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A SIMPLE DERIVATION OF PRELEC'S PROBABILITY WEIGHTING FUNCTION

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Working Paper No. 05/20
July 2005

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June 2005

Abstract

Since Kahneman and Tversky (1979), it has been generally recognized that decision makers overweight low probabilities and underweight high probabilities. Of the several weighting functions that have been proposed, that of Prelec (1998) has the attractions that it is parsimonious, consistent with much of the available empirical evidence and has an axiomatic foundation. Luce (2001) provided a simpler derivation based on *reduction invariance*, rather than *compound invariance* of Prelec (1998). This note gives a simpler form of *reduction invariance*, which we call *power invariance*. A more direct derivation of Prelec's function is given, achieving a further simplification.

Keywords: Decision making under risk, Prelec's probability weighting function, Compound invariance, Reduction invariance, Power invariance, Prospect theory, Algebraic functional equations.

JEL Classification: C60, D81.

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1. Introduction

It is now generally accepted that decision makers overweight low probabilities and underweight high probabilities. Thus they behave as if they transform the objective (cumulative) probability distribution using an inverted S-shaped weighting function (Tversky and Kahneman, 1992). A number of weighting functions have been proposed. However, the first axiomatically derived weighting function was that of Prelec (1998). His main axiom was *compound invariance*. The importance of this axiom is as follows. In expected utility theory, the product rule for probabilities allows us to reduce a compound lottery to a simple lottery of the same expected utility. Once we depart from expected utility theory, we need a rule that plays an analogous role. *Compound invariance* is a candidate for such a rule. Luce (2001) proposed the simpler and more easily testable assumption of *reduction invariance*. He also provided a simpler derivation of Prelec's function.

In this note, we provide a more direct proof and achieve a further simplification. In the main part of his proof, Luce transforms the problem to be able to apply Cauchy's functional equations $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x) f(y)$. We proceed as follows. We start from a simpler form of *reduction invariance* which we call *power invariance*. We give a simple direct proof that if the weighting function satisfies *power invariance*, then it must satisfy $w(p^\lambda) = (w(p))^{\phi(\lambda)}$ where p is the cumulative probability function and φ is some function of $\lambda, \lambda > 0$.We then use an appropriate functional equation to derive Prelec's function. Theorem 2 of section 4 is our main result. For ease of reference, sections 2 and 3 give fairly standard results and definitions.

2. Lotteries, probability weighting functions, decision weights and the value of lotteries

Assume n mutually exclusive states of the world, $s_1, s_2, ..., s_n$, where $n \in \mathbf{N}$ is a fixed natural number. State s_i occurs with probability p_i, $p_i > 0, \Sigma_{i=1}^{n} p_i = 1$. An *event* is a subset of $S = \{s_1, s_2, ..., s_n\}$. The *probability* of an event $A \subset S$ is $p(A) = \Sigma \{p_i : s_i \in A\}$. Let $\{A_i\}_{i=1}^{m}$, $m \in \mathbf{N}$, be a partition of S, i.e., $A_i \neq \phi, i \neq j \Longrightarrow A_i \cap A_j = \phi, \cup_{i=1}^{m} A_i = S$. Let $x_1, x_2, ..., x_m \in \mathbf{R}.$ Then $x = \{(x_1, A_1), (x_2, A_2), ..., (x_m, A_m)\}$ is a *simple lottery* that pays x_i if event A_i occurs. Hence, the lottery x pays x_i with probability $p(A_i)$. Consider an event $A \subset S$. Let $\{A_i\}_{i=1}^{l}$, $l \in \mathbf{N}$, be a partition of A. Then $x = \{(x_1, A_1), (x_2, A_2), ..., (x_l, A_l)\}$ stands for the simple lottery

$$\{(x_1, A_1), (x_2, A_2), ..., (x_l, A_l), (0, S - A)\}$$

that pays x_i if event A_i occurs, but pays 0 if none of the events $A_1, A_2, ..., A_l$ occur. We shall describe such a lottery as *conditional* (on A occurring). A compound lottery, which is a lottery of lotteries, can be given the following recursive definition.

Definition 1 : *(i) L_0 is the set of simple lotteries, defined above.*
(ii) Let $L_{k+1} = L_k \cup \{\{(y_1, A_1), (y_2, A_2), ..., (y_m, A_m)\} : y_i \in \mathbf{L}_k$ and is conditional on A_i and $\{A_i\}_{i=1}^{m}$is any partition of $S\}$.
(iii) $L = \cup_{k=0}^{\infty} L_k$. L is the set of compound lotteries or, more simply, lotteries.

