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6

A capital asset pricing model with time-varying betas: some results from the London Stock Exchange

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6.1 INTRODUCTION

According to the capital asset pricing model (CAPM) developed originally by Sharpe (1964) and Lintner (1965), the required excess return on a risky asset is proportional to its non-diversifiable risk, for which a sufficient statistic is the covariance of the asset return with the return on the market portfolio. In the case where this covariance is zero, the risk is completely diversifiable and the required excess return over the safe rate of return is zero. Empirical tests of the CAPM have generally yielded results unfavourable to the model in its simplest form. In particular, variables other than the covariance of the return with the market return have been found to be significant in explaining the excess return—variables ranging from the own return variance to seasonal dummies (see Jensen (1972) or Schwert (1983) for surveys).

Following doubts cast on such empirical studies by Fama (1977, Chapter 9), Roll (1977) demonstrated that a theoretically correct implementation of the CAPM is practically impossible. This is because, in the theoretical derivation of the CAPM, the market portfolio is assumed to be mean-variance efficient in the sense of Markowitz (1952). In particular, such a portfolio would involve investment in every individual existing asset with an optimal weight. In practice, researchers have generally employed stock (e.g housing, human capital). According to Roll's critique, virtually no empirical study is capable of rejecting the CAPM.

Another explanation for these empirical findings is simply that the CAPM is incorrect and that another pricing theory should be used such as the arbitrage pricing theory (APT) of Ross (1976) or the consumption-based CAPM due to Breeden (1979).

In this chapter, we pursue a third line of inquiry into the empirical failure of the CAPM—namely that the relevant distributional moments for the CAPM are not the unconditional covariances of returns, but the *conditional* covariances. Thus, asset betas—the ratio of the covariance of the asset return with the market return to the variance of the market return—may be random and time varying rather than constant as in the traditional formulation. It seems reasonable, since agents must form expectations concerning these moments, that conditional covariances, given information up to the time that the expectation is formed, should be the relevant measures of risk for the CAPM. If, in addition, excess returns are characterized by conditional heteroscedasticity (e.g. Klemkosky and Martin, 1975), time variation in asset betas is implied.

In what follows, we model the conditional heteroscedasticity of excess returns using the autoregressive conditional heteroscedasticity (ARCH) model of Engle (1982) and its generalization—the generalized ARCH (GARCH) formulation—due to Bollerslev (1986). The models are estimated on monthly UK data for the period March 1975 to June 1987. To some extent our analysis is similar to the US study by Bollerslev *et al.* (1988). However, we also test this version of the CAPM against a more general model which allows expected returns to depend on time-varying conditional covariances with consumption. We are able to reject both the traditional CAPM and the consumption-based CAPM of Breeden in favour of a model which makes the risk premiums depend on a composite measure of non-diversifiable risk.

6.2 THE CAPITAL ASSET PRICING MODEL

The Sharpe–Lintner formulation of the CAPM states that the expected or required excess return on an asset over the safe asset return is proportional to the required excess return on the market portfolio, with the factor of proportionality being equal to the asset beta:

$$E(R_i) - r = \beta_i\{E(R_m) - r\} \qquad (6.1)$$

where R_i and R_m are the one-period returns on the asset and the market portfolio respectively and r is the one-period safe rate of return. A natural extension of the model is to condition the moments of 6.1 and 6.2 on information available to agents at the end of period $t - 1$ when formulating required returns during period t:

$$E(R_{it}|\Omega_{t-1}) - r_{t-1} = \beta_{it}\{E(R_{mt}|\Omega_{t-1}) - r_{t-1}\} \qquad (6.3)$$

$$\beta_{it} = \frac{\operatorname{cov}(R_{it}, R_{mt}|\Omega_{t-1})}{\operatorname{var}(R_{mt}|\Omega_{t-1})} \qquad (6.4)$$

where Ω_{t-1} is the information set at time $t - 1$.

