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SEQUENTIAL VS. SINGLE-ROUND UNIFORM-PRICE AUCTIONS[1]

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**Abstract**

We study sequential and single-round uniform-price auctions with affiliated values. We derive symmetric equilibrium for the auction in which $k_1$ objects are sold in the first round and $k_2$ in the second round, with and without revelation of the first-round winning bids. We demonstrate that auctioning objects in sequence generates a *lowballing effect* that reduces the first-round price. Total revenue is greater in a single-round, uniform auction for $k = k_1 + k_2$ objects than in a sequential uniform auction with no bid announcement. When the first-round winning bids are announced, we also identify a positive *informational effect* on the second-round price. Total expected revenue in a sequential uniform auction with winning-bids announcement may be greater or smaller than in a single-round uniform auction, depending on the model's parameters.

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# 1 Introduction

Uniform-price auctions are widely used to sell identical, or quite similar, objects. Sometimes sellers auction all objects together in a single round, while other times they auction them separately in a sequence of rounds. For example, cattle, fish, vegetables, timber, tobacco, and wine typically are sold in sequence, while government securities and mineral rights are sold in a single round. When using sequential auctions, the seller must decide what information to release after each round of bidding.

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Two important questions arise. Do sequential sales raise the seller's revenue, or is revenue maximized in a simultaneous auction? How do equilibrium prices in each round of a sequential auction depend on the information that the seller reveals about bidding in earlier rounds? To address these questions, we suppose that the seller owns $k$ identical objects. Each buyer demands only one object.[2] Buyers' value estimates, or signals, are affiliated random variables, as in Milgrom and Weber (1982, 2000).

We first derive the equilibrium bidding strategies of two versions of a sequential uniform-price auction, in which $k_1$ objects are sold in the first round and $k_2 = k - k_1$ are sold in the second round. In both versions, the price in a given round is equal to the highest losing bid in that round. (The extension of our results to more than two rounds is discussed in Section 3.) The two versions differ in the information policy followed by the seller. Under the first policy, the seller does not reveal any information after the first round. Under the second policy, the seller announces all the first-round winning bids before the second round.[3]

While we are the first to study and provide an equilibrium for the sequential, uniform-price auction with winning-bids announcement, the equilibrium of the sequential auction with no bid announcement (when only one object is sold in each round) was conjectured by Milgrom and Weber (2000), first circulated as a working paper in 1982. In a forward and bracketed comments, Milgrom and Weber (2000) explain that the delay in publishing their work was due to the proofs of this and other related results having "refused to come together" (p. 179). They add that the conjectured equilibrium "should be regarded as being in doubt" (p. 188). In Theorem 1, we have been able to prove (for the case of two rounds) that the equilibrium conjectured by Milgrom and Weber (extended to more than just one object per round) is indeed an equilibrium.

After deriving the equilibrium bidding strategies, we compare prices and revenues in the two sequential auctions and the single-round uniform-price auction (Milgrom and Weber, 2000, obtained the equilibrium of the single-round auction). We show that with no bid

[2] Government-run auctions often limit each bidder to bid for at most one asset. This has been the case, for example, in the spectrum auctions of many European countries in the last few years.

[3] As we point out in the concluding section, it is simple to see that intermediate information policies like the policy of revealing only the lowest first-round winning bid (an approximation of the policy of revealing the winning price) yield lower revenue than the policy of revealing all winning bids.

announcement there is a *lowballing effect* at play in the first round. The expected price is lower in the first than in the second round. Furthermore, in the second round the bidding function is the same as in a single-round uniform auction for $k$ objects; bidders bid as if they are tied with the $k$-th highest bid. As a result, total revenue is greater in a single-round auction than in a sequential auction with no bid announcement.

When the first-round winning bids are announced the lowballing effect is still present, but there is also a positive *informational effect* on second-round bids. The informational effect is closely related to the revenue enhancing effect at work in the single-item model, when the seller reveals information that is affiliated with the bidders' signals. Because of the combination of the lowballing and the informational effect, total revenue in a sequential uniform auction with winning-bids announcement could be greater or smaller than in a single-round uniform auction. The ranking depends, in a complicated way, on the signal distribution, the number of bidders and their payoff functions, and the number of objects sold in each round. When the sequential auction with winning-bids announcement yields higher revenue than the uniform auction, it may also yield higher revenue than the English auction, and hence any standard simultaneous auction.[4]

Intuition derived from the single-item model had led Milgrom and Weber (2000) to conjecture that auctioning items in sequence would raise greater revenue than a single-round auction for $k$ objects. This conjecture was based on intuition derived from the single-unit, affiliated-values model, where public revelation of information raises revenue (see Milgrom and Weber, 1982). As we pointed out, this conjecture is incorrect; only when the winning bids are announced, there are model parameters under which the sequential auction raises greater revenue than the single-round auction.

While there are many papers on sequential auctions with bidders having independent private values (see Klemperer, 1999, and Krishna, 2002, for surveys), sequential auctions with affiliated values have been little studied. Two papers related to our work are Ortega Reichert (1968) and Hausch (1986). In both papers, bidders demand more than one object.

[4]Milgrom and Weber (1982, 2000) have shown that the ascending (English) auction raises the highest revenue among standard simultaneous auctions. When there are three bidders and two objects, the single-round uniform-price auction is equivalent to an ascending auction.

Ortega Reichert (1968) studies a two-bidder, two-period, sequential first-price auction with positive correlation of bidders' valuations across periods and across bidders. He shows that there is a deception effect. Compared to a one-shot auction, bidders reduce their first-round bids to induce rivals to hold more pessimistic beliefs about their valuations for the second object. Hausch (1986) studies a special discrete case of a two-bidder, two-unit demand, two-signal, two-period, common-value, sequential first-price auction in which both the losing and the winning bids are announced after the first round. Besides the deception effect, he shows that there is an opposite informational effect that raises the seller's revenue. In our model, bidders have unit-demand, so there is no deception effect; with no bid announcement first-round bids are lower because bidders condition on being tied with the price setter, not because they want to deceive their opponents. Furthermore, when the seller reveals the first-round winning bids, there are informational effects on both first and second-round bidding.

We introduce the model in the next section. Section 3 studies the symmetric equilibria of the sequential auction with and without winning-bids announcement. Section 4 compares the price sequences and revenues in the sequential and single-round auctions. Section 5 concludes. The proofs of the theorems reported in Section 3 are in the Appendix.

## 2 The Model

We consider the standard affiliated-value model of Milgrom and Weber (1982, 2000). A seller owns $k$ identical objects. There are $n$ bidders participating in the auction, every bidder desiring only one object. Before the auction, bidder $i$, $i = 1, 2, \ldots, n$, observes the realization $x_i$ of a signal $X_i$. Let $s_1, ..., s_m$ be the realizations of additional signals $S_1, ..., S_m$ unobservable to the bidders, and denote with $w$ the vector of signal realizations $(s_1, ..., s_m, x_1, ..., x_n)$. Let $w \vee w'$ be the component-wise maximum and $w \wedge w'$ be the component-wise minimum of $w$ and $w'$. Signals are drawn from a distribution with a joint pdf $f(w)$, which is symmetric in its last $n$ arguments (the signals $x_i$) and satisfies the affiliation property:

$$f(w \vee w')f(w \wedge w') \geq f(w)f(w') \qquad \text{for all } w, w'. \tag{1}$$

The support of $f$ is $[\underline{s}, \overline{s}]^m \times [\underline{x}, \overline{x}]^n$, with $-\infty \leq \underline{s} < \overline{s} \leq +\infty$, and $-\infty \leq \underline{x} < \overline{x} \leq +\infty$.

The value of one object for bidder $i$ is given by $V_i = u(S_1, ..., S_m, X_i, \{X_j\}_{j \neq i})$, where the function $u(\cdot)$ satisfies the following assumption.