Often, it suffices to give the probability of each outcome, rather than fully specify the probability space on which a lottery is defined. For example, if $p(A_i) = q_i$, it is often sufficient to indicate the lottery $\{(x_1, A_1), (x_2, A_2), ..., (x_m, A_m)\}$ by $\{(x_1, q_1), (x_2, q_2), ..., (x_m, q_m)\}$. We shall also use the following standard short-hand notation. (x) is the lottery $\{(x, 1)\}$ that pays 1 with certainty, (x, p) is the lottery $\{(x, p), (0, 1-p)\}$ that pays x with probability p but 0 otherwise and $((x, p), q)$ is the compound lottery $\{(\{(x, p), (0, 1-p)\}, q), (0, 1-q)\}$ that 'pays' (x, p) with probability q but 0 otherwise. Our Definition 1 may appear more formal than necessary. However, it does facilitate an extension of the concept of the value of a simple lottery (Definition 5) to that of a compound lottery (Definition 6).

Definition 2 : *By a value function we mean a strictly increasing function $v : \mathbf{R} \longrightarrow \mathbf{R}$ such that $v(0) = 0$[1].*

Definition 3 : *By a probability weighting function we mean a strictly increasing function $w : [0,1] \overset{onto}{\longrightarrow} [0,1]$.*

Note that a probability weighting function, w, has a unique inverse, $w^{-1} : [0,1] \overset{onto}{\longrightarrow} [0,1]$ and that w^{-1} is strictly increasing. Hence, w^{-1} is also a probability weighting function. Furthermore, it follows that w and w^{-1} are continuous and must satisfy $w(0) = w^{-1}(0) = 0$ and $w(1) = w^{-1}(1) = 1$.

Definition 4 : *(Prelec, 1998). By the Prelec function we mean the probability weighting function $w : [0,1] \overset{onto}{\longrightarrow} [0,1]$ given by*

$$w(p) = e^{-\beta(-\ln p)^{\alpha}}, \alpha > 0, \beta > 0 \tag{2.1}$$

Definition 5 : *Let $\{(x_1, A_1), (x_2, A_2), ..., (x_m, A_m)\}$ be a simple lottery, where $x_1 \leq x_2 \leq ... \leq x_k < 0 \leq x_{k+1} \leq ... \leq x_m$. Let w^- be the probability weighting function associated with losses ($x_i < 0$) and w^+ the probability weighting function associated with gains ($x_i \geq 0$). We define the decision weights $\pi_1, \pi_2, ..., \pi_m$ as follows,*
$\pi_1 = w^-(p(A_1))$, *if $k \geq 1$*
$\pi_j = w^-\left(\Sigma_{i=1}^{j} p(A_i)\right) - w^-\left(\Sigma_{i=1}^{j-1} p(A_i)\right); j = 2, 3, ..., k$; *if $k > 1$*

[1]The value function was introduced by Kahnemann and Tversky (1979). They interpret $x_i = z_i - r$, where z_i is the i^{th} outcome and r is some reference point for the individual. When $z_i = r$, a natural normalization is $v(r - r) = v(0) = 0$.

$\pi_j = w^+ \left(\Sigma_{i=j}^m p\left(A_i\right)\right) - w^+ \left(\Sigma_{i=j+1}^m p\left(A_i\right)\right); j = k+1, ..., m-1;$ *if* $k < m-1$
$\pi_m = w^+ \left(p\left(A_m\right)\right),$ *if* $k < m$
Let $v : \mathbf{R} \longrightarrow \mathbf{R}$ *be a value function, i.e., a strictly increasing function such that* $v(0) = 0,$ *then the value of the simple lottery* $x = \{(x_1, A_1), (x_2, A_2), ..., (x_m, A_m)\}$ *to the decision maker is given by* $V(x) = \Sigma_{j=1}^m \pi_j v(x_j)$.

Definition 6 [2]*: Let* $y = \{(y_1, A_1), (y_2, A_2), ..., (y_m, A_m)\} \in L_{k+1}$*, so* $y_i \in L_k$*, and is conditional on* $A_i, i = 1, 2, ..., m$*. Let the value of* y_i *be* $V(y_i)$*. Assume that* $V(y_1) \leq V(y_2) \leq ... \leq V(y_k) < 0 \leq V(y_{k+1}) \leq ... \leq V(y_n)$*. Let* w^- *be the probability weighting function associated with losses and* w^+ *the probability weighting function associated with gains. We define the decision weights* $\pi_1, \pi_2, ..., \pi_n$ *as in definition 5:*
$\pi_1 = w^-(p_1),$ *if* $k \geq 1$
$\pi_j = w^-\left(\Sigma_{i=1}^j p_i\right) - w^-\left(\Sigma_{i=1}^{j-1} p_i\right); j = 2, 3, ..., k;$ *if* $k > 1$
$\pi_j = w^+\left(\Sigma_{i=j}^n p_i\right) - w^+\left(\Sigma_{i=j+1}^n p_i\right); j = k+1, ..., m-1;$ *if* $k < m-1$
$\pi_m = w^+(p_m),$ *if* $k < m$
Then the value of y *is* $V(y) = \Sigma_{j=1}^m \pi_j V(y_j)$.

