More strategies, more Nash equilibria: Proof available upon request

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In this note, we formally state and prove a discrete counterpart of Theorem 1 in [1]. This discrete version of Theorem 1 answers a question raised by Prof. Abdou during a seminar in Paris on January 2005. This note is almost self-contained.

1 A discrete counterpart of Theorem 1

Let $G := \langle N, (Y_i, u_i)_{i\in N}\rangle$ be a strategic-form game with N the (finite) set of players, Y_i the set of strategies available to player i, and $u_i : \times_{i\in N} Y_i \to \mathbb{R}$ the payoff function of player i. Denote $Y := \times_{i\in N} Y_i$, $Y_{-i} := \times_{j\in N\setminus\{i\}} Y_j$, and y_{-i} a generic element of Y_{-i}. Let $\mathcal{G}^\star$ be the class of strategic-form games such that $N = \{1, 2\}$, and for all $i \in N$,

- $Y_i = \{1, 2, \cdots, m_i\}$, Y_i together with the usual order $\geq$ is a completely ordered set,

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- u_i is strictly quasi-concave in y_i, that is, for any $y_{-i} \in Y_{-i}$, for any $y_i \geq y_i'$, $y_i \neq y_i'$,

  $$u_i(y_i'', y_{-i}) > \min(u_i(y_i, y_{-i}), u_i(y_i', y_{-i})),$$

  for any $y_i \geq y_i'' \geq y_i'$, $y_i'' \neq y_i'$, $y_i'' \neq y_i$.

- u_i has increasing differences in (y_i, y_{-i}), that is, for any $y_i \geq y_i'$, $y_{-i} \geq y_{-i}'$,

  $$u_i(y_i, y_{-i}) - u_i(y_i', y_{-i}) \geq u_i(y_i, y_{-i}') - u_i(y_i', y_{-i}'),$$

- the game has generic payoffs, that is, for any player $i \in N$, for all $y \in Y$, $y' \in Y$, $y \neq y'$, $u_i(y) \neq u_i(y')$.

An interval $[\underline{x}_i, \overline{x}_i]$ in Y_i is a set of the form $\{x_i \in Y_i : \overline{x}_i \geq x_i \geq \underline{x}_i\}$. For instance, if $Y_i = \{1, 2, 3, 4, 5\}$, $\{2, 3, 4\}$ is an interval of Y_i while $\{2, 4, 5\}$ is not.

We say that the game $g := \langle N, (X_i, v_i)_{i \in N} \rangle$ is a restriction of G if for each player $i \in N$, X_i is an interval of Y_i, and $v_i(x) = u_i(x)$ for all $x \in X$.

We denote the set of *pure* Nash equilibria of g and G by $N(g)$ and $N(G)$, respectively.

Theorem 1 *Let g and G be two games in $\mathcal{G}^\star$ such that g is a restriction of G. There exists an injective map from $N(g)$ to $N(G)$.*

2 Proof

Let g and G be two games in $\mathcal{G}^\star$ with g a restriction of G. As a first observation, note that $N(g)$ and $N(G)$ are non-empty sets since both g and G are games with strategic complementarities (see Theorem 4.2.1. in Topkis [2, p. 181]). Second, observe that we can easily construct an injective mapping from the set of strategy profiles, which are equilibria in both the game G and its restriction g, to the set of equilibria of G; the identity mapping will do. Therefore, the crucial part of the proof consists in showing that there exists an injective mapping that associates to every equilibrium in $N(g) \setminus N(G)$ an equilibrium in $N(G) \setminus N(g)$.

Characterization of $N(g) \setminus N(G)$***.***

The characterization of $N(g) \setminus N(G)$ is almost identical to the characterization presented in [1]. One main difference is that the existence of an equilibrium in g and G is essentially a consequence of assuming payoffs having increasing differences (hence non-decreasing best-reply maps). A second main difference is that single-valuedness of the best-reply maps comes from the payoff genericity. For completeness, we restate the characterization, however.

