Implementation and communication networks

Ludovic Renou

University of Leicester

Nash implementation

Implementation theory is concerned with the design of institutions (mechanisms) to achieve socially desirable outcomes.

Formally, let X be a set of alternatives e.g., candidates, project, allocation of goods, etc.

$N = \{1, \ldots, n\}$ is the set of individuals. Individual i has preference over X represented by $u_i : X \times \Theta \to \mathbb{R}$ parameterized by $\theta \in \Theta$.

In this paper, we assume that θ is commonly known $\Rightarrow$ environment with complete information. (If not, Bayesian implementation.)

Nash implementation

Define $f : \Theta \to 2^X \setminus \emptyset$ the social choice correspondence.

For any $\theta \in \Theta$, $f(\theta) \subseteq X$ represents the set of socially desirable alternatives that one wishes to implement in state $\theta \in \Theta$.

A mechanism is a tuple $\langle (M_i)_{i \in N}, g \rangle$ where M_i is the set of actions of individual i and $g : \times_i M_i \to X$.

Note that the mechanism $\langle (M_i)_{i \in N}, g \rangle$ together with a profile of utilities indexed by θ induces the strategic-form game

$$G(\theta) := \langle N, (M_i, u_i(g(\cdot), \theta)_{i \in N} \rangle.$$

Nash implementation (Maskin(1999))

The mechanism $\langle (M_i)_{i \in N}, g \rangle$ implements the social choice correspondence f in *Nash equilibrium* if for each $\theta \in \Theta$,

- For each $x \in f(\theta)$, there is a *pure* equilibrium s^* of $G(\theta)$ such that $g(s^*) = x$, and
- if σ^* is a mixed equilibrium of $G(\theta)$, then $g(m^*) \in f(\theta)$ for any m^* in the support of σ^*.

Example 1: Hiring a candidate

There are two candidates, a and b, for a professorship in Economics and the Vice Chancellor wishes to hire the candidate that is preferred by a majority of the Economics department composed of 1, 2 and 3. The department has two possible (strict) preference profiles θ and θ':

θ			θ'		
1	2	3	1	2	3
a	b	a	b	a	b
b	a	b	a	b	a

The social choice correspondence is therefore $f(\theta) = \{a\}$ and $f(\theta') = \{b\}$.

Can we implement f?

Example 1 Continued

YES. A simple mechanism to implement f is as follows:

- Ask each individual to report either a or b i.e., $M_i = \{a, b\}$.
- Select the alternative that individual 1 announces i.e., $g(m_1, m_2, m_3) = m_1$. (1 is a dictator.)

More generally, we have:

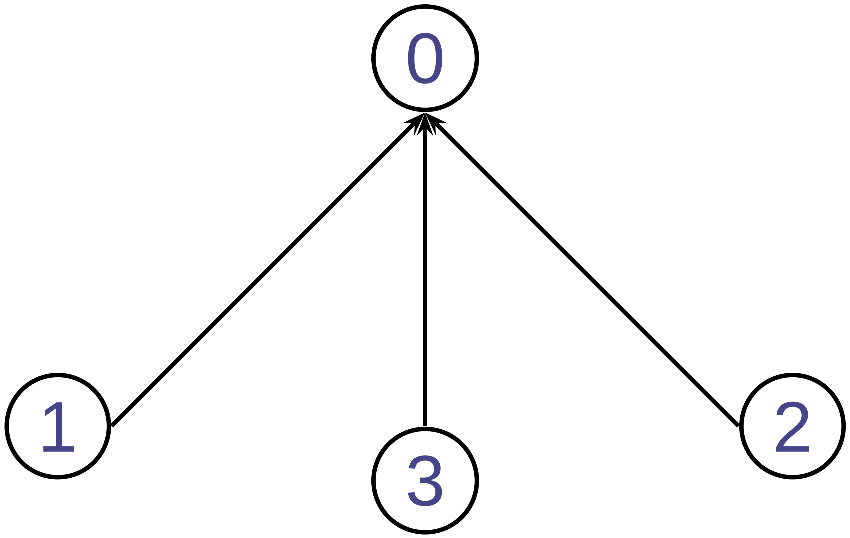
Theorem 1 (Maskin (1999)) *Let $n \geq 3$. If the social choice correspondence f satisfies monotonicity and no-veto power, then f is implementable in Nash equilibrium.*

For necessary and sufficient conditions: Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991).