**Assumption 1.** *$V_i = u(S_1, ..., S_m, X_i, \{X_j\}_{j \neq i})$ is non-negative, bounded, continuous, increasing in each variable, and symmetric in the other bidders' signals $X_j$, $j \neq i$.*

We compare two standard auction formats. In a *single-round uniform auction* (see Vickrey, 1961) the seller auctions all objects simultaneously in a single round. The bidders with the $k$ highest bids win one object each at a price equal to the $(k+1)$-st highest bid. In a *sequential uniform auction*, the seller auctions the objects in two rounds, $k_1$ objects in the first round and $k_2 = k - k_1$ in the second round. In round $t$, $t = 1, 2$, the bidders with the $k_t$ highest bids win one object each at a price equal to the $(k_t + 1)$-st highest bid. Since bidders have unit demand, only the $n - k_1$ first-round losers participate in the second round. For the sequential auction, we consider two information policies. According to the first policy, the seller does not reveal any information after the first round. This is referred to as the *no-bid-announcement* policy. The second policy prescribes that the seller announces the first-round winning bids (i.e., the $k_1$ highest bids), before the second round bids are submitted. This is referred to as the *winning-bids-announcement* policy.

# 3 Symmetric Equilibria

To derive the symmetric equilibrium bidding functions in each of the auction formats, it is useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$, and to consider the order statistics associated with the signals of all other bidders. We denote with $Y^m$ the $m$-th highest signal of bidders $2, 3, ..., n$ (i.e., all bidders except bidder 1).

An important implication of affiliation is that if $H(\cdot)$ is an increasing function, then $E\left[H\left(X_1, Y^1, ..., Y^k\right) | c_1 \leq Y^1 \leq d_1, ..., c_k \leq Y^k \leq d_k\right]$ is increasing in all its arguments (Milgrom and Weber, 1982, Theorem 5). We use this property repeatedly in our proofs; when we refer to affiliation, we refer to this property.

We denote with $b^s(\cdot)$ the symmetric equilibrium bidding function of the single-round uniform auction; $b_t^n(\cdot)$ and $b_t^a(\cdot)$ are the symmetric equilibrium bidding functions in round $t$, $t = 1, 2$, of the sequential uniform auction with no bid announcement and with winning-bids announcement, respectively.

We begin by recalling (see Milgrom and Weber, 1982, 2000) that a symmetric equilibrium bidding function in the single-round uniform auction is:

$$b^s(x) = E\left[V_1 | X_1 = x, Y^k = x\right]. \tag{2}$$

Due to affiliation and Assumption 1, $b^s(x)$ is an increasing function of $x$. Bidder 1 bids the expected value of an object conditional on his own signal, $X_1 = x$, and on his signal being just high enough to guarantee winning (i.e., being equal to the $k$-th highest signal among all other bidders' signals).

**Theorem 1.** *A symmetric equilibrium bidding function in the sequential uniform auction with no bid announcement is given by*

$$b_2^n(x) = E\left[V_1 | X_1 = x, Y^k = x\right], \tag{3}$$

$$b_1^n(x) = E\left[b_2^n\left(Y^k\right) | X_1 = x, Y^{k_1} = x\right]. \tag{4}$$

In an auction with no bid announcement, bids in both rounds depend only on a bidder's own signal. In a sequential auction with winning-bids announcement, the second-round bid must also depend on the first-round winning bids. If the first-round symmetric-equilibrium bidding function is increasing (as shown below), announcing the winning bids is equivalent to announcing the $k_1$ highest signals. Taking the point of view of a bidder who is bidding in the second round, without loss of generality bidder 1, the announced bids reveal the realizations $y_1, ..., y_{k_1}$ of $Y^1, ..., Y^{k_1}$, the $k_1$ highest signals among bidders $2, ..., n$.

**Theorem 2.** *Let $y_1, ..., y_{k_1}$ be the realizations of the signals that correspond to the winning bids in the first round. A symmetric equilibrium bidding function in the sequential uniform* *auction with winning-bids announcement is given by*

$$b_2^a(x; y_1, ..., y_{k_1}) = E\left[V_1 | X_1 = x, Y^1 = y_1, ..., Y^{k_1} = y_{k_1}, Y^k = x\right], \tag{5}$$

$$b_1^a(x) = E\left[b_2^a\left(Y^k; Y^1, ..., Y^{k_1-1}, x\right) | X_1 = x, Y^{k_1} = x\right]. \tag{6}$$

The proofs are in the Appendix; here we discuss the underlying intuition.

First, note that affiliation and Assumption 1 imply that the bidding functions (3), (4), and (6) are increasing in $x$, while the bidding function (5) is increasing in $x$ and $y_1, ..., y_{k_1}$.

Second, observe that the bidding function in the second round of the auction with no bid announcement (3) coincides with the bidding function in the single-round uniform auction (2). Intuitively, this makes sense. With no announcement, the only additional information that bidders have in the second round is that in the first round $k_1$ bidders bid higher than they did. Since the first-round bid function is increasing, this implies that the remaining bidders know that $k_1$ of the signals of the other bidders are higher than their own. Thus, a bidder bids the expected value of an object conditional on (a) his own signal, (b) the fact that the $k_1$ first-round winners have higher signals, and (c) his own signal being just high enough to win (i.e., being equal to the $(k - k_1)$-th highest signal of the $n - 1 - k_1$ remaining opponents). This is equivalent to saying that a bidder conditions on his own signal and on his signal being equal to the $k$-th highest signal of the other $n - 1$ bidders, which yields the same equilibrium bidding function as in a single-round uniform auction.

Third, the second-round bidding function for the case in which the first-round winning bids are announced must also condition on the signals revealed by this announcement. In this case, each remaining bidder bids the expected value of an object conditional on (a) his own signal, (b) his own signal being just high enough to win (i.e., being equal to the $k$-th highest signal of his opponents), and (c) the revealed signal values of the first-round winning bidders.

Finally, a bidder knows that if he loses in the first round of a sequential auction with or without winning-bids announcement, then he will get another chance to win the object. Hence, he does not want to pay more than what he expects to pay in the second round.

He bids the expected second-round price conditional on the observed value of his own signal and his own signal being just high enough to win in the first round (i.e., being equal to the $k_1$-th highest signal of the opponents). The second-round price is the second-round bid of the opponent with the $k$-th highest signal: $b_2^n(Y^k)$ in an auction with no bid announcement and $b_2^a\left(Y^k; Y^1, ..., Y^{k_1}\right)$ in an auction with winning-bids announcement.

As we shall see in the Appendix, it is simpler to prove Theorem 2 than Theorem 1. There are two steps in the proof of Theorem 2. Assuming that all other bidders follow the bidding functions $b_1^a(\cdot)$ and $b_2^a(\cdot)$, first we show that, no matter what bidder 1 did in the first round, in the second round it is optimal for him to bid according to $b_2^a(\cdot)$. Then we show that in the first round it is optimal to follow $b_1^a(\cdot)$. This method of proof does not fully generalize to the case of no bid announcement. In this case, it is optimal for bidder 1 with signal $x$ to bid according to $b_2^n(x)$ in the second round if and only if he has bid according to $b_1^n(x)$, or lower, in the first round. On the contrary, if bidder 1 has bid higher than $b_1^n(x)$ in the first round and lost, he will want to bid higher than $b_2^n(x)$ in the second round. This, in turn, makes it difficult to show that it is optimal to bid according to $b_1^n(x)$ in the first round. As Milgrom and Weber (2000, p. 182) point out, the difficulty in proving equilibrium existence in this case is in ruling out that "a bidder might choose to bid a bit higher in the first round in order to have a better estimate of the winning bid, should he lose." Our proof of Theorem 1 overcomes this difficulty.

We conclude this section with a remark about extending Theorems 1 and 2 to more than two rounds of bidding.

