Technical appendix of "Debt contracts with *ex-ante* and *ex-post* asymmetric information: an example."

G. Carlier [*], L. Renou [†]

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Computation and properties of the density g_μ

In the sequel for $x := (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}$ we shall simply write $x := (x_1, x')$. Taking a test-function $\varphi \in C^0([\alpha_1, 0], \mathbb{R})$, using the change of variables $p = (p_1, p') \mapsto (\alpha \cdot p, p')$ and Fubini's formula we get:

$$\int_{\Sigma_{N-1}} \varphi(\alpha \cdot p)\rho(p)dp = -\int_{\alpha_1}^{0} \frac{\varphi(t)}{\alpha_1} \left(\int_{\Sigma_t} \rho(\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p') dp' \right) dt,$$

where:

$$\Sigma_t := \left\{ p' \in \Sigma_{N-2} : (\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p') \in \Sigma_{N-1} \right\}.$$

This proves that μ is absolutely continuous with respect to Lebesgue's measure on $[\alpha_1, 0]$ and admits the density:

$$g_\mu(t) = -\frac{1}{\alpha_1} \int_{\Sigma_t} \rho(\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p') dp'.$$

g_μ is trivially positive since α_1 is negative. We shall write g_μ in a more compact way as:

$$g_\mu(t) = \int_{\Sigma_t} \sigma(t, p')dp' \text{ with } \sigma(t, p') = -\frac{1}{\alpha_1} \rho(\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p').$$

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[*] Université Paris IX-Dauphine, CEREMADE, Place du Maréchal De Lattre De Tassigny, 75775 Paris CEDEX 16, France, `carlier@ceremade.dauphine.fr`.

[†] School of Economics, Room G52, Napier Building, University of Adelaide, 5005 Australia, `ludovic.renou@adelaide.edu.au`.

If ρ is continuous so is g_μ (use Lebesgue dominated convergence theorem and the fact that, if t_n converge to t, the indicator function of Σ_{t_n} converges to that of Σ_t a.e. with respect to the $N-2$ dimensional Lebesgue's measure). Since $\Sigma_0 = \Sigma_{\alpha_1} = 0_{\mathbb{R}^{N-2}}$, we have $g_\mu(0) = g_\mu(\alpha_1) = 0$. Moreover, if ρ is strictly positive on the interior of Σ_{N-1}, so is g_μ on $(\alpha_1, 0)$.

Lemma 1 *If $\rho \in C^1(\Sigma_{N-1}, \mathbb{R})$ and ρ is strictly positive on Σ_{N-1}, then there exists a neighborhood of 0 on which g_μ is nonincreasing.*

Proof. First, note that Σ_t can be written as

$$\Sigma_t = \left\{ p' \in \Sigma_{N-2} : \sum_{n=2}^{N-1} \alpha_n p_n \geq t \text{ and } \sum_{n=2}^{N-1} (1 - \frac{\alpha_n}{\alpha_1}) p_n \leq 1 - \frac{t}{\alpha_1} \right\}.$$

For t sufficiently close to 0, the requirements $p' \in \mathbb{R}_+^{N-2}$ and $\sum \alpha_n p_n \geq t$ imply $\sum_{n=2}^{N-1}(1 - \frac{\alpha_n}{\alpha_1})p_n \leq 1 - \frac{t}{\alpha_1}$ and $\sum_{n=2}^{N-1} p_n \leq 1$ so that there exists t_0 such that for $t_0 \leq t < 0$

$$\Sigma_t = \left\{ p' \in \mathbb{R}_+^{N-2} : \sum_{n=2}^{N-1} \alpha_n p_n \geq t \right\} = -tA_0, \tag{1}$$

where A_0 is the (fixed) set

$$A_0 := \left\{ p' \in \mathbb{R}_+^{N-2} : \sum_{n=2}^{N-1} \alpha_n p_n \geq -1 \right\}.$$

Hence, for $t \in (t_0, 0)$, $g_\mu(t)$ can be computed as

$$g_\mu(t) = (-t)^{N-2} \int_{A_0} \sigma(t, -tq) dq.$$

Differentiating this expression yields:

$$\dot{g_\mu}(t) = -(N-2)(-t)^{N-3} \int_{A_0} \sigma(t, -tq) dq$$

$$+(-t)^{N-2} \int_{A_0} (\partial_t \sigma(t, -tq) - \partial_q \sigma(t, -tq) \cdot q) dq$$

so that $\dot{g_\mu}(t) < 0$ for sufficiently close to 0. □

The case of observable lender's type

As a benchmark, let us consider the optimal (type-dependent) contract when the lender's type $p \in \Sigma_{N-1}$ is observable by the borrower. If p is observable, for each p, the borrower determines $\overline{\omega}$ and C by maximizing the profit:

$$\pi(\overline{\omega}, C) := \sum_{n=1}^{N} \frac{q_n}{2\varepsilon} \int_{\overline{\omega})}^{\theta_n+\varepsilon} (\omega - \overline{\omega}) d\omega - C \tag{2}$$

subject to $\overline{\omega} \in [\underline{\delta}, \overline{\delta}]$ and the participation constraint:

$$U(p, \overline{\omega}, C) = (\alpha \cdot p)\overline{\omega} + \beta \cdot p + z(\overline{\omega}) + C \geq r \tag{3}$$

Since clearly (3) is binding, one gets:

$$C = r - (\alpha \cdot p)\overline{\omega} - \beta \cdot p - z(\overline{\omega}).$$

Replacing in (2) and defining the reduced type $t := \alpha \cdot p$, elementary calculus yields:

$$\pi(p, \overline{\omega}, C) = \overline{\omega}\,(t - \tau) + k'$$

where k' is some constant and:

$$\tau = -\frac{1}{2\varepsilon}\left(\theta_N - \sum_{n=1}^{N} \theta_n q_n - \gamma\right).$$

Hence, when p is observable, the borrower offers the contract $\underline{\delta}$ to lenders with types $t \in [\alpha_1, 0]$ with $t \leq \tau$ and contract $\overline{\delta}$ to lenders with type $t \geq \tau$. It is easy to check that $\tau \geq \alpha_1$ and that pooling occurs ($\underline{\delta}$ offered to all the lenders) exactly when $\tau \geq 0$ which is the same condition as in the adverse selection case.