Definition 7 *: Two lotteries,* x *and* y*, are equivalent if, and only if, they have the same value :* $V(x) = V(y)$.

Example 1 *As an illustration, consider Problems 5 and 6 from Kahneman and Tversky (1984). Problem 5 asks a decision maker to choose between the two compound lotteries* $A = ((30, 1), 0.25)$ *and* $B = ((45, 0.8), 0.25)$*. Problem 6 asks the decision maker to choose between the simple lotteries* $C = (30, 0.25)$ *and* $D = (45, 0.2)$*. Note that in expected utility theory* $A \sim C$ *and* $B \sim D$*. However empirical evidence shows that, for most decision makers,* $A \succ B$ *but* $C \prec D$*. On the other hand, using the value function* $v(x) = x^{0.6}$ *and the Prelec probability weighting function* $w(p) = -e^{-(-\ln p)^{0.65}}$ *(from Prelec, 1998), gives*
$V(A) = V((30, 1), 0.25) = v(30)\, w(0.25) = 30^{0.6} e^{-(-\ln 0.25)^{0.65}} = 2.2349$
$V(B) = V((45, 0.80), 0.25) = [v(45)\, w(0.80)]\, w(0.25) = 45^{0.6} e^{-(-\ln 0.8)^{0.65}} e^{-(-\ln 0.25)^{0.65}} = 1.9547$
$V(C) = V(30, 0.25) = v(30)\, w(0.25) = 30^{0.6} e^{-(-\ln 0.25)^{0.65}} = 2.2349$
$V(D) = V(45, 0.20) = v(45)\, w(0.20) = 45^{0.6} e^{-(-\ln 0.2)^{0.65}} = 2.513$
Hence, for prospect theory, $A \succ B$ *and* $C \prec D$*, which is in agreement with the empirical evidence.*

[2]Our Definition 6 extends the standard definition of the value of a simple lottery to compound lotteries. It is in agreement with the usage of Luce (2001) for the special case of compound lotteries of the form $((x, p), q)$, where (x, p) is the simple lottery that pays x with probability p. To the best of our knowledge, this definition is new.

3. Reduction invariance and Luce's derivation of Prelec's function

Definition 8 *: (Luce, 2001). Let $\lambda > 0$. The probability weighting function, w, satisfies λ-reduction invariance if whenever $((x,p),q) \sim (x,r)$, then $\left((x,p^{\lambda}),q^{\lambda}\right) \sim \left(x,r^{\lambda}\right)$. The probability weighting function, w, satisfies reduction invariance if it satisfies λ-reduction invariance for all $\lambda > 0$.*

Theorem 1 : (Luce, 2001). The following are equivalent

(i) The probability weighting function, w, satisfies λ-*reduction invariance* for $\lambda \in \{2,3\}$.

(ii)The probability weighting function, w, satisfies *reduction invariance.*

(iii) The probability weighting function, w, is the *Prelec function.*

4. 'Power invariance' and a simple derivation of Prelec's function

Definition 9 ***(PI):*** *The probability weighting function w satisfies power invariance (**PI**) if, for all $p,q \in [0,1], \lambda \in (0,\infty)$ and $m \in \mathbf{N}$, $(w(p))^m = w(q) \Longrightarrow \left(w\left(p^{\lambda}\right)\right)^m = w\left(q^{\lambda}\right)$.*

Definition 10 ***(PIPI):*** *The probability weighting function w satisfies probability independent power invariance (**PIPI**) if there is a function $\varphi : R_{++} \longrightarrow R$, such that for all $p \in [0,1], \lambda \in (0,\infty)$, the equality $w\left(p^{\lambda}\right) = (w(p))^{\phi(\lambda)}$ holds*[3].

Theorem 2 : The following are equivalent

(i) The probability weighting function w satisfies **PI**.

(ii) The probability weighting function w satisfies **PIPI**.

