Microeconomics 5, Part II
Thursday 11th November 2004.

General instructions:

- This is a closed-book examination. Put away your books, handouts, notes, cellular phones, etc.
- The exam has **4** pages. Make sure you have them all.
- Print your name clearly on the front cover of the exam. Number your pages.
- Each questions indicates its total point score, and the suggested time for your answer. Allocate your time optimally. You have **15 minutes** to read the exam and **two hours** to write your answers.
- Write clearly. Illegible answers will cost you points. Show the step of your math. Use well-labelled diagrams and explain any symbols or notation you introduce.

Good luck.

Question 1: (20 points, 15 minutes).

Define the following terms, using one or two sentences each, and a mathematical formula or a simple diagram where appropriate.

(a) incomplete information

(b) perfect information

(c) Nash equilibrium

(d) correlated equilibrium

(e) subgame

Question 2: (20 points, 25 minutes)

State whether the following statements are **true, false or uncertain**, and in a brief sentence or two, and a mathematical formula or a simple diagram where appropriate, give the reason for your answers. The reasoning is more important than the correct classification.

(a) A rationalizable outcome is a Nash equilibrium.

(b) The set of Nash equilibria of a finite strategic-form game is non-empty.

(c) Let G be a two-player symmetric game with for all $i \in \{1,2\}$, $X_i = [0,1]$, and

$$u_i(x_i, x_{-i}) = (\frac{1}{2} - x_i - x_{-i})x_i,$$

the (real) payoff of player i. The best-reply BR_i of player i in G is

$$BR_i(x_{-i}) = \frac{1 - 2x_{-i}}{4}.$$

(d) In any Nash equilibrium of a finitely repeated prisoner dilemma, the terminal history is $h^* = (\emptyset, (C,C), (C,C), \ldots, (C,C))$.

(e) Let Γ be a finite extensive-form game. Γ has a unique subgame-perfect equilibrium.

Question 3: (20 points, 30 minutes)

Prove **two** out of the four following statements.

(1) Let $G :=< \{1,2\}, (A_i, u_i)_{i \in \{1,2\}} >$ be a finite strategic-form game. For any $a_i^* \in A_i$, the following statements are equivalent:

(a) a_i^* is strictly dominant.
(b) a_i^* strictly dominates every mixed action $\alpha_i \neq \alpha_i^*$, with α_i^* the mixed

action corresponding to the pure action a_i^*.
(c) For every conjecture $\mu^i \in \Delta(A_{-i})$, a_i^* is the unique best-reply to μ^i.

(2) Let G be a strategic-form game. A profile of strategies is a subgame perfect equilibrium of the δ-discounted infinitely repeated game of G if and only if no player can gain by deviating in a single period after any history.

(3) We say that the strategic-form game $G :< N, (X_i, u_i)_{i \in N} >$ is a simple game with strategic complements if for each player $i \in N$, (i) X_i is a compact, non-empty subset of $\mathbb{R}$, (ii) u_i is continuous, (iii) the best-reply $BR_i : X_{-i} \to 2^{X_i}$ is single-valued, and (iv) for all $x_i \in X_i$, $x'_i \in X_i$ such that $x_i \geq x'_i$, for all $x_{-i} \in X_{-i}$, $x'_{-i} \in X_{-i}$ such that $x_{-i} \geq x'_{-i}$, we have

$$u_i(x_i, x_{-i}) - u_i(x'_i, x_{-i}) \geq u_i(x_i, x'_{-i}) - u_i(x'_i, x'_{-i}). \tag{SC}$$

Show that the set of pure Nash equilibria of any simple game with strategic complements is non-empty. [HINT: you might want to first prove that the best-reply BR_i of any player $i \in N$ is non-decreasing.]

(4) Consider a simple game G with strategic complements as in (3), but with $\sharp N = 3$. Suppose that, in addition, G satisfies that for all $i \in N$, for all $x_{-i} \in X_{-i}$, $x'_{-i} \in X_{-i}$ such that $x_{-i} \geq x'_{-i}$, for all $x_i \in X_i$, we have

$$u_i(x_i, x_{-i}) \geq u_i(x_i, x'_{-i}). \tag{ND}$$

Show that the set of pure Nash equilibria of G is Pareto-ordered. [HINT: you might want to use the property that best-replies are non-decreasing in a game with strategic complements to prove that the set of Nash equilibria is completely ordered, that is, for any two Nash equilibria x^* and x^{**} of G, we have either $x^* \geq x^{**}$ or $x^* \leq x^{**}$.]

Question 4: (40 points, 50 minutes)

A simple strategy reduction game. Consider a two-stage game Γ with observable actions (almost-perfect information) and two players. Each player has $[0,1]$ as their set of available actions. In the first stage, the two players simultaneously commit to a subset of their available actions i.e., a subset of $[0,1]$. We assume that player i can commit either to a singleton $\{x_i\} \subset [0,1]$ or to $[0,1]$, but cannot commit to something else. For instance, a player cannot commit to $[\frac{1}{4}, \frac{3}{4}]$ while he can commit to $\{\frac{1}{4}\}$. In the second stage, players simultaneously choose an action in the set of their first-stage committed actions. Let (x_i, x_{-i}) be a profile of actions played in the second-stage game of Γ. The payoff of player i is then $u_i(x_i, x_{-i})$. We assume that u_i is continuous and strictly quasi-concave in x_i.

For later use, denote BR_i the best-reply of player i in the strategic-form game

$$G :=< \{1,2\}, ([0,1], u_i)_{i \in \{1,2\}} > .$$

(1) [6 points]. Define carefully Γ. What is a strategy in Γ?

(2) [8 points]. Prove that Γ has a subgame perfect equilibrium.

(3) [8 points]. Prove that, at any **equilibrium path** of Γ, for at least one player $i \in N$, his second-stage equilibrium action is the best-reply $(BR_i(x_{-i}))$ in G to his opponent second-stage equilibrium action x_{-i}.

(4) [12 points]. Find all subgame perfect equilibria of Γ.

(5) [6 points]. Discuss your results.

Bonus question : (20 points)

Prove the two statements you didn't prove in question 3.