Communication networks

A key assumption in most of the implementation literature is that each individual can **directly** communicate with the designer.

The communication network is the star network $\mathcal{N}^{\star}$ with the designer as the center.



The aim of this project to completely characterize the social choice correspondences that are implementable on general communication networks $\mathcal{N}$ (hierarchies in organizations).

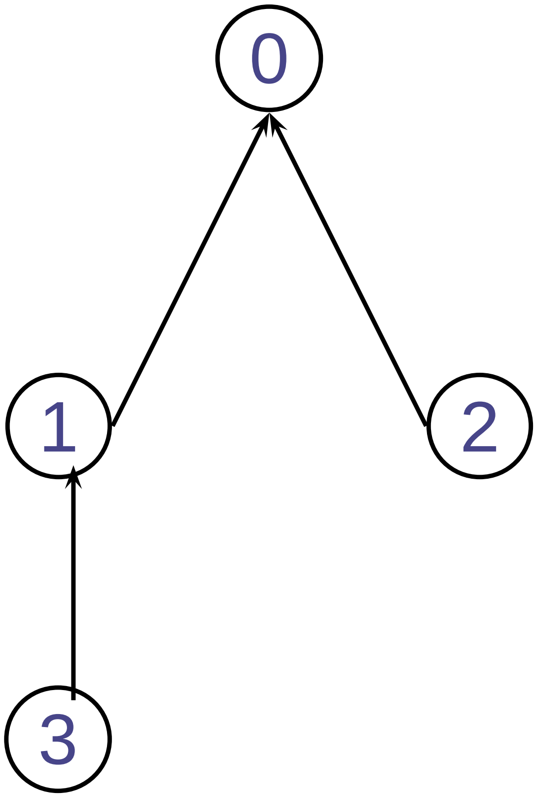
How to define general communication networks?

Communication network

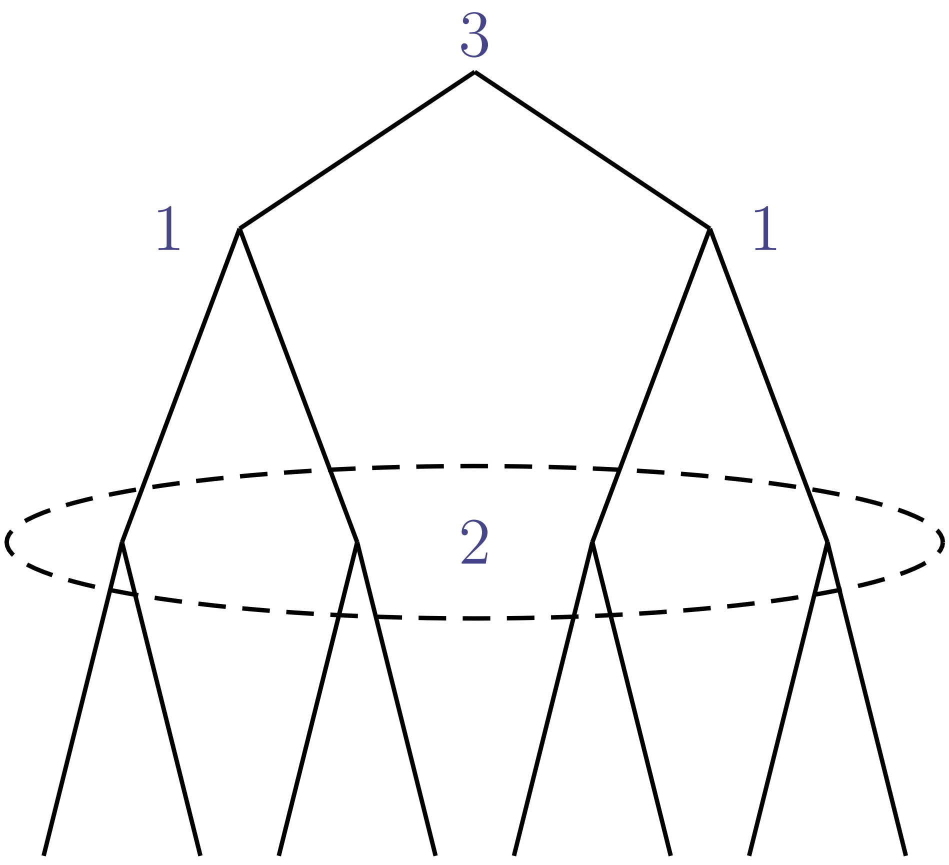
- $\mathcal{N}$ is a directed graph with vertices $N \cup \{0\}$; the designer is 0.
- An edge between i and j is denoted ij.
- $C(i) = \{j \in N \cup \{0\} : ij \in \mathcal{N}\}$.
- $D(i) = \{j \in N \cup \{0\} : ji \in \mathcal{N}\}$.
- The graph $\mathcal{N}$ is acyclic. No self-loops.
- $D(0) \neq \emptyset$, $C(0) = \emptyset$