Apart from the problem of parameterizing the conditional covariances in 6.4, which we shall address in the next section, 6.3 is non-operational because of the lack of an observed series for the required or expected market rate of return one period ahead. If, however, we assume that the 'market price of risk' λ, defined as

$$\lambda = \frac{E(R_{mt}|\Omega_{t-1}) - r_{t-1}}{\operatorname{var}(R_{mt}|\Omega_{t-1})} \qquad (6.5)$$

is a constant, we have

$$E(R_{mt}|\Omega_{t-1}) - r_{t-1} = \lambda \operatorname{var}(R_{mt}|\Omega_{t-1}) \qquad (6.6)$$

so that parameterization of the process generating the conditional second moments will be sufficient.

If the conditional variance of the market return is a sufficient statistic for the risk attached to the market portfolio, then λ simply measures how many units of excess market return are required to compensate for one unit of market risk. It is in this sense that it measures the market price of risk. Under certain regularity conditions, Merton (1980) shows that the market price of risk is the weighed harmonic mean of investors' coefficients of relative risk aversion, with the weights given by each investor's share in aggregate wealth. Thus, in a simple representative agent model, λ would itself be the coefficient of relative risk aversion. Throughout this chapter we assume λ to be a constant. This considerably simplifies the analysis and seems reasonable given the relatively short data period—roughly 10 years—that we examine.

Using 6.3, 6.4 and 6.6, we can write

$$R_{it} = r_{t-1} + \lambda \operatorname{cov}(R_{it}, R_{mt}|\Omega_{t-1}) + \varepsilon_{it} \qquad (6.7)$$

$$R_{mt} = r_{t-1} + \lambda \operatorname{var}(R_{mt}|\Omega_{t-1}) + v_t \qquad (6.8)$$

errors

$$\varepsilon_{it} = R_{it} - E(R_{it}|\Omega_{t-1}) \qquad (6.9)$$

$$v_t = R_{mt} - E(R_{mt}|\Omega_{t-1}) \qquad (6.10)$$

It is clear from 6.9 and 6.10 that the relevant conditional second moments are themselves equal to the forecast error variances and covariances:

$$\text{var}(R_{mt}|\Omega_{t-1}) = E(v_t^2|\Omega_{t-1}) \qquad (6.11)$$

$$\text{cov}(R_{it}, R_{mt}|\Omega_{t-1}) = E(\varepsilon_{it}v_t|\Omega_{t-1}) \qquad (6.12)$$

Equations 6.7, 6.8, 6.11 and 6.12 form the basis of the variant of the CAPM estimated in this chapter.

In the next section we suggest a tractable parameterization of the conditional second moments.

6.3 PARAMETERIZING THE CONDITIONAL SECOND MOMENTS

Engle *et al.* (1987) suggest an extension of Engle's (1982) ARCH model whereby the conditional first moment of a time series itself becomes a function of the conditional second moment, which follows an ARCH process:

$$Y_t = \alpha' x_t + \delta h_t^2 + \varepsilon_t$$

$$h_t^2 = E(\varepsilon_t^2|\Omega_{t-1}) = \gamma_0 + \sum_{i=1}^{n} \gamma_i \varepsilon_{t-i}^2 + \kappa' z_t$$

where x_t and z_t are vectors of weakly exogenous conditioning variables. Engle *et al.* (1987) term this kind of model ARCH in mean or ARCH-M.

A straightforward multivariate extension of the ARCH-M model can be applied to the CAPM formulation of the previous section as follows. We establish the following notation:

$$R_t = (R_{it} - R_{mt})'$$

$$\omega_t = (\varepsilon_t\ v_t)'$$

$$\iota = (1\ 1)'$$

$$e = (0\ 1)'$$

$$H_t = \begin{bmatrix} \text{var}(R_{it}|\Omega_{t-1}) & \text{cov}(R_{it}, R_{mt}|\Omega_{t-1}) \\ \text{cov}(R_{it}, R_{mt}|\Omega_{t-1}) & \text{var}(R_{mt}|\Omega_{t-1}) \end{bmatrix}$$

Then the ARCH-M formulation of 6.7, 6.8, 6.11 and 6.12 is

$$\text{vech}(H_t) = A_0 + \sum_{i=1} A_i \text{vech}(\omega_{t-i}\omega'_{t-i}) \qquad (6.14)$$

where vech( ) denotes the column-stacking operator of the lower triangular portion of a symmetric matrix and the A_i are 3×3 coefficient matrices except for A_0 which is a 3×1 coefficient vector. The system 6.13 and 6.14 describes a multivariate ARCH-M(n) model.