REMARK. Theorem 2 and its proof readily generalize to the case of any finite number of rounds. Suppose that there are $T$ rounds of bidding and $k_t$ objects are sold in round $t$. Let $m_t = \sum_{\tau=1}^{t} k_\tau$. Then the symmetric equilibrium bidding functions of the sequential auction with winning-bids announcement are

$$b_T^a(x; y_1, ..., y_{m_{T-1}}) = E\left[V_1 | X_1 = x, Y^1 = y_1, ..., Y^{m_{T-1}} = y_{m_{T-1}}, Y^{m_T} = x\right],$$

$$b_t^a(x; y_1, ..., y_{m_{t-1}}) =$$
$$= E\left[b_{t+1}^a\left(Y^{m_{t+1}}; y_1, ..., y_{m_{t-1}}, Y^{m_{t-1}+1}, ..., Y^{m_t-1}, x\right)|X_1 = x, Y^{m_t} = x, Y^1 = y_1, ..., Y^{m_{t-1}} = y_{m_{t-1}}\right].$$

In the case of no bid announcement, this extension presents a technical difficulty. We can show that if a symmetric increasing equilibrium exists, then the bidding functions must have the following form:[5]

$$b_T^n(x) = E\left[V_1|X_1 = x, Y^{m_T} = x\right],$$

$$b_t^n(x) = E\left[b_{t+1}^a\left(Y^{m_{t+1}}\right)|X_1 = x, Y^{m_t} = x\right].$$

However, we have not been able to generalize the existence part of the proof of Theorem 1 to more than two rounds.

# 4 Properties of Sequential Auctions

This section establishes the properties and compares the equilibrium bidding strategies of the single-round and the sequential uniform auctions with and without winning-bids announcement. It is now convenient to take the point of view of the seller, or of an outside observer, and consider the order statistics of the signals of all $n$ bidders. Denote with $Z^m$ the $m$-th highest signal among all $n$ bidders.

We first look at the sequential auction with no bid announcement. Let $P_t^n$ be the price in round $t$ of such an a auction. Prices are random variables: $P_1^n = b_1^n(Z^{k_1+1})$, and $P_2^n = b_2^n(Z^{k+1})$. We show that, conditional on the realization $p_1^n$ of $P_1^n$, the expected second-round price is higher than $p_1^n$.

**Theorem 3.** *In a sequential uniform auction with no bid announcement, the expected second-round price conditional on the realized first-round price is higher than the realized first-round price:* $E\left[P_2^n|p_1^n\right] \geq p_1^n$.

**Proof.** The realized price in the first round is given by $p_1^n = b_1^n(z_{k_1+1})$, where $z_{k_1+1}$ is the

[5] Milgrom and Weber (2000) conjectured that this is an equilibrium (for the case $m_1 = ... = m_T = 1$).

realized value of $Z^{k_1+1}$, the $(k_1+1)$-st highest out of $n$ signals. Thus, conditioning on $p_1^n$ is the same as conditioning on $Z^{k_1+1} = z_{k_1+1}$. The price in the second round is $P_2^n = b_2^n(Z^{k+1})$.

Since conditioning on the event $\{Z^{k_1+1} = z_{k_1+1}\}$ is equivalent to conditioning on the event $\{Y^{k_1} \geq X_1 = z_{k_1+1} \geq Y^{k_1+1}\}$,[6] the expected price in the second round conditional on $p_1^n$ is

$$\begin{aligned} E[P_2^n|p_1^n] &= E[b_2^n(Z^{k+1})|Z^{k_1+1} = z_{k_1+1}] = E[b_2^n(Y^k)|Y^{k_1} \geq X_1 = z_{k_1+1} \geq Y^{k_1+1}] \\ &\geq E[b_2^n(Y^k)|Y^{k_1} = X_1 = z_{k_1+1}] = b_1^n(z_{k_1+1}) = p_1^n, \end{aligned}$$

where the inequality follows from affiliation. ∎

We will call the difference between the expected first-round price and the expected second-round price the *lowballing effect,* $L^n = E[P_1^n] - E[P_2^n]$. Theorem 3 implies that if signals are strictly affiliated, then $L^n < 0$; that is, the expected price in the second round is higher than the expected price in the first round. If, on the other hand, signals are independent, then $L^n = 0$.

Two observations are useful for an intuitive understanding of the lowballing effect. First, in a uniform auction a bidder's payoff only varies with a small change in his first-round bid if he is the first-round price setter and his bid is tied with one of the winners' bids (hence they have the same signals). Optimal bidding requires that, conditional on such an event, a bidder is indifferent between winning in the first or in the second round. In other words, conditional on being the first-round price setter and his bid being tied with the bid of one of the first-round winners, a bidder expects the first and second round prices to be equal.

Second, consider the event, call it event $\Theta$, in which bidder 1 is the first-round price setter and he and a first-round winner have the same realized signal value. By the first observation, conditional on $\Theta$, the expected first-round price equals the expected second-round price. The expected first-round price, conditional on $\Theta$, is a correct estimate of the

[6] Because of the symmetry of signals, conditioning on the event $\{Z^{m+1} = x\}$ is equivalent to conditioning on the event $\{X_1 \geq x, Y^m = x\}$, or on the event $\{Y^m \geq X_1 = x \geq Y^{m+1}\}$. In other words, the event that the $(m+1)$-st highest signal is $x$ is equivalent to the event that one bidder, without loss of generality bidder 1, has a signal higher than or equal to $x$, and the $m$-th highest signal among all other bidders' signals is $x$. It is also equivalent to the event that bidder 1 has signal $x$ and the $m$-th highest signal among all other bidders is greater than $x$, while the $(m+1)$-st highest signal is smaller than $x$.

first-round price conditional on bidder 1 being the first-round price setter. On the other hand, the expected second-round price, conditional on $\Theta$, is a correct estimate of the second-round price conditional on bidder 1 being the first-round price setter only if signals are independent, while it is an underestimate if signals are strictly affiliated. This is because, when bidder 1 is a price setter in the first round, all first-round winners have higher signal values with probability one. It follows that, conditional on bidder 1 being the first-round price setter (and hence also unconditionally, by the law of iterated expectations), the first-round expected price is strictly lower than the second-round expected price when signals are strictly affiliated. This is the lowballing effect. With independent signals, the expected prices in the two rounds coincide and the price sequence is a martingale.[7]

As shown in the next proposition, it is a direct consequence of the lowballing effect, $L^n \leq 0$, and the equality of the second-round price with the price in a single-round auction, that the seller's expected revenue is higher in a single-round uniform auction than in the sequential uniform auction with no bid announcement.

**Proposition 4.** *The seller's expected revenue in a sequential uniform auction with no bid announcement is lower than in a single-round uniform auction for $k$ objects.*

**Proof.** The second round bidding function $b_2^n(x)$, given by (3), is the same as the bidding function in a single-round auction $b^s(x)$, given by (2). Therefore, the expected second-round price in the sequential auction is the same as the expected price in the single-round auction. By Theorem 3, the expected price in the first round of a sequential auction is lower than the expected price in the second round. Thus expected revenue in the sequential auction is lower. ▮

We now study the sequential auction with winning-bids announcement. Let the random variable $P_t^a$ be the price in round $t$: $P_1^a = b_1^a(Z^{k_1+1})$, $P_2^a = b_2^a(Z^{k+1}; Z^1, ..., Z^{k_1})$. We first show that prices drift upward in this case as well.

[7] Milgrom and Weber (2000) and Weber (1983) discuss this result in the standard independent private value model. In a recent paper, Feng and Chatterjee (2005) compare equilibrium prices and revenue in single-round and sequential auctions with independent private values, under the assumption that bidders discount future payoffs and there is uncertainty about the number of items being sold.

**Theorem 5.** *In a sequential uniform auction with winning-bids announcement, the expected second-round price conditional on the realized first-round price is higher than the realized first-round price:* $E\left[P_2^a|p_1^a\right] \geq p_1^a$.

**Proof.** The proof is analogous to the proof of Theorem 3. Letting $p_1^a = b_1^a(z_{k_1+1})$, where $z_{k_1+1}$ is the realized value of $Z^{k_1+1}$, we have

$$\begin{aligned} E[P_2^a|p_1^a] &= E[b_2^a(Z^{k+1}; Z^1, ..., Z^{k_1})|Z^{k_1+1} = z_{k_1+1}] \\ &= E[b_2^a(Y^k; Y^1, ..., Y^{k_1})|Y^{k_1} \geq X_1 = z_{k_1+1} \geq Y^{k_1+1}] \\ &\geq E[b_2^a(Y^k; Y^1, ..., Y^{k_1-1}, z_{k_1+1})|Y^{k_1} = X_1 = z_{k_1+1}] = b_1^a(z_{k_1+1}) = p_1^a. \quad \blacksquare \end{aligned}$$

When the winning bids are announced, we can define the lowballing effect as $L^a = E[P_1^a] - E[P_2^a]$; it is $L^a \leq 0$, with the inequality being strict if signals are strictly affiliated. As in the case of no bid announcement, the lowballing effect stems from the first-round price setter's bidding so as to equate a correct estimate of the first-round price and an underestimate of the second-round price. The lowballing effects under the two different announcement policies cannot be easily compared; their difference depends on the details of the signal distribution, the number of bidders, the payoff function, and the number of objects auctioned in each round.