Acyclicity of $\mathcal{N}$ implies that there exists a timing structure: we can associate with each network $\mathcal{N}$ an extensive-form game $G_{\mathcal{N}}$.

An example

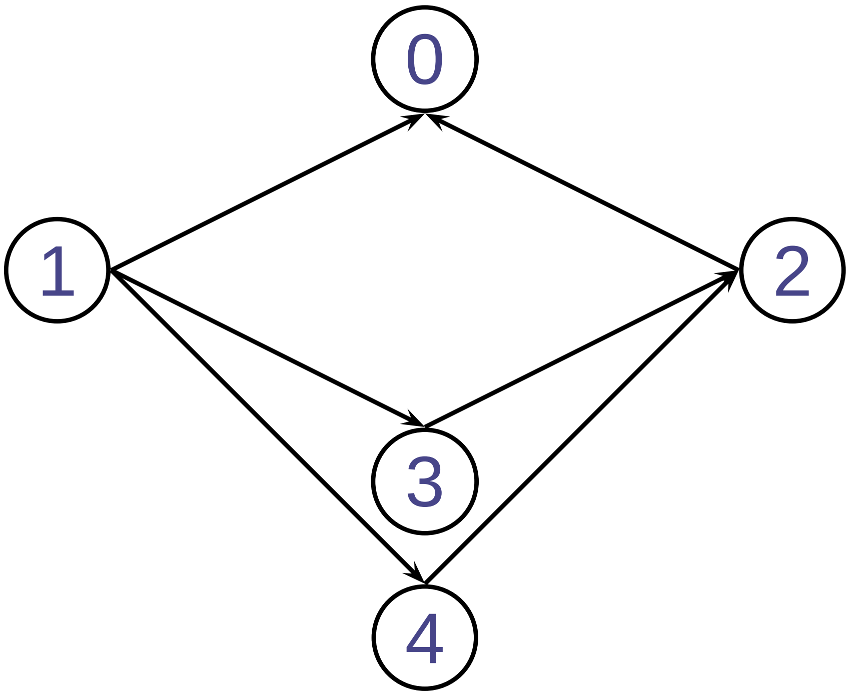


$\mathcal{N}$

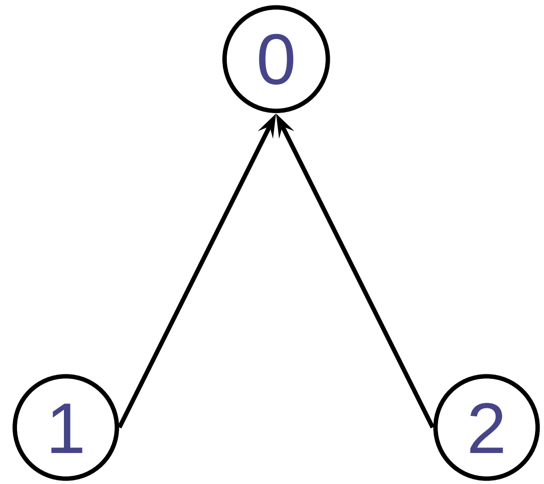


$G_{\mathcal{N}}$

For any network $\mathcal{N}$, define $\mathcal{N}(D(0))$ as the star network with $|D(0)| + 1$ vertices and 0 as the center.