A further extension of the ARCH formulation, which imposes smoother behaviour on the conditional second moments, has been suggested by Bollerslev (1986). In Bollerslev's GARCH formulation, the conditional second moments are functions of their own lagged values as well as the squares and cross-products of lagged forecast errors. Thus, for example, the GARCH-M(n, p) formulation of the above model would consist of 6.13 and

$$\text{vech}(H_t) = A_0 + \sum_{i=1}^{n} A_i \text{vech}(\omega_{t-i}\omega'_{t-i}) + \sum_{i=1}^{p} B_i \text{vech}(H_{t-i}) \qquad (6.15)$$

where the B_i are 3×3 coefficient matrices. Below, we present estimates of both ARCH-M and GARCH-M formulations of CAPM.

It should be noted that, although there now exists considerable empirical evidence that financial asset prices are characterized by ARCH behaviour (e.g. Dickens, 1987), the ARCH parameterization of the conditional second moments does not appear to have any immediate economic rationale. It should therefore be interpreted in much the same spirit as autoregressive moving-average (ARMA) time series models, i.e. as a convenient and parsimonious representation of the behaviour of time series data.

Stacking all the parameters of the system into a single vector

$$\mu = (\lambda, (A_0)', \text{vech}(A_1)', \ldots, \text{vech}(A_n)', \text{vech}(B_1)', \ldots, \text{vech}(B_p)')'$$

and applying Schweppe's (1965) prediction error decomposition form of the likelihood function, we obtain the log likelihood for a sample of T observations (conditional on initial values) as

$$L(\mu) = \sum_{t=1}^{T} \log|H_t(\mu)| - \sum_{t=1}^{T} \omega_t' H_t^{-1}(\mu)\omega_t \qquad (6.16)$$

where we have assumed normality of the forecast errors.

Although the analytic derivatives of 6.16 can be computed (Engle *et al.*, 1987) variable-metric algorithms which employ numerical derivative are simpler to use and easily allow changes in specification, and this approach was applied here. Under the usual regularity conditions (Crowder, 1976), maximization of 6.16 will yield maximum likelihood estimates with the usual properties.

As they stand, the above formulations are very general and contain a large number of parameters to be estimated, which may be problematic given the non-linearities of the system. As a first step, it seems reasonable to assume that each covariance depends only upon its own past values and surprises. Accordingly, we assume that the A_i and B_i matrices are diagonal. In addition, since inversion of any GARCH model of non-zero order implies an infinitely long memory with respect to past surprises, we limit the estimated GARCH models to first order, i.e. GARCH-M(1, 1). For the ARCH-M model, we set $n = 8$ and, following Engle (1982), assume that agents linearly discount past surprises in forming expectations of future forecast error variances and, furthermore, that these discount factors are the same in each of the three ARCH equations:

$$\text{vech}(H_t) = A_0 + A_1\{c_i \,\text{vech}(\omega_{t-i}\omega'_{t-i})\} \tag{6.17}$$

$$c_i = \frac{9-i}{36} \qquad i = 1, 2, \ldots, 8$$

or, for the GARCH model,

$$\text{vech}(H_t) = A_0 + A_1\,\text{vech}(\omega_{t-1}\omega'_{t-1}) + B_1\,\text{vech}(H_{t-1}) \tag{6.18}$$

Apart from estimating the ARCH-M and GARCH-M formulations of the CAPM, we also subject the CAPM specification to a range of diagnostic tests.

6.4. DATA

We used beginning-of-month data on the share prices of firms quoted on the London Stock Exchange to define one-month *ex post* rates of return. Data on dividend payments per share were used to adjust the change in share prices to give the one-month holding-period returns. Value-weighted data on the returns from companies in particular sectors were aggregated to form the returns on four portfolios. The four portfolios were for the following sectors (numbers of individual stocks in each portfolio in parentheses): mechanical engineering (58), financial (122), electrical (12) and chemical (20).