Revealing the winning bids has an impact on the equilibrium prices. In particular, in the second round bidders have more information on which to base their bids. Define the *informational effect* on second-round prices, $I_2^a$, as the difference between second-round expected prices with and without winning-bids announcement: $I_2^a = E[P_2^a] - E[P_2^n]$.

We now show that when the first-round winning bids are announced the informational effect on second-round bids is positive; that is, the expected second-round price is higher when the first-round winning bids are announced than when they are not.

**Proposition 6.** *In the sequential auction with winning-bids announcement the expected second-round price is higher than the expected second-round price in a sequential auction with no bid announcement:* $I_2^a \geq 0$.

**Proof.** Let $z_{k+1}$ be the realization of the $(k+1)$-st highest signal among the $n$ bidders. The expected price in the second round of the sequential auction with winning-bids announcement, conditional on $Z^{k+1} = z_{k+1}$, is

$$\begin{aligned} E[P_2^a | Z^{k+1} = z_{k+1}] &= E[b_2^a(z_{k+1}; Y^1, ..., Y^{k_1}) | X_1 \geq Y^k = z_{k+1}] \\ &\geq E[b_2^a(z_{k+1}; Y^1, ..., Y^{k_1}) | X_1 = Y^k = z_{k+1}] \\ &= E[E[V_1 | X_1 = Y^k = z_{k+1}, Y^1, ..., Y^{k_1}] | X_1 = Y^k = z_{k+1}] \\ &= E[V_1 | X_1 = Y^k = z_{k+1}] \\ &= b_2^n(z_{k+1}) = E[P_2^n | Z^{k+1} = z_{k+1}], \end{aligned}$$

where the inequality follows from affiliation. Taking expectations over $Z^{k+1}$ concludes the proof. ∎

Recall that the lowballing effect can be explained by focusing on the behavior of the first-round price setter. To understand the intuition behind the informational effect we must look at the *second-round* price setter, say bidder 1. Under both policies, in the second round bidder 1 bids his expected value for the object conditional on (i) being the price setter, (ii) his bid being tied with the bid of a winner, and (iii) any additional available information. By (i) and (ii), in both sequential auction formats the second-round price setter's bid is an underestimate of his true value for the object when signals are affiliated. By (iii), when the first-round bids are announced the price setter conditions on them, and so his bid is a better estimate of the object's value; that is, the bid is closer to the second-round price setter's expected value, and hence the second-round price is higher than when there is no bid announcement.

Announcing the first-round winning bids also changes the first-round bids. Let $I_1^a$ be the difference between first-round expected prices with and without winning-bids announcement: $I_1^a = E[P_1^a] - E[P_1^n]$. It is useful to think of $I_1^a$ as being the sum of two components, the second-round informational effect and the difference in lowballing effects: $I_1^a = I_2^a + (L^a - L^n)$.

For the same reason that it is not possible to say anything general about $L^a - L^n$, no general result about $I_1^a$ is available. As the numerical example in the next subsection will show, $I_1^a$ could be either positive or negative.

All auction formats discussed in this paper are efficient and, as shown by Proposition 4, the single-round uniform auction yields higher revenue than the sequential auction with no bid announcement. The revenue comparison between the single-round uniform auction and the sequential auction with winning-bids announcement, on the other hand, is ambiguous. The numerical example in Section 4.1 shows that either could yield higher revenue. Let $E[R^s]$ be the expected revenue in the single-round auction,

$$E[R^s] = kE[P^s] = kE[P_2^n].$$

Let $E[R^a]$ be the expected revenue in the sequential auction with winning-bids announcement,

$$E[R^a] = k_1 E[P_1^a] + (k - k_1)E[P_2^a] = k_1 L^a + kE[P_2^a].$$

It follows that

$$E[R^a] - E[R^s] = k_1 L^a + kI_2^a;$$

the first component of the revenue difference, the lowballing effect, is negative, while the second component, the information effect, is positive. Intuitively, increasing the number of objects sold in the first round reduces revenue because of the lowballing effect ($L^a \leq 0$), and it increases revenue due to the informational effect. In general, the second-round expected price $E[P_2^a]$ and, as a consequence, the informational effect $I_2^a$, is increasing in $k_1$, because as more objects are sold in the first round, more information filters to the second round when the winning bids are announced.[8] However, the rate of change in the informational effect with respect to $k_1$ depends in a complicated way on the signal distribution, the number of bidders and objects, and the payoff function. Intuitively, it depends on the informational content of revealing one additional order statistic; it could well be a non-monotone function of $k_1$. Similarly, the lowballing effect and its rate of change with respect to $k_1$ could increase

[8]The proof follows the same lines as the proof of Proposition 6.

or decrease and in general are not monotone functions of $k_1$. It follows that, if $k$ objects are to be auctioned, the number of objects $k_1$ that should be sold in the first round to maximize revenue depends on the signal distribution, the number of bidders and objects, and the bidders' payoff function.

There is a special version of the model that yields unambiguous first-round price and revenue ranking. It is the case in which values are private: $V_i = X_i$. In the second round of a sequential auction with affiliated private values, a bidder bids his own value; that is, the second-round bid coincides with the bid in a single-round uniform auction, irrespective of whether the first-round winning bids are revealed. It follows that the first-round bidding function is independent of the information policy. Because of the lowballing effect, with strictly affiliated signals the first-round expected price is lower than the second-round expected price. Thus, with affiliated private values, auctioning the objects in a single round yields the seller higher revenue, independently of the information policy that he follows.

**Proposition 7.** *With affiliated private values, the seller's expected revenue in a sequential uniform auction with winning-bids announcement is the same as in a sequential uniform auction with no bid announcement, and is lower than in a single-round uniform auction for $k$ objects.*

## 4.1 A Numerical Example

We now present a numerical example that shows that revenue and first-round price comparisons are ambiguous, when values are not purely private. The example is constructed to make the numerical calculations as simple as possible, not to be realistic. We assume that there are three bidders, two objects, each bidder has the same value for one object, and that each bidder's signal is a conditionally independent estimate of this common value. This is a special case of our model in which $n = 3$, $k_1 = k_2 = 1$, $u(V, X_1, X_2, X_3) = V$, and each $X_i$, $i = 1, 2, 3$, is independently drawn from a conditional density $f(x|v)$. The common value has a discrete distribution: its value is either $v_1 = 0$ or $v_2 = 0.5$, with equal probability. The pdf