$\mathcal{N}$



$\mathcal{N}(D(0))$

Strategies and implementation

Fix a communication network $\mathcal{N}$ and mechanism $\langle (M_i)_{i \in N}, g \rangle$ with $g : \times_{i \in D(0)} M_i \rightarrow X$. The allocation rule g can only depend on the messages the designer receives.

A pure strategy for player $s_i : M_{D(i)} \rightarrow M_i$. Let S_i be the set of strategies of player i.

Together with a profile of utilities indexed by θ, the mechanism and the communication network induces the the strategic-form game

$$G_{\mathcal{N}}(\theta) := \langle N, (S_i, u_i(g(\cdot), \theta))_{i \in N} \rangle.$$

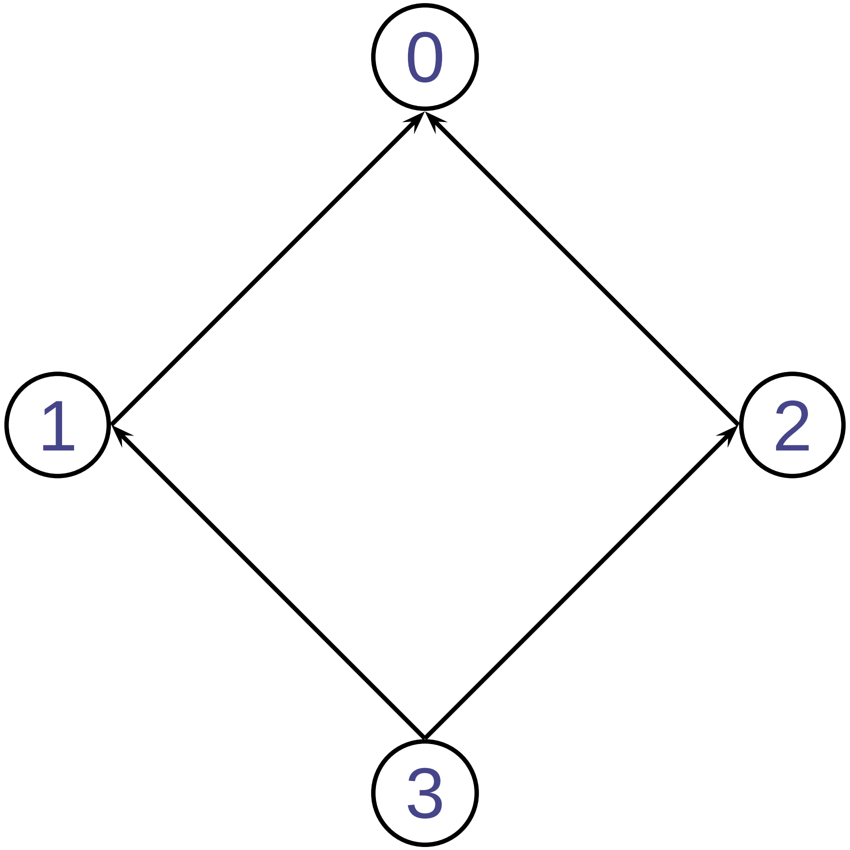
Implementation is then defined as in Maskin (1999).

A remark

Player i can potentially influence the chosen alternative in two different ways:

- Directly if $i \in D(0)$.
- Indirectly if there exists a path from i to some players in $D(0)$, the set of players directly connected to the designer.

Example 1: Continued



$\mathcal{N}$

Example 1: Continued

Can f be implemented on $\mathcal{N}(D(0))$?

The answer is YES.

- Ask players 1 and 2 to report either a or b i.e., $M_1 = M_2 = \{a, b\}$.
- Implement the alternative that 1 announces i.e., $g(m_1, m_2) = m_1$ for any m_1.

Can f be implemented on $\mathcal{N}$?