The return on the market portfolio was taken as the percentage change in the FT 500 share index (beginning month to beginning month) adjusted for dividend payments. The dividends adjustment is crude because data on the exact timing of dividend payments by each firm in the index are not easily available. Annual average (value-weighted) dividend yields of the firms in the index were used to calculate estimates of the monthly dividend yields assuming that the timing of payments was uniformly distributed throughout the year. We used the one-month yield on the UK Treasury Bill as our measure of the safe rate of interest.

Data were collected from March 1975 to June 1987. Allowing for lags this gives an estimation period beginning in December 1975, yielding 139 observations.

6.5 EMPIRICAL RESULTS

The model to be estimated comprises the CAPM equation 6.13 and the accompanying ARCH or GARCH error process 6.17 or 6.18. We shall present two sets of estimates here. First we present a set of estimates for the four sectors chemical (C), electrical (E), mechanical (M) and financial (F), based on a set of two-equation ARCH-M models. These results do not incorporate the restriction that the price of risk λ should be constant across all four sectors. We shall then present estimates for a five-equation system, comprising all four sectors and the market equation, using a GARCH-M error process. This imposes the cross-equation parameter restriction on all sectors simultaneously and also allows a free estimate of the rate at which individuals discount observed past errors rather than imposing a fixed eight-month period, as in the case of our ARCH-M estimates.

We begin, in Table 6.1, by reporting the estimates of the ARCH-M model for the four sectors estimated independently. The lower half of the table presents a range of statistical tests for normality of the errors and for serial correlation. All the models passed all the tests for normality and for absence of serial correlation in the errors. However, only one of the estimated models demonstrates a significant ARCH coefficient: A_1 for the mechanical sector is significantly different from zero while in the other three sectors it is insignificant and numerically very small, which suggests that the conditional covariance matrix of the error term may not be time varying. Apart from this result the performance of the models is fairly satisfactory.

One possible explanation of the finding of a constant conditional covariance matrix may be that the imposed linear weighting pattern over eight-months is in fact too restrictive. In particular, this weighting pattern puts a very large emphasis on the first two or three months in the lag structure; in fact the mean lag is only about two and a half months. Thus, in these estimates we are effectively forcing the model to choose between a very short memory, which produces a rapidly changing conditional covariance matrix, or a constant covariance matrix, but nothing between the two. It may be that the conditional covariance matrix is changing over time but only fairly slowly. To investigate this we need to estimate the type of lag structure in a less restrictive way. The GARCH-M model allows us to do this in a reasonably parsimonious way. Furthermore, there is an intuitive interpretation of the GARCH(1, 1) model given by 6.18: expectations of conditional moments are updated in the light of new information with the weight given to the latest outcome equal to A_1. In additon, taking advantage of the parsimony of the GARCH formulation, we also estimate all four sectors simultaneously so that the price of risk λ is constrained to be equal across all the markets.[1]

We impose the restriction that A_1 and B_1 in 6.18 are scalars rather than vectors, but that A_0 is a full vector of 15 constants. In words, we assume

Table 6.1 ARCH-M estimates of the four sectors

	Chemical		*Electrical*		*Mechanical*		*Financial*	
λ	5.0	(7.8)	4.9	(7.6)	3.1	(4.4)	5.1	(7.8)
$A_0(\varepsilon_t, \varepsilon_t)$	0.0033	(18.5)	0.0042	(23.8)	0.0002	(0.9)	0.0028	(16.3)
$A_0(\varepsilon_t, v_t)$	0.0022	(8.6)	0.0007	(10.5)	0.0001	(0.3)	0.00044	(7.5)
$A_0(v_t, v_t)$	0.0023	(18.6)	0.0024	(23.5)	0.0005	(6.4)	0.0022	(15.8)
A_1	0.047	(1.4)	0.0000001	(0.00)	0.985	(6.0)	0.06	(1.7)
SK_1	0.39		0.09		−0.33		−0.01	
$KURT_1$	0.35		0.18		0.24		0.66	
BJ_1	4.64		0.44		3.1		2.6	
$LB(1)_1$	0.19		1.8		1.9		0.04	
$LB(2)_1$	1.05		1.8		2.4		1.11	
$LB(4)_1$	1.54		3.6		5.5		1.98	
$LB(8)_1$	8.00		6.8		18.1		4.5	
$LB(16)_1$	12.73		18.6		21.6		8.23	
SK_2	−0.2		−0.19		−0.19		−0.2	
$KURT_2$	0.32		0.32		0.24		0.32	
BJ_2	1.63		1.68		1.34		1.63	
$LB(1)_2$	0.32		0.33		0.31		0.32	
$LB(2)_2$	0.70		0.69		0.81		0.71	
$LB(4)_2$	3.05		3.02		3.08		3.06	
$LB(8)_2$	9.62		9.67		8.34		9.59	
$LB(16)_2$	14.59		14.58		13.49		14.6	