(iii) The probability weighting function, w, is the *Prelec function.*

Proof: $(i) \Longrightarrow (ii)$. Let $p,q \in [0,1], \lambda \in (0,\infty)$ *and* $n \in \mathbf{N}$ and assume that

$$(w(p))^n = w(q) \tag{4.1}$$

Hence, by *power invariance*,

$$\left(w\left(p^{\lambda}\right)\right)^n = w\left(q^{\lambda}\right) \tag{4.2}$$

Eliminating q from (4.1) and (4.2), we get[4]

$$\left(w\left(p^{\lambda}\right)\right)^n = w\left(w^{-1}(w(p))^n\right)^{\lambda} \tag{4.3}$$

Taking logs and reversing sign, gives

$$-n\ln w\left(p^{\lambda}\right) = -\ln w\left(w^{-1}(w(p))^n\right)^{\lambda} \tag{4.4}$$

[3]Notice that $\phi(\lambda) = \frac{\ln w\left(p^{\lambda}\right)}{\ln w(p)}$ is independent of the probability p, hence, the name.

[4]Here we use the standard notation, $w(z)^n = w((z)^n)$.

For given $\lambda \in (0, \infty)$, define the function (note that this is essentially the RHS of (4.4)) :

$$f(y) = -\ln w\left(w^{-1}\left(e^{-y}\right)\right)^{\lambda}, y \geq 0 \tag{4.5}$$

Hence,

$$f(-n \ln w(p)) = -n \ln w\left(p^{\lambda}\right) \tag{4.6}$$

In particular, for $n = 1$,

$$f(-\ln w(p)) = -\ln w\left(p^{\lambda}\right) \tag{4.7}$$

From (4.6) and (4.7),

$$f(n(-\ln w(p))) = nf(-\ln w(p)) \tag{4.8}$$

As p varies from 1 down to 0, $-\ln w(p)$ varies from 0 up to ∞. Hence, (4.8) gives, for all $n \in \mathbf{N}$ and all $y \in (0, \infty)$,

$$f(ny) = nf(y) \tag{4.9}$$

Extend (4.9) from natural numbers to positive reals in the usual way. Let $x = \frac{1}{n}y$. Then $f(y) = f(nx) = nf(x) = nf\left(\frac{1}{n}y\right)$. Hence $f\left(\frac{1}{n}y\right) = \frac{1}{n}f(y)$ and, hence, $f\left(\frac{m}{n}y\right) = mf\left(\frac{1}{n}y\right) = \frac{m}{n}f(y)$. Since f is continuous, it follows that, for all $x, y \in (0, \infty)$, $f(xy) = xf(y)$. In particular, for $y = 1$, $f(x) = xf(1)$. Letting $\phi = f(1)$, we get, for all $x \in (0, \infty)$,

$$f(x) = \phi x \tag{4.10}$$

Recall that the above derivation is conditional on a given $\lambda \in (0, \infty)$. Hence, ϕ is a function of $\lambda, \phi = \phi(\lambda)$. (4.7) and (4.10) give, for all $p \in [0, 1], \lambda \in (0, \infty)$,

$$w\left(p^{\lambda}\right) = (w(p))^{\phi(\lambda)} \tag{4.11}$$

We now show that $(ii) \implies (iii)$. Introduce the new function

$$g(x) = \ln\left(-\ln w\left(e^{-x}\right)\right), x > 0 \tag{4.12}$$

From (4.11) and (4.12), we get that for all $\lambda, x \in (0, \infty), g(\lambda x) = g(x) + \ln \phi(\lambda)$. Since g is strictly monotonic, this functional equation has the unique solution $g(x) = \ln(\beta x^{\alpha}), \alpha \neq 0, \beta > 0$ (see, for example, Theorem 2.7.3 of Eichhorn (1978)[5]). Substituting from (4.12) gives Prelec's function

$$w(p) = e^{-\beta(-\ln p)^{\alpha}}, \alpha > 0, \beta > 0 \tag{4.13}$$

where α is taken to be positive to ensure that $w(p)$ is strictly increasing, rather than just strictly monotonic.

Finally, $(iii) \implies (i)$ follows by direct calculation. ■

[5] There is a minor error on p42 of Eichhorn (1978): In (2.7.4), the first occurrence of λ should be γ.

From Theorem 1, *reduction invariance* is equivalent to the probability weighting function being the Prelec function. From Theorem 2, *power invariance* is also equivalent to the probability weighting function being the Prelec function. Hence, we get the following result.

Corollary 1 : *Power invariance is equivalent to reduction invariance of Luce (2001).*

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