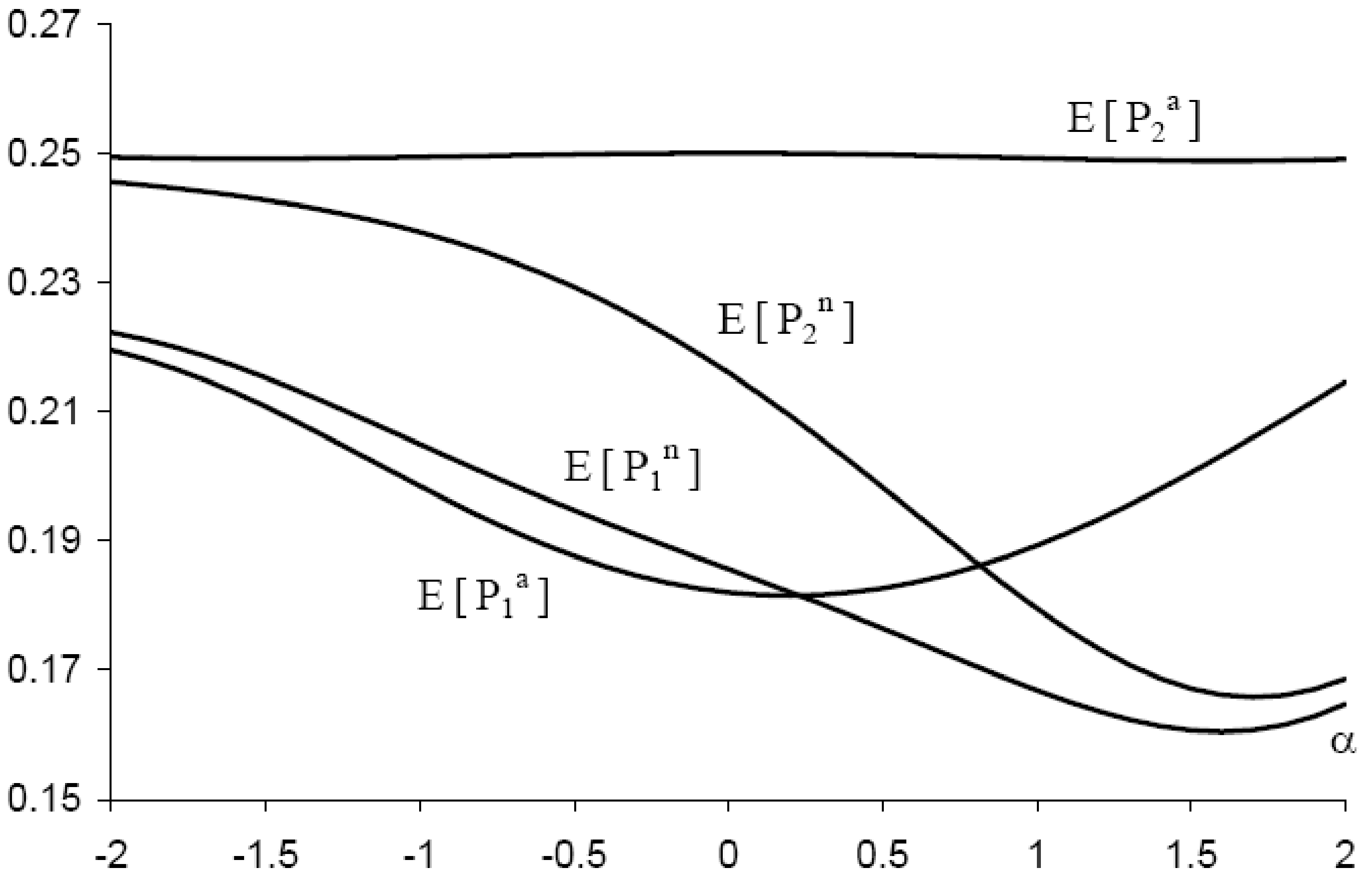


Figure 1: Prices as functions of the parameter $\alpha$.

$f(x|v)$ is given by

$$f(x|v)=\begin{cases}1+\alpha\,(x-v) & \text{if} \quad x\in\left[v-\frac{1}{2},v+\frac{1}{2}\right], \\ 0 & \text{if} \quad x\notin\left[v-\frac{1}{2},v+\frac{1}{2}\right],\end{cases} \tag{7}$$

where $\alpha\in[-2,2]$.[9]

Figure 1 plots the expected prices $E\left[P_1^n\right]$, $E\left[P_2^n\right]$, $E\left[P_1^a\right]$, and $E\left[P_2^a\right]$ as functions of $\alpha$. Recall that, by (2) and (3), $E\left[P_2^n\right]$ is equal to $E\left[P^s\right]$. The following conclusions can be drawn from the figure.

First, as claimed by Theorems 3 and 5, $E\left[P_1^n\right]$ is always less than $E\left[P_2^n\right]$, and $E\left[P_1^a\right]$ is

[9] In such a case, the affiliation property can be written as $f(x|v)f(x'|v')\geq f(x|v')f(x'|v)$ for all $x,x',v,v'$ such that $x\geq x'$ and $v\geq v'$. The signal distribution (7) satisfies the affiliation property (1). All the results in the previous sections extend to the model with a common value $V$ having a discrete distribution.

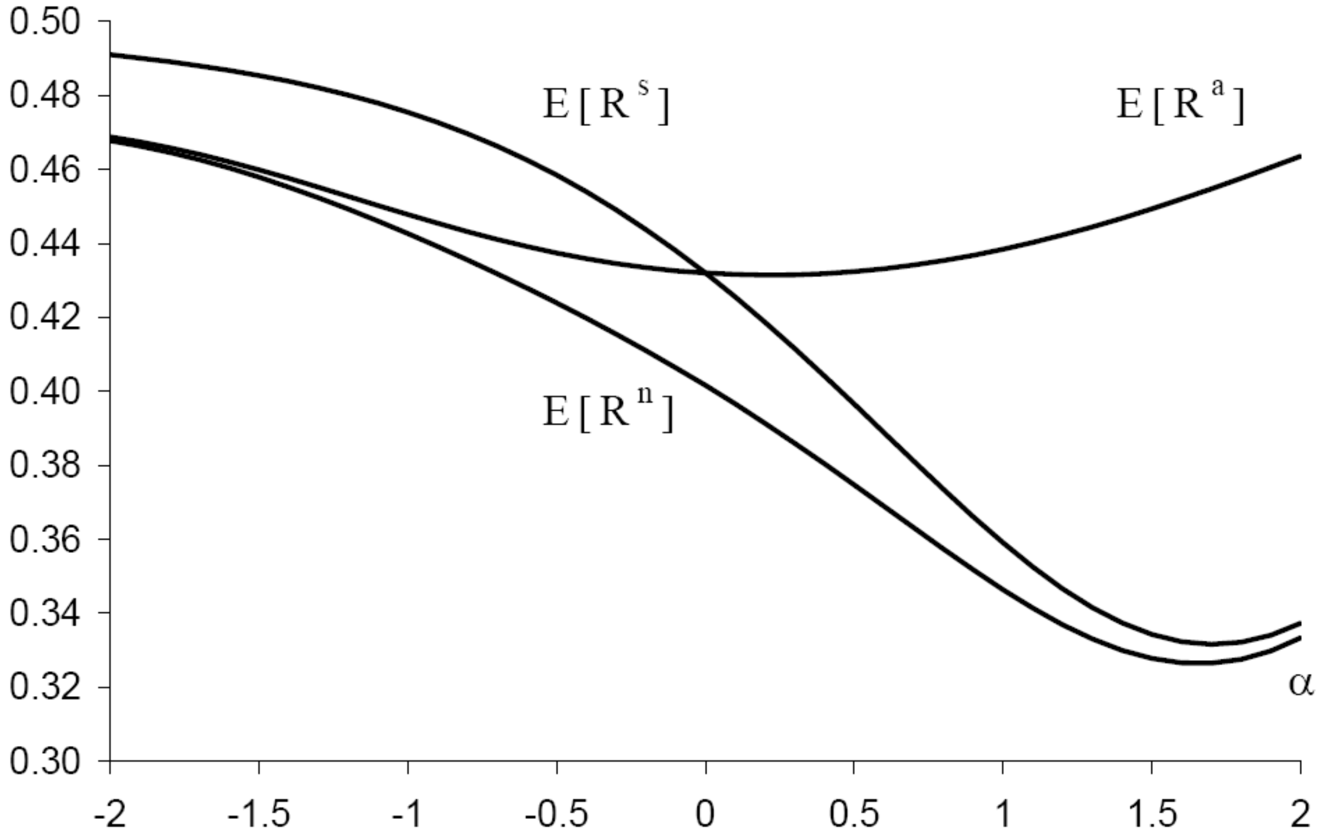


Figure 2: Revenues as functions of the parameter $\alpha$.

always less than $E[P_2^a]$. Second, as stated in Proposition 6, $E[P_2^a]$ is always higher than $E[P_2^n]$. Third, $E[P_1^a]$ may be smaller or greater than $E[P_1^n]$, depending on whether $\alpha$ is smaller or greater than 0.25. Fourth, $E[P_1^a]$ may be smaller or greater than $E[P_2^n]$, depending on whether $\alpha$ is smaller or greater than 0.85.