SK, coefficient of skewness, critical value (95%) ≈ 0.41.
KURT, excess kurtosis, critical value (95%) ≈ 0.81.
BJ, Bera–Jarque normality test ($\sim\chi^2(2)$ under null).
LB(N), Ljung–Box statistic for serial correlation ($\sim\chi^2(N)$ under null).
Subscripts 1, 2, sector equation and market equation respectively.
The numbers in parentheses are the asymptotic t ratios.

that across a range of stocks people attach the same relative importance to past events in forming expectations about volatility of prices, i.e. people use similar forecasting rules for similar forecasting problems. A common assumption made in GARCH models is that the parameter matrices are diagonal; this is not appropriate in this model as the covariance terms are central to the analysis.

When we allow unrestricted parameter matrices, the question of the positive semi-definiteness of H_t arises. This is discussed by Baba *et al.* (1987), who suggest the following restriction to impose positive semi-definiteness on the model:

$$H_t = A_0'A_0 + A_1'\omega_{t-1}\omega_{t-1}'A_1 + B_1'H_{t-1}B_1$$

where A_0, A_1 and B_1 are symmetric-parameter matrices. In our case A_1 and B_1 are scalars rather than vectors, so that we only need restrict A_0.

Estimation of this five-equation (four sectors and the market index) model by maximum likelihood then gives the following set of parameter estimates:

$$\hat{\lambda} = 3.24 \quad (3.1)$$
$$\hat{A}_1 = 0.027 \quad (4.1)$$
$$\hat{B}_1 = 0.956 \quad (119.6)$$

$$\hat{A}_0 = \begin{bmatrix} -0.006\ (7.5) & & & & \\ 0.001\ (0.9) & 0.005\ (4.0) & & & \\ -0.0004\ (0.1) & 0.0003\ (0.4) & 0.005\ (8.4) & & \\ -0.0009\ (0.5) & 0.000006\ (0.0) & -0.0005\ (1.1) & 0.003\ (2.6) & \\ -0.001\ (22.2) & 0.0002\ (0.9) & 0.0003\ (1.7) & -0.02\ (1.0) & -0.002\ (4.6) \end{bmatrix}$$

where the asymptotic t ratios are in parentheses.

Table 6.2 presents evidence on the properties of the model's five sets of residuals. In this model the estimate of the price of risk λ is reasonably close

[1] In order to interpret the GARCH CAPM model 6.13 and 6.18 for this multi-equation system, simply re-interpret some of the notation. In particular,

$$R_t = (R_{1t}\ R_{2t}\ R_{3t}\ R_{4t}\ R_{mt})'$$
$$\omega_t = (\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\ \varepsilon_{4t}\ v_t)'$$
$$\iota = (1\ 1\ 1\ 1\ 1)'$$
$$e = (0\ 0\ 0\ 0\ 1)'$$
$$H_t = \mathrm{cov}(R_t|\Omega_{t-1})$$

where the numbers in the subscripts refer to the four sectors examined.