Trivially, the answer is again YES: since f is implementable on $\mathcal{N}(D(0))$, it can be implemented on $\mathcal{N}$ by simply ignoring the players in $N \setminus D(0)$.

What about the converse?

Example 2

Same as Example 1, but with three candidates and

θ			θ'		
1	2	3	1	2	3
a	c	a	c	b	b
c	b	c	a	c	c
b	a	b	b	a	a

Therefore, $f(\theta) = \{a\}$ and $f(\theta') = \{b\}$.

Example 2: Continued

Can f be implemented on $\mathcal{N}(D(0))$?

The answer is NO.
Suppose that f is implementable on $\mathcal{N}(D(0))$:

- We must have a *pure* Nash equilibrium (m_1^*, m_2^*) of $G_{\mathcal{N}(D(0))}(\theta)$ with $g(m_1^*, m_2^*) = a$, and $u_2(g(m_1^*, m_2^*), \theta) \geq u_2(g(m_1^*, m_2), \theta)$ for all $m_2 \in M_2$.
- Similarly, we must have a pure Nash equilibrium (m_1^{**}, m_2^{**}) of $G_{\mathcal{N}(D(0))}(\theta')$ with $g(m_1^{**}, m_2^{**}) = b$, and $u_1(g(m_1^{**}, m_2^{**}), \theta') \geq u_1(g(m_1, m_2^{**}), \theta')$ for all $m_1 \in M_1$.

This implies that $g(m_1^*, m_2) = a$ for all $m_2 \in M_2$ and $g(m_1, m_2^{**}) = b$ for all $m_1 \in M_1$, which is impossible.

Example 2: Continued

Can f be implemented on $\mathcal{N}$?

Suppose YES. This implies that there exist

- a pure Nash equilibrium (s_1^*, s_2^*, s_3^*) of $G_{\mathcal{N}}(\theta)$ with $g(s_1^*(s_{31}^*), s_2^*(s_{32}^*)) = a$
- a pure Nash equilibrium $(s_1^{**}, s_2^{**}, s_3^{**})$ of $G_{\mathcal{N}}(\theta')$ with $g(s_1^{**}(s_{31}^{**}), s_2^{**}(s_{32}^{**})) = b$.

Let $s_i^*(s_{3i}^*) = m_i^*$ and $s_i^{**}(s_{3i}^{**}) = m_i^{**}$ for $i \in \{1, 2\}$.

It follows that m^* and m^{**} are pure Nash equilibria of $G_{\mathcal{N}(D(0))}(\theta)$ and $G_{\mathcal{N}(D(0))}(\theta')$, respectively, with $g(m^*) = a$ and $g(m^{**}) = b$, a contradiction.

Example 2: Continued

This suggests that if f is implementable on $\mathcal{N}$, then it is implementable on $\mathcal{N}(D(0))$.

Henceforth, the communication network is largely irrelevant despites the rich possibilities of communication (and correlation) it introduces.

Note that Example 2 relies on the impossibility of implementing a and b as pure Nash equilibria of a game between players 1 and 2.

In general e.g., if $|D(0)| \geq 3$, this impossibility does not arise. Use "majority" mechanisms: if $|D(0)| - 1$ players announces a, select a regardless of the remaining player announcement.

The main result

Theorem 2 *The social choice correspondence f is implementable on $\mathcal{N}$ if and only if it is implementable on $\mathcal{N}(D(0))$.*

The intuition for this result is two-fold:

- Communication can only expand the set of equilibrium outcomes: any equilibrium outcome of $G_{\mathcal{N}(D(0))}(\theta)$ is also an equilibrium outcome of $G_{\mathcal{N}}(\theta)$ for any $\theta \in \Theta$: the players in $D(0)$ can simply play as in $G_{\mathcal{N}(D(0))}(\theta)$ regardless of the messages they might receive. These are the babbling equilibria of the cheap talk literature.

- Implementation requires that for each preference profile θ and each alternative $x \in f(\theta)$, there exists a *pure* Nash equilibrium s^* of $G_{\mathcal{N}}(\theta)$ with equilibrium outcome x. Since the actions played by the players directly connected to the designer in a pure Nash equilibrium of $G_{\mathcal{N}}(\theta)$ constitute a *pure* Nash equilibrium of $G_{\mathcal{N}(D(0))}(\theta)$, it follows from the implementation of f on $\mathcal{N}$ that the set of equilibrium outcomes of $G_{\mathcal{N}(D(0))}(\theta)$ contains at least $f(\theta)$ for each $\theta \in \Theta$.