Figure 2 plots expected auction revenues for the sequential auction with no bid announcement, $E[R^n]$, the sequential auction with winning-bids announcement, $E[R^a]$, and the single-round auction for two objects, $E[R^s]$. As we know from Theorem 4, the expected revenue of a sequential auction with no bid announcement is always lower than the expected revenue of a single-round auction. Figure 2 shows that the expected revenue of a sequential auction with winning-bids announcement may be smaller or greater than the expected revenue of a single-round auction, depending on whether $\alpha$ is smaller or greater than zero.[10]

[10] In our example, for all values of $\alpha$, expected revenue in a sequential auction is higher with winning-bids

# 5 Concluding Remarks

We have derived the symmetric equilibrium bidding functions for the sequential uniform auction with and without winning-bids announcement (Theorems 1 and 2), and we have identified two effects on revenue of auctioning objects sequentially, rather than simultaneously: a lowballing effect and an informational effect. The lowballing effect reduces bids in the first round (Theorems 3 and 5). When there are no bid announcements (or values are private), only the lowballing effect is at work and both the first-round expected price and the seller's revenue are lower than in a single-round auction (Propositions 4 and 7). When the first-round winning bids are announced, the informational effect raises the expected second-round price above the price in a single-round auction (Proposition 6). Because of the combination of the lowballing and the informational effect, the first-round expected price in the sequential auction with winning-bids announcement could be lower or higher then the expected price in the single-round auction.[11]

We know from Milgrom and Weber (1982, 2000) that the ascending (English) auction raises the highest revenue among the standard auctions in which all objects are sold simultaneously. Since the ascending auction is equivalent to the single-round uniform auction when there are only three bidders for two objects, we can conclude that in some cases the sequential uniform-price auction with winning-bids announcement raises greater revenue than any standard simultaneous auction.[12]

Milgrom and Weber (2000) were the first to report, for their conjectured equilibrium, that with no bid announcement prices drift upward in a sequential uniform auction. We have extended this result to the case of winning-bids announcement. Surprisingly, even though

announcement than with no bid announcement. In other words, even though $I_1^a$ may be negative, $I_1^a + I_2^a$ turns out to be positive. We have obtained the same result in all other numerical examples we have tried, but we have not been able to provide a formal proof.

[11] As shown in the example of Section 4.1, the first-round expected price in the sequential auction with winning-bids announcement could even be lower than the first-round expected price with no bid announcement.

[12] When it raises greater revenue than an ascending auction, the sequential uniform auction with winning-bids announcement also raises greater revenue than a sequence of ascending auctions. As as shown by Milgrom and Weber (2000), the equilibrium outcome of a sequence of ascending auctions is the same as the equilibrium outcome of an ascending auction in which all objects are sold simultaneously.

they noticed that the price sequence is upward drifting, Milgrom and Weber (2000, p. 193) conjectured that the sequential auction with no bid announcement yields greater revenue than the single-round uniform auction. We have shown that this is incorrect. Because of the lowballing effect, the sequential auction with no bid announcement yields a lower revenue than the single-round auction.

The linkage principle is a mathematical result for single-item auctions, first obtained by Milgrom and Weber (1982), which has been broadly interpreted as saying that public revelation of information raises prices and revenue.[13] Our result that sequential auctions may yield lower revenue than a single-round auction, even if the winning bids are announced, could be interpreted as a failure of the broad interpretation of the "linkage principle" in multi-unit, sequential auctions.[14] On the other hand, one could also argue that our result that the informational effect raises second-round prices is consistent with the broad interpretation of the linkage principle. It is to be noted that in the derivation of both results we make no use of the narrower, mathematical definition of the linkage principle.

In the auctions we see in practice, it is common to announce only the winning price. Unfortunately, as Milgrom and Weber (2000, pp. 181-182) pointed out, analyzing such a policy in a sequential uniform auction involves technical complications. If the first round bidding function is increasing, then announcing the winning price reveals the price-setter's signal (i.e., the highest losing signal). In the second round, the first-round price-setter will find himself at a disadvantage, and in an asymmetric position with respect to all other remaining bidders.[15] An approximation of the policy of announcing the first-round price is to announce the lowest first-round winning bid; such a bid converges to the first-round

[13] See Krishna (2002, pp.103-110) for a survey and nice discussion.

[14] In our model bidders have unit-demand. See Perry and Reny (1999) for an example of a "failure of the linkage principle," in a multi-unit, single-round auction, in which bidders have multi-unit demand.

[15] de Frutos and Rosenthal (1998) studied an example with three bidders and two objects, in which the objects' common value is the sum of the bidders' signals, and signals are independent random variables that can only take values 0 or 1. They showed that, whether or not the winning price is announced, the expected second-round price, conditional on the first-round price, can be either higher or lower than the first-round price. This result does not contradict our results. In our paper, affiliation in the bidders' signal is the reason why the price sequence is a submartingale. In de Frutos and Rosenthal (1998), signals are independent and discrete, and it is because equilibrium is in mixed strategies that the price sequence can be either increasing or decreasing.

price as the number of losing bidders increases (see Mezzetti and Tsetlin, 2007). For such a policy, it is straightforward to establish equilibrium existence and to derive the equilibrium bidding function, using arguments analogous to those in the proof of Theorem 2. It is also simple to show that the policy studied in this paper of announcing all first-round winning bids yields higher expected prices in both rounds. More generally, regardless of whether any other winning bids are released, analogues of Theorem 2 and all results in Section 4 still hold, as long as the lowest winning bid is announced after the first round, and no losing bids are revealed.[16] Once some of the losing bids are announced, a signaling motive is introduced in bidders' first-round behavior, besides the potential asymmetry of the bidders in the second round mentioned above. When a signaling motive is present, bidders will want to conceal their information and first-round pooling of bids typically occurs.

We have considered two-rounds uniform auctions. As we point out at the end of Section 3, Theorem 2, that deals with the equilibrium bidding functions when all winning bids are announced, easily extends to auctions with any number of rounds. While we have so far been unable to generalize the existence part in the proof of Theorem 1 to more than two rounds, we conjecture that this generalization is possible, and that the bidding functions displayed at the end of Section 3 correspond to an equilibrium. With this caveat, all theorems about revenue ranking and comparison of prices in different rounds and auction formats generalize to more than two rounds. The lowballing effect and informational effect on the last-round price generalize as well. Expected prices increase over time because, by the lowballing effect, the price setter in round $t$ bids so as to equate a correct estimate of the price in round $t$ and an underestimate of the price in round $t+1$. In the last round, if the winning bids have been announced, then, by the informational effect, the price setter's bid is a better estimate of the objects's value; that is, the expected price is higher than in a single-round auction.

[16] Revealing the lowest winning bid eliminates the difficulty mentioned by Milgrom and Weber (2000, p.182) of proving that it is not profitable to bid higher in order to have better information in subsequent rounds.

# Appendix

**Proof of Theorem 2.** First, note that the last round of the sequential uniform auction is equivalent to a single-round uniform auction in which (i) the first $k_1$ signals have been revealed, (ii) there are $n - k_1$ bidders and $k - k_1$ objects left. Define

$$v_2(x; y_1, ..., y_{k_1}; y_k) = E\left[V_1 | X_1 = x, Y^1 = y_1, Y^2 = y_2, ..., Y^{k_1} = y_{k_1}, Y^k = y_k\right]. \tag{8}$$

The results in Milgrom and Weber (1982, 2000) imply that

$$b_2^a(x; y_1, ..., y_{k_1}) = v_2(x; y_1, ..., y_{k_1}; x). \tag{9}$$

Now consider the first round. Assume that all bidders other than bidder 1 use the bidding function $b_1^a(\cdot)$. Suppose that bidder 1 observes signal $x$ and bids $\beta_1$. Note that bidding below $\min(b_1^a)$ in the first round yields bidder 1 the same payoff as bidding $\min(b_1^a)$ - in this case bidder 1 never wins - while bidding above $\max(b_1^a)$ yields the same payoff as bidding $\max(b_1^a)$ - in this case bidder 1 always wins. Since the bidding functions are continuous, this implies that we can define $\sigma_1$ such that $b_1^a(\sigma_1) = \beta_1$; that is, we can think that bidder 1 uses the same bidding function as all other bidders, but in the first round he bids as if he had observed signal $\sigma_1$. We need to show that $\sigma_1 = x$. Let

$$\begin{aligned} v_1(x; y_{k_1}; y_k) &= E\left[V_1 | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right] \\ &= E\left[v_2(x; Y^1, ..., Y^{k_1 - 1}, y_{k_1}; y_k) | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right], \end{aligned} \tag{10}$$

$$b_2^*(y_k; y_{k_1} | x) = E\left[b_2^a(y_k; Y^1, ..., Y^{k_1 - 1}, y_{k_1}) | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right], \tag{11}$$

where the second equality in (10) follows from (8).