Table 6.2 Diagnostics for multivariate GARCH residuals

	Chemical	*Electrical*	*Mechanical*	*Financial*	*Market*
SK	0.37	0.06	− 0.35	−0.18	− 0.26
KURT	0.38	0.11	0.21	0.39	0.14
BJ	4.33	0.19	3.3	1.83	1.88
LB(1)	0.33	1.49	2.0	0.05	0.63
LB(2)	0.91	1.51	2.7	0.39	0.76
LB(4)	1.37	3.77	5.6	0.94	2.33
LB(8)	6.34	7.40	18.0	2.77	7.09
LB(16)	11.08	20.03	22.2	5.50	12.42

See Table 6.1 for definitions of diagnostic test statistics.

to those in the simpler ARCH model, although somewhat lower than the average across the four sectors. Our estimates of the price of risk are also reasonably close to those obtained for US data by French *et al.* (1987), who in effect estimate only the market equation 6.8, allowing for univariate GARCH-M. They are at variance, however, with the point estimate of 0.499 obtained by Bollerslev *et al.* using US data, although their estimate does seem intuitively rather small. The residual diagnostics show that all the error processes are consistent with normality with no serious problems of serial correlation. We now find a highly significant role for the time-varying part of the error process. This is demonstrated by the t statistics on A_1 and B_1. The sum of A_1 and B_1 is close to, but below, unity which gives finite long-run forecasts for moments (Engle, 1987). The mean lag is estimated to be around 22 months, which confirms the suggestion made earlier that a much longer time horizon was needed in the ARCH process.

The estimates of the matrix of constants A_0 has the interesting property that it is almost diagonal; all the diagonal elements are significant while only one of the off-diagonal elements is significant. This suggests that the unconditional covariance matrix is actually diagonal. This amounts to the statement that agents expect a non-zero variance on all sectors even in the long run but that they do not have a prior expectation about the long-run covariances of the system. The model actually performs fairly well if the complete A_0 matrix is dropped, although this restriction is clearly rejected by the data (likelihood ratio test statistic of 77.5 ($\sim\chi^2(15)$ under the null). However, we decided to investigate the possibility of a diagonal constant matrix A_0. The likelihood ratio test for restricting the A_0 matrix to be diagonal is 6.8 ($\sim\chi^2(10)$ under the null) which is easily accepted. Therefore this form of the GARCH error process is our preferred model. The parameter estimates for the restricted model are

$$\hat{\lambda} = 3.25\ (3.0)$$

$$\hat{A}_1 = 0.031\ (4.8)$$

$$\hat{B}_1 = 0.955\ (142.0)$$

$$\hat{A}_0 = \begin{bmatrix} 0.004\ (5.2) & & & & \\ & 0.002\ (4.4) & & & \\ & & 0.006\ (8.8) & & \\ & & & 0.005\ (5.5) & \\ & & & & 0.005\ (8.1) \end{bmatrix}$$

where the asymptotic t ratios are in parentheses. The diagnostics for the model residuals are given in Table 6.3.

Table 6.3 Diagnostics for restricted multivariate GARCH residuals

	Chemical	*Electrical*	*Mechanical*	*Financial*	*Market*
SK	0.37	0.07	− 0.34	−0.02	− 0.26
KURT	0.38	0.11	0.22	0.4	0.15
BJ	4.37	0.2	3.3	1.8	1.86
LB(1)	0.33	1.47	2.0	0.04	0.62
LB(2)	0.92	1.50	2.7	0.4	0.75
LB(4)	1.38	3.79	5.6	0.94	2.34
LB(8)	6.34	7.42	18.0	2.78	7.11
LB(16)	11.9	20.0	22.2	5.52	12.46

See Table 6.1 for definitions of diagnostic test statistics.

The residual diagnostics again indicate that the model is free from serial correlation and has a normally distributed error process. The parameter estimates have changed very little from the unrestricted model. This model shows an important element of time variation in the determination of the covariance matrix and again the mean lag is just under 2 years.