Define $BR_i : Y_{-i} \to Y_i$ the best-reply map of player i in the game G with for all $y_{-i} \in Y_{-i}$,

$$BR_i(y_{-i}) := \{y_i \in Y_i : u_i(y_i, y_{-i}) \geq u_i(y_i', y_{-i}) \, \text{for all} \, y_i' \in Y_i\}.$$

Analogously, we denote the best-reply map of player i in the game g by $br_i : X_{-i} \to X_i$. From Theorem 2.8.3 of Topkis [2, p.77] and payoff genericity, it follows that best-reply maps are non-empty, single-valued and monotone non-decreasing. For any non-empty interval Z, we denote $\overline{z}$ its least upper bound and $\underline{z}$ its greatest lower bound. We also denote $>$ for the asymmetric part of the order $\geq$.

Lemma 1 *Player i's best-reply function* $br_i : X_{-i} \to X_i$ *in* g *is given by*

$$br_i(x_{-i}) = \begin{cases} \underline{x}_i & \textit{if} \;\; BR_i(x_{-i}) < \underline{x}_i \\ BR_i(x_{-i}) & \textit{if} \;\; \underline{x}_i \leq BR_i(x_{-i}) \leq \overline{x}_i \\ \overline{x}_i & \textit{if} \;\; \overline{x}_i > BR_i(x_{-i}) \end{cases}.$$

Proof First, we clearly have that $br_i(x_{-i}) = BR_i(x_{-i})$ for any x_{-i} such that $BR_i(x_{-i}) \in X_i$. Second, choose a $x_{-i} \in X_{-i}$ such that $BR_i(x_{-i}) < \underline{x}_i$, and suppose that $br_i(x_{-i}) > \underline{x}_i$. The single-valuedness of the best-reply maps implies that $u(BR_i(x_{-i}), x_{-i}) > u(\underline{x}_i, x_{-i})$ and $u(br_i(x_{-i}), x_{-i}) > u(\underline{x}_i, x_{-i})$. It follows that $(BR_i(x_{-i}), x_{-i})$ and $(br_i(x_{-i}), x_{-i})$ both belong to the strict upper contour set of $(\underline{x}_i, x_{-i})$. Since $BR_i(x_{-i}) < \underline{x}_i < br_i(x_{-i})$, we have a contradiction with the strict quasi-concavity of u_i. Analogous arguments hold if we choose a $x_{-i} \in X_{-i}$ such that $BR_i(x_{-i}) > \overline{x}_i$, and suppose that $br_i(x_{-i}) < \overline{x}_i$. $\square$

In words, the best-reply map br_i of the restricted game g agrees with the best-reply map BR_i of the game G on the set $\{x_{-i} \in X_{-i} : BR_i(x_{-i}) \in X_i\}$, and is on the boundary of X_i, otherwise.

Lemma 2 *If $y^* \in N(G) \cap X$, then $y^* \in N(g)$.*

Proof Since y^* is a Nash equilibrium of G and $y_i^* \in X_i \subseteq Y_i$ for all $i \in N$, we have that $u_i(y_i^*, y_{-i}^*) \geq u_i(x_i, y_{-i}^*)$, for all $x_i \in X_i \subseteq Y_i$ and all $i \in N$, hence $y^* \in N(g)$. $\square$

Lemma 2 states that any equilibrium of G, which belongs to the restricted set of strategies X, is also an equilibrium of g. The converse is not true. However, we can prove that any equilibrium of g, which is not on the *boundary* $\partial_Y X$ *of* X *in* Y, is also an equilibrium of G.

Lemma 3 *If $x^* \in N(g) \setminus \partial_Y X$, then $x^* \in N(G)$.*

Proof Let $x^* \in N(g) \setminus \partial_Y X$. First, if $x^* \notin \partial X$, then for all $i \in N$, $\underline{x}_i < x_i^* < \overline{x}_i$. Moreover, since $x^* \in N(g)$, we have that $x_i^* = br_i(x_{-i}^*)$ for all $i \in N$. From Lemma 1, it follows that $br_i(x_{-i}^*) = BR_i(x_{-i}^*)$ for all $i \in N$, hence $x^* \in N(G)$. Second, if $x^* \in \partial X \cap \partial Y$, then either $x_i^* = \underline{y}_i$ or $x_i^* = \overline{y}_i$ for at least one player $i \in N$. Suppose $x_i^* = \underline{y}_i$. Then, it follows from Lemma 1 and compactness of Y_i that $\underline{y}_i = br_i(x_{-i}^*) \geq BR_i(x_{-i}^*) \geq \underline{y}_i$. It then trivially follows that $x^* \in N(G)$. $\square$

From Lemmata 2 and 3, it immediately follows that the equilibria of g, which are not equilibria of G, are on the *boundary* of X in Y. The next proposition formally states this result.