Let $h(y_{k_1}, y_k | x)$ be the joint density of $Y^{k_1}$ and $Y^k$ conditional on $X_1 = x$. Bidder 1's total expected profit at the beginning of the first round is

$$\Pi^a(x;\sigma_1) = \int_{\underline{x}}^{\sigma_1}\int_{\underline{x}}^{y_{k_1}} \Big(v_1(x;y_{k_1};y_k) - b_1^a(y_{k_1})\Big)h(y_{k_1},y_k|x)dy_k dy_{k_1} + \int_{\sigma_1}^{\overline{x}}\int_{\underline{x}}^{x} \Big(v_1(x;y_{k_1};y_k) - b_2^*(y_k;y_{k_1}|x)\Big)h(y_{k_1},y_k|x)dy_k dy_{k_1},$$

where the first (second) term is the profit from the first (second) round. Differentiating $\Pi^a(x;\sigma_1)$ with respect to $\sigma_1$ yields

$$\begin{aligned}
\frac{\partial \Pi^a(x;\sigma_1)}{\partial \sigma_1} &= \int_{\underline{x}}^{\sigma_1}\Big(v_1(x;\sigma_1;y_k) - b_1^a(\sigma_1)\Big)h(\sigma_1,y_k|x)dy_k \\
&\quad -\int_{\underline{x}}^{x}\Big(v_1(x;\sigma_1;y_k) - b_2^*(y_k;\sigma_1|x)\Big)h(\sigma_1,y_k|x)dy_k \\
&= \int_{\underline{x}}^{\sigma_1}\Big(b_2^*(y_k;\sigma_1|x) - b_1^a(\sigma_1)\Big)h(\sigma_1,y_k|x)dy_k \\
&\quad +\int_{x}^{\sigma_1}\Big(v_1(x;\sigma_1;y_k) - b_2^*(y_k;\sigma_1|x)\Big)h(\sigma_1,y_k|x)dy_k.
\end{aligned} \tag{12}$$

To prove that $\Pi^a(x;\sigma_1)$ is maximized at $\sigma_1 = x$, we will show that $\frac{\partial \Pi^a(x;\sigma_1)}{\partial \sigma_1}$ has the same sign as $(x-\sigma_1)$. The second term in (12) is zero for $\sigma_1 < x$ (by definition, $Y^{k_1} \geq Y^k$ and hence $h(\sigma_1,y_k|x) = 0$ for $\sigma_1 < x$), while for $\sigma_1 > x$ it is negative because

$$\begin{aligned}
&\int_{x}^{\sigma_1}\Big(v_1(x;\sigma_1;y_k) - b_2^*(y_k;\sigma_1|x)\Big)h(\sigma_1,y_k|x)dy_k \\
= &\int_{x}^{\sigma_1} E\left[v_2\left(x;Y^1,...,Y^{k_1-1},\sigma_1;y_k\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right]h(\sigma_1,y_k|x)dy_k \\
&-\int_{x}^{\sigma_1} E\left[b_2^a\left(y_k;Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right]h(\sigma_1,y_k|x)dy_k \\
= &\int_{x}^{\sigma_1} E\left[v_2\left(x;Y^1,...,Y^{k_1-1},\sigma_1;y_k\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right]h(\sigma_1,y_k|x)dy_k \\
&-\int_{x}^{\sigma_1} E\left[v_2\left(y_k;Y^1,...,Y^{k_1-1},\sigma_1;y_k\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right]h(\sigma_1,y_k|x)dy_k \leq 0,
\end{aligned}$$

where the first equality follows from (10) and (11), the second equality follows from (9), and the inequality follows from affiliation and $y_k \geq x$.

By (11) and (6), the first term in (12) is equal to

$$\begin{aligned}
&\int_{\underline{x}}^{\sigma_1} E\left[b_2^a\left(y_k; Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1=x, Y^{k_1}=\sigma_1, Y^k=y_k\right] h(\sigma_1, y_k|x)dy_k \\
&-b_1^a(\sigma_1)\int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x)dy_k \\
= \; & E\left[b_2^a\left(Y^k; Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1=x, Y^{k_1}=\sigma_1\right]\int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x)dy_k \\
&-E\left[b_2^a\left(Y^k; Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1=\sigma_1, Y^{k_1}=\sigma_1\right]\int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x)dy_k.
\end{aligned}$$

Because of affiliation, this difference has the same sign as $(x-\sigma_1)$. Hence $\frac{\partial \Pi^a(x;\sigma_1)}{\partial \sigma_1}$ is positive for $\sigma_1 < x$ and negative for $\sigma_1 > x$; the expected profit of bidder 1 is maximized at $\sigma_1 = x$. This implies that bidder 1's optimal first-round bid is $b_1^a(x)$ and concludes the proof. ▮

In the proof of Theorem 1 we use the following lemma.

**Lemma 1.** *Let $D(s)$ be an integrable function defined on $[0,S]$. Let $a(s)$ be a non-decreasing positive function, defined on $[0,S]$. If $\int_0^x D(s)a(s)ds \leq 0$ for all $x \in [0,S]$, then $\int_0^x D(s)ds \leq 0$ for all $x \in [0,S]$.*

**Proof of Lemma 1.** Define $F(x) = \int_0^x D(s)a(s)ds$ and $F'(x) = D(x)a(x)$ for all $x \in [0,S]$. Then, using integration by parts, we have

$$\begin{aligned}
\int_0^x D(s)ds &= \int_0^x D(s)a(s)\frac{1}{a(s)}ds \\
&= \int_0^x F'(s)\frac{1}{a(s)}ds \\
&= F(x)\frac{1}{a(x)} - F(0)\frac{1}{a(0)} + \int_0^x F(s)\frac{a'(s)}{a^2(s)}ds.
\end{aligned}$$

The first term on the third line is non-positive because $F(x) \leq 0$ and $a(x) > 0$ by assumption. The second term is zero, because $F(0) = 0$. The third term is non-positive because $F(s) \leq 0$ and $a'(s) \geq 0$ ($a(s)$ is differentiable a.e.). Therefore $\int_0^x D(s)ds \leq 0$, as claimed. ▮

**Proof of Theorem 1.** Assume that all bidders other than bidder 1 use the bidding functions $b_1^n(\cdot)$ and $b_2^n(\cdot)$ given by (4) and (3). We want to show that it is also optimal for bidder 1 to use them. Suppose that bidder 1 observes signal $x$ and bids $\beta_1$ in the first round and $\beta_2$ in the second round. As argued in the proof of Theorem 2, we can define $\sigma_1$ and $\sigma_2$ such that $b_1^n(\sigma_1) = \beta_1$ and $b_2^n(\sigma_2) = \beta_2$.