It follows that if f is implementable on $\mathcal{N}$, then f is implementable on $\mathcal{N}(D(0))$.

The converse being clearly true, this completes the "proof."

Further remarks

Consider the class of communication networks with direct communication i.e., $C(i) = \{0\}$ for any $i \in D(0)$. This class of communication networks is important because players in $N \setminus D(0)$ serve the role of correlation devices.

Lemma 1 *Let $\mathcal{N}$ be a communication network with direct communication and $\langle (M_i)_{i \in N}, g \rangle$ a mechanism. For any preference profile $\theta \in \Theta$, a mixed equilibrium of $G_{\mathcal{N}}(\theta)$ is a correlated equilibrium of $G_{\mathcal{N}(D(0))}(\theta)$.*

Intuition: consider the communication network of examples 1 and 2.

Implementation in correlated equilibrium

Theorem 2 together with Lemma 1 implies the following corollary.

Corollary 1 *Let $\mathcal{N}$ be a direct communication network. If the social choice correspondence f is implementable on $\mathcal{N}$ in Nash equilibrium, then f is implementable on $\mathcal{N}(D(0))$ in correlated equilibrium for a subset of correlation devices.*

From Theorem 2, if f is implementable on $\mathcal{N}$, then f is implementable on all communication networks that are equivalent to $\mathcal{N}$ i.e., with the same set $D(0)$. The subset of correlation devices is therefore composed of all correlation devices induced by any mixed equilibrium of any mechanism that implements f on any communication network equivalent to $\mathcal{N}$.

Two concluding remarks

First, the irrelevance result obtained in this paper crucially depends on Maskin's definition of implementation. An alternative definition is:

The mechanism $\langle (M_i)_{i \in N}, g \rangle$ implements the social choice correspondence f if for each $\theta \in \Theta$,

- For each $x \in f(\theta)$, there is a (possibly mixed) equilibrium σ^* of $G_{\mathcal{N}}(\theta)$ such that $g(m^*) = x$ for some m^* in the support of σ^*, and
- if σ^* is a mixed equilibrium of $G_{\mathcal{N}}(\theta)$, then $g(m^*) \in f(\theta)$ for any m^* in the support of σ^*.

This definition respects the spirit of implementation theory: only desirable outcomes can be observed by the designer.

An example

Consider two profiles θ and θ' of strict preferences, the star network $\mathcal{N}^*$, and $f(\theta) = \{a\}$ and $f(\theta') = \{a, b, c, d\}$.

θ		θ'	
1	2	1	2
b	c	a	c
a	a	d	d
c	b	c	a
d	d	b	b

f is **not** monotonic, hence it is not implementable (Maskin (1999)).

However, it is implementable with this new definition of implementation by the mechanism represented below.

	m_1	m_2
m_1	a	b
m_2	d	c

This implies that if f is implementable on $\mathcal{N}$, it might not be implementable on $\mathcal{N}(D(0))$ as a mixed equilibrium on $\mathcal{N}$ might not be a mixed equilibrium on $\mathcal{N}(D(0))$. Rather, it is a correlated equilibrium. (See Mezzetti, Mutsuwami and Renou.)

Second, in Bayesian environments, this equivalence breaks down (Renou and Tomala, “Bayesian implementation and communication networks” in progress).

- Social choice sets implementable on $\mathcal{N}(D(0))$ cannot depend on the types of players in $N \setminus D(0)$ since this is unknown to players in $D(0)$.
- If f is implementable on $\mathcal{N}$, then it is incentive compatible, Bayesian monotonic and satisfies the closure property. Henceforth, it satisfies the necessary conditions for implementation on $\mathcal{N}^\star$.
- If $\mathcal{N}$ is 2-connected, then a f implementable on $\mathcal{N}^\star$ is weakly implementable on $\mathcal{N}$.