Proposition 1 *If $x^* \in N(g) \setminus N(G)$, then $x^* \in \partial_Y X$.*

The injective mapping.

The construction of an injective mapping is once again almost identical to the one in [1].

Lemma 4 *Let g and G be two games in $\mathcal{G}$ such that g is a restriction of G with $X_1 = [\underline{y}_1, \overline{x}_1]$ and $X_2 = Y_2$. Then, there exists an injective map from $N(g)$ to $N(G)$.*

Proof First, if $N(g) \setminus N(G)$ is empty, then all equilibria of g are also equilibria of G, hence the identity mapping injectively maps $N(g)$ to $N(G)$. Second, suppose that $N(g) \setminus N(G)$ is non-empty. From Proposition 1, it follows that any equilibrium in $N(g) \setminus N(G)$ must lie on $\partial_Y X = \{(x_1, x_2) \in Y : x_1 = \overline{x_1}\}$. Observe that there is exactly one equilibrium in that set, since player 2 has a unique best reply to $\overline{x_1}$. Therefore, $N(g) \setminus N(G) = \{(\overline{x_1}, br_2(\overline{x_1}))\}$.

Define $f : Y_1 \to Y_1$ with $f(y_1) = BR_1(BR_2(y_1))$. Since $(\overline{x_1}, br_2(\overline{x_1})) \notin N(G)$, we have $f(\overline{x_1}) \neq \overline{x}_1$. Furthermore, since $(\overline{x_1}, br_2(\overline{x_1}))$ is a Nash equilibrium of g, we have that $\overline{x_1} = br_1(br_2(\overline{x_1}))$. By Lemma 1, br_2 is the restriction of BR_2 to the domain $X_1 \subseteq Y_1$, hence $br_2(\overline{x_1}) = BR_2(\overline{x_1})$. From previous arguments, it follows that $f(\overline{x}_1) > \overline{x}_1$. For otherwise, we would have $BR_1(BR_2(\overline{x_1})) < \overline{x_1} = br_1(BR_2(\overline{x_1}))$, a contradiction with Lemma 1.

Since $f : [\overline{x}_1, \overline{y}_1] \to [\overline{x}_1, \overline{y}_1]$ is a non-decreasing self-map, it follows from Tarski's fixed point Theorem (Theorem 2.5.1. in Topkis [2, p. 39]) that there exists a y_1^* in $(\overline{x_1}, \overline{y}_1]$ such that $f(y_1^*) = y_1^*$. Thus, we can associate to $(\overline{x_1}, br_2(\overline{x_1})) \in N(g) \setminus N(G)$ the equilibrium $y^* = (y_1^*, BR_2(y_1^*))$ in $N(G) \setminus N(g)$. Therefore, we can construct a mapping that projects all elements of $N(g) \cap N(G)$ onto themselves and projects the unique element of $N(g) \setminus N(G)$ to an element of $N(G) \setminus N(g)$. $\square$

To complete the proof of Theorem 1, proceed as in [1].

Finally, it is easy to construct counter-examples that parallel counter-examples presented in Section 4 of [1]. For instance, note that if restricted strategy spaces are not interval, then Theorem 1 does not hold as the following example shows.

	1	2
1	$2, 2$	$0, 0$
2	$5, 1$	$4, 4$
3	$-\frac{1}{2}, -\frac{1}{2}$	$3, 3$

G

	1	2
1	$2, 2$	$0, 0$
3	$-\frac{1}{2}, -\frac{1}{2}$	$3, 3$

g

Figure 1: Finite games.

References

[1] S. Bade, G. Haeringer, L. Renou, More Strategies, More Nash Equilibria, Forthcoming Journal of Economic Theory.

[2] D. Topkis, Supermodularity and Complementarity, 1998, MIT Press, Cambridge, Massachussets.