)$. Since each signal has a positive marginal density on its support, $G(x|\zeta)$ is strictly increasing in x for a positive-measure set of ζ's. This implies that $\left(\frac{G(x_k-\varepsilon|\zeta)}{G(x_k|\zeta)}\right)^{n_r-k_r}$, and hence $P_r(X_r^{k_r+1} \leq x_k - \varepsilon | X_r^{k_r} = x_k)$, goes to zero as $n_r - k_r$ goes to infinity. This concludes the proof.[5] ∎

Define an auction as being *competitive* if the price converges to the value of an object to the marginal buyer as the number of losers grows. Theorem 3 shows that the k_r-th and (k_r+1)-st uniform-price auction are competitive.[6]

[5]Note that the k_r-th and the (k_r+1)-st order statistic converge even if $n_r - k_r$ does not go to infinity, provided that n_r converges to infinity. If n_r goes to infinity, but $n_r - k_r$ does not, then k_r must converge to infinity. We can then write an expression for $P_r(X_r^{k_r} \geq x+\varepsilon | X_r^{k_r+1} = x)$ similar to (17):

$$P_r(X_r^{k_r} \geq x+\varepsilon | X_r^{k_r+1} = x) = \frac{\int_{-\infty}^{+\infty} \left(\frac{1-G(x+\varepsilon|\zeta)}{1-G(x|\zeta)}\right)^{k_r} (1-G(x|\zeta))^{k_r} g(x|\zeta) G(x|\zeta)^{n_r-k_r-1} dH(\zeta)}{\int_{-\infty}^{+\infty} (1-G(x|\zeta))^{k_r} g(x|\zeta) G(x|\zeta)^{n_r-k_r-1} dH(\zeta)},$$

which converges to zero as k_r goes to infinity.

[6]Our definition of a competitive auction is different from the definition in Kremer (2002). In a model with pure common values, he calls an auction competitive if the expected price converges to the expected value of the object. Our definition conforms more closely with the standard definition of a competitive market by economists and applies beyond the common-value setting.

5 Conclusions

This paper provides a link between the highest-losing-bid auctions, which have been extensively studied by theorists, and the lowest-winning-bid auctions that are used in practice. We have shown that the symmetric equilibrium bidding function of the lowest-winning-bid auction converges to the bidding function of the highest-losing-bid auction if and only if the number of losing bidders gets large. When the number of losers grows large, the two bidding functions converge at a linear rate and prices in the two auctions converge in probability to the willingness to pay of the marginal bidder (his expected value for an object).

In a pure common value model with signals that are independent conditional on the common value, Pesendorfer and Swinkels (1997) showed that the $(k+1)$-st price auction aggregates information (i.e., the price converges to the common value in probability) if and only if the number of objects k and the number of losers $n-k$ go to infinity. The results in this paper, specialized to such a pure common value model, imply that the k-th price auction aggregates information under the same conditions. In particular, if $n-k$ goes to infinity, but k stays finite, then the expected value of an object to the marginal winner does not converge, in probability, to the object's common value.

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