If bidder 1 does not win an object in the first round, he knows that $y_{k_1} > \sigma_1$. Then his expected second-round profit conditional on $\sigma_1$, $\sigma_2$, and $X_1 = x$ can be written as

$$\pi_2^n(x; \sigma_1, \sigma_2) = \int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\sigma_2} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k)\Big) \frac{h(y_{k_1}, y_k|x)}{\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\widetilde{y}_{k_1}} h(\widetilde{y}_{k_1}, \widetilde{y}_k|x) d\widetilde{y}_k d\widetilde{y}_{k_1}} dy_k dy_{k_1}, \quad (13)$$

where $v_1(\cdot)$ is given by (10) and $h(y_{k_1}, y_k|x)$ is the joint density of $Y^{k_1}$ and $Y^k$ conditional on $X_1 = x$.[17] Differentiating $\pi_2^n(x; \sigma_1, \sigma_2)$ with respect to $\sigma_2$, we obtain

$$\frac{\partial \pi_2^n(x; \sigma_1, \sigma_2)}{\partial \sigma_2} = \int_{\sigma_1}^{\overline{x}} \Big(v_1(x; y_{k_1}; \sigma_2) - b_2^n(\sigma_2)\Big) \frac{h(y_{k_1}, \sigma_2|x)}{\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\widetilde{y}_{k_1}} h(\widetilde{y}_{k_1}, \widetilde{y}_k|x) d\widetilde{y}_k d\widetilde{y}_{k_1}} dy_{k_1},$$

which is equal to

$$\Big(E[V_1|X_1 = x, Y^{k_1} \geq \sigma_1, Y^k = \sigma_2] - E[V_1|X_1 = \sigma_2, Y^k = \sigma_2]\Big) \frac{\int_{\sigma_1}^{\overline{x}} h(\widetilde{y}_{k_1}, \sigma_2|x) d\widetilde{y}_{k_1}}{\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\widetilde{y}_{k_1}} h(\widetilde{y}_{k_1}, \widetilde{y}_k|x) d\widetilde{y}_k d\widetilde{y}_{k_1}}.$$

It follows from affiliation and Assumption 1 that, when $\sigma_1 \leq x$, $\frac{\partial \pi_2^n(x;\sigma_1,\sigma_2)}{\partial \sigma_2}$ has the same sign as $(x - \sigma_2)$. This shows that if bidder 1 bids less than or equal to $b_1^n(x)$ in the first round (i.e., if $\sigma_1 \leq x$), then his optimal bid in the second round is $b_2^n(x)$ (i.e., $\sigma_2 = x$).

We complete the proof by showing that it is optimal for bidder 1 to bid $b_1^n(x)$ in the first round (i.e., $\sigma_1 = x$). Let $\sigma_2^*(\sigma_1)$ be the value of $\sigma_2$ that maximizes $\pi_2^n(x; \sigma_1, \sigma_2)$; we have already shown that $\sigma_2^*(\sigma_1) = x$ for $\sigma_1 \leq x$. Using (10) and (13), bidder 1's total expected profit at the beginning of the first round is

$$\begin{aligned} \Pi^n(x; \sigma_1) &= \int_{\underline{x}}^{\sigma_1} \int_{\underline{x}}^{y_{k_1}} \Big(v_1(x; y_{k_1}; y_k) - b_1^n(y_{k_1})\Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1} \\ &\quad + \int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\sigma_2^*(\sigma_1)} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k)\Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1}. \end{aligned}$$

[17] The second-round profit depends on $\sigma_1$, contrary to the case in which the winning bids are announced.

Differentiating with respect to $\sigma_1$ and applying the envelope theorem, we obtain

$$\begin{aligned}\frac{\partial \Pi^n(x;\sigma_1)}{\partial \sigma_1} &= \int_{\underline{x}}^{\sigma_1} \Big(v_1(x;\sigma_1;y_k) - b_1^n(\sigma_1)\Big) h(\sigma_1, y_k|x) dy_k \\ &\quad - \int_{\underline{x}}^{\sigma_2^*(\sigma_1)} \Big(v_1(x;\sigma_1;y_k) - b_2^n(y_k)\Big) h(\sigma_1, y_k|x) dy_k \\ &= \int_{\underline{x}}^{\sigma_1} \Big(b_2^n(y_k) - b_1^n(\sigma_1)\Big) h(\sigma_1, y_k|x) dy_k \qquad (14) \\ &\quad + \int_{\sigma_2^*(\sigma_1)}^{\sigma_1} \Big(v_1(x;\sigma_1;y_k) - b_2^n(y_k)\Big) h(\sigma_1, y_k|x) dy_k.\end{aligned}$$

The first term in equation (14), using (4), is

$$\Big(\int_{\underline{x}}^{\sigma_1} \Big(b_2^n(y_k) - b_1^n(\sigma_1)\Big) \frac{h(\sigma_1, y_k|x)}{\int_{\underline{x}}^{\sigma_1} h(\sigma_1, \widetilde{y}_k|x) d\widetilde{y}_k} dy_k\Big) \int_{\underline{x}}^{\sigma_1} h(\sigma_1, \widetilde{y}_k|x) d\widetilde{y}_k =$$

$$\Big(E\big[b_2^n(Y^k)\big|X_1 = x, Y^{k_1} = \sigma_1\big] - E\big[b_2^n(Y^k)\big|X_1 = \sigma_1, Y^{k_1} = \sigma_1\big]\Big) \int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x) dy_k.$$

By affiliation, this term has the same sign as $(x - \sigma_1)$.

It only remains to show that the second term in (14) is non-negative for $\sigma_1 < x$ and non-positive for $\sigma_1 > x$.

First, observe that if $\sigma_2^*(\sigma_1) \geq \sigma_1$ then the second term in (14) is zero, since, by definition, $h(\sigma_1, y_k|x) = 0$ for $y_k > \sigma_1$. Therefore, since $\sigma_2^*(\sigma_1) = x$ for $\sigma_1 \leq x$, the second term in (14) is zero for $\sigma_1 \leq x$. To show that in the case $\sigma_1 > x$ and $\sigma_2^*(\sigma_1) < \sigma_1$ the second term in (14) is non-positive, we will use Lemma 1.[18] Define

$$D(y_k) = \left(\frac{\int_{\sigma_1}^{\overline{x}} v_1(x;y_{k_1};y_k) h(y_{k_1}, y_k|x) dy_{k_1}}{\int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1}} - b_2^n(y_k)\right) h(\sigma_1, y_k|x), \qquad (15)$$

and note that, by affiliation, the second term in (14) is smaller than $\int_{\sigma_2^*(\sigma_1)}^{\sigma_1} D(y_k) dy_k$. Then a sufficient condition for the second term in (14) to be negative is

$$\int_{\sigma_2^*(\sigma_1)}^{\sigma_1} D(y_k) dy_k \leq 0. \qquad (16)$$

[18] This is precisely where lies the difficulty mentioned by Milgrom and Weber (2000). We need to show that in the first round bidder 1 does not want to bid as if his signal were higher than $x$.

Define

$$a(y_k) = \frac{\int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1}}{h(\sigma_1, y_k|x)}. \tag{17}$$

Since $\sigma_2^*(\sigma_1)$ maximizes $\pi_2^n(x; \sigma_1, \sigma_2)$, by (13) we have, for all $\sigma_2$,

$$\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\sigma_2^*(\sigma_1)} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k)\Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1} \geq$$

$$\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\sigma_2} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k)\Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1}.$$

Rearranging terms yields, for all $\sigma_2$,

$$\int_{\sigma_2^*(\sigma_1)}^{\sigma_2} \int_{\sigma_1}^{\overline{x}} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k)\Big) h(y_{k_1}, y_k|x) dy_{k_1} dy_k \leq 0. \tag{18}$$

Using the definitions (15) and (17) of $D(y_k)$ and $a(y_k)$, expression (18) can be rewritten as

$$\int_{\sigma_2^*(\sigma_1)}^{\sigma_2} \left( \frac{\int_{\sigma_1}^{\overline{x}} v_1(x; y_{k_1}; y_k) h(y_{k_1}, y_k|x) dy_{k_1}}{\int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1}} - b_2^n(y_k) \right) \left( \int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1} \right) dy_k$$

$$= \int_{\sigma_2^*(\sigma_1)}^{\sigma_2} D(y_k) a(y_k) dy_k \leq 0 \qquad \text{for all } \sigma_2. \tag{19}$$

By affiliation $a(y_k)$, defined in (17), is positive and increasing. Then, by Lemma 1, equation (19) implies that $\int_{\sigma_2^*(\sigma_1)}^{z} D(y_k) dy_k \leq 0$ for all $z \geq \sigma_2^*(\sigma_1)$; in particular, $\int_{\sigma_2^*(\sigma_1)}^{\sigma_1} D(y_k) dy_k \leq 0$, and (16) holds. Thus, the second term in (14) is negative. This concludes the proof. ∎

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