Multi-lender coalition in CSV models: A Mathematical Tool

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In this technical appendix, we prove a theorem used in the paper "Multi-lender coalition in CSV models" to prove the existence of a non-trivial equilibriium.

1 A mathematical tool

Degree theory is a natural tool to establish the existence of zeros of a mapping, and more broadly the existence of Nash equilibrium (Wilson (1971), Harsanyi (1973), Govindan and Wilson (1997)) or competitive equilibrium (e.g. Debreu (1974)). Loosely speaking, the degree of a mapping at 0 with respect to a bounded, open set counts the solution in that set in a particular way. Two continuous mappings have the same degree at 0 if they do not point into opposite directions at the boundary. The following theorem states that a continuous real-valued function on a compact product set $X = \prod_{i \in N} X_i$ with X_i a non-empty compact, convex subset of the real line for each $i \in N$, which satisfies some boundary conditions, has a zero on the interior X. Without loss of generality, we assume that each X_i is the unit interval $[0, 1]$. We refer the reader to Zeidler (1986), chapters 12-15 for the concepts introduced in the proof of Theorem A.

Theorem A *Let* $f : \mathrm{int}\,[0,1]^n \to \mathbb{R}^n$ *be a continuous mapping. If for any* $x = (x_1, \ldots, x_i, \ldots, x_n) \in [0,1]^n$ *such that* $x_i = 0$*,* $f_i(x) \geq 0$*, for any*

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$x = (x_1, \ldots, x_i, \ldots, x_n) \in [0,1]^n$ such that $x_i = 1$, $f_i(x) \leq 0$, then f has a zero in the interior of $[0,1]^n$.

Proof First, let us define the mapping $g : [0,1]^n \to \mathbb{R}^n$ with $g_i(x) = \frac{1}{2} - x_i$ for all $i \in \{1, \ldots, n\}$. Obviously, g_i is continuous, if $x_i = 0$, $g_i(x) = 1/2 > 0$, and if $x_i = 1$, $g_i(x) = -1/2 < 0$. It is also clear that g admits a unique zero $x = 1/2$ in the interior of $[0,1]^n$.

Second, define the homotopy $h(x, \lambda) : [0,1]^n \times [0,1] \to \mathbb{R}^n$ with $h(x, \lambda) = \lambda f(x) + (1 - \lambda) g(x)$ and the set $Z = \{x \in \operatorname{int}[0,1]^n : h(x, \lambda) = 0, \lambda \in [0,1]\}$. We want to show that the closure of Z, $\operatorname{cl} Z$ does not intersect the boundary $\partial [0,1]^n$ of $[0,1]^n$. By contradiction. Consider a sequence $(x_n, \lambda_n)_{n \in \mathbb{N}} \subset Z \times [0,1]$ converging to a point $(x^*, \lambda^*) \in \partial Z \times [0,1]$. Since $x^* \in \partial [0,1]^n$, x^* has at least one component which is either 0 or 1. Suppose $x_i = 0$. It follows that $h_i(x^*, \lambda^*) < 0$, hence by continuity, there exists a $\bar{n}$ such that for all $n \geq \bar{n}$, $h(x_n, \lambda_n) < 0$, contradicting $x_n \in Z$. A similar applies if $x_i = 1$. Hence, we have that $\operatorname{cl} Z \cap \partial [0,1]^n = \emptyset$.

Third, let us choose the open set U with $\operatorname{cl} Z \subset U$ and $\operatorname{cl} U \subset \operatorname{int} X$. As $\partial U \cap Z = \emptyset$, we have that $d(f, 0, U) = d(g, 0, U)$, where $d(f, 0, U)$ is the degree of f at 0 with respect to the open set U. Since $x = 1/2 \in Z \subset U$, we have $d(g, 0, U) = +1$, hence f also admits a zero.

□

As a direct corollary, we have:

Corollary A *Let $f : \operatorname{int}[0,1]^n \to \mathbb{R}^n$ be a continuous mapping. If for any $x = (x_1, \ldots, x_i, \ldots, x_n) \in [0,1]^n$ such that $x_i = 0$, $f_i(x) \leq 0$, for any $x = (x_1, \ldots, x_i, \ldots, x_n) \in [0,1]^n$ such that $x_i = 1$, $f_i(x) \geq 0$, then f has a zero in the interior of $[0,1]^n$.*

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