Figures 6.1–6.4 show the time-varying betas for each of the four sectors, i.e. the ratio of the expected covariance between the sector error and the market error divided by the expected variance of the market error at each point in time (Equation 6.2). Figure 6.5 shows the expected variability of our diversified portfolio—the FT 500. These figures show that, despite the rather long lag structure, the conditional covariance matrix can still change in a fairly sudden fashion, although the actual range of the fluctuations is not very great. Figure 6.5 reveals that 1977 and 1981–2 were perceived as being particularly risky periods with the expected variances from holding the diversified portfolio around twice as high as the average for the whole period. In the 1980s there appears to have been a significant reduction in the correlation between returns in the financial sector and for the whole portfolio. Further, by the end of the period the risk from holding a financial

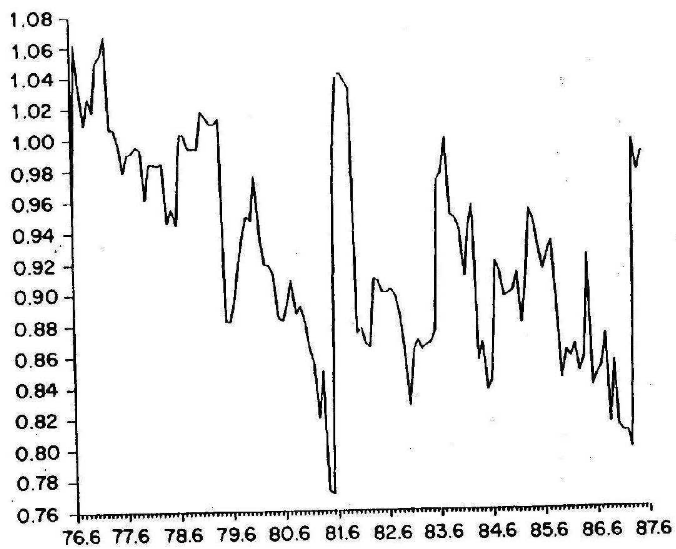

Fig. 6.1 Chemical sector beta.

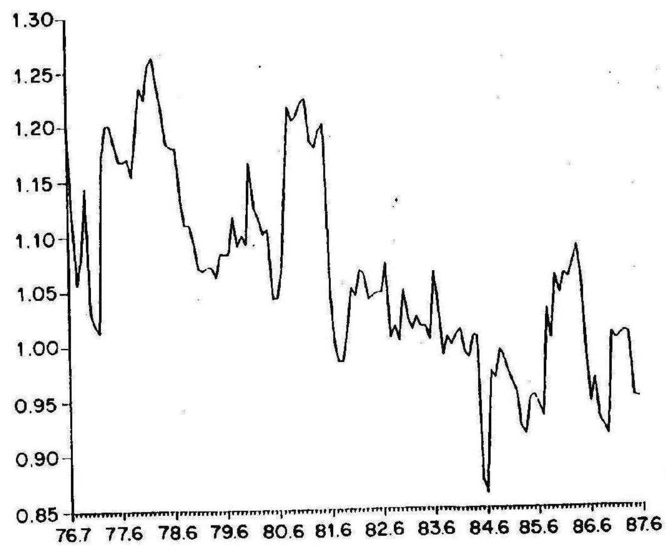

Fig. 6.2 Electrical sector beta.

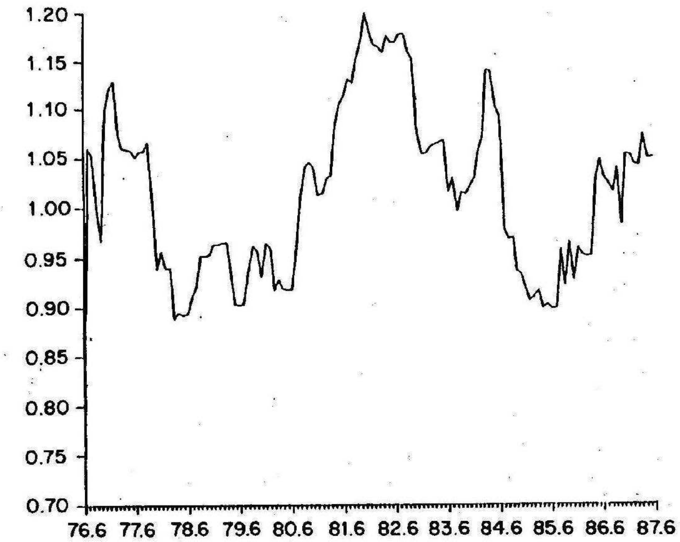

Fig. 6.3 Mechanical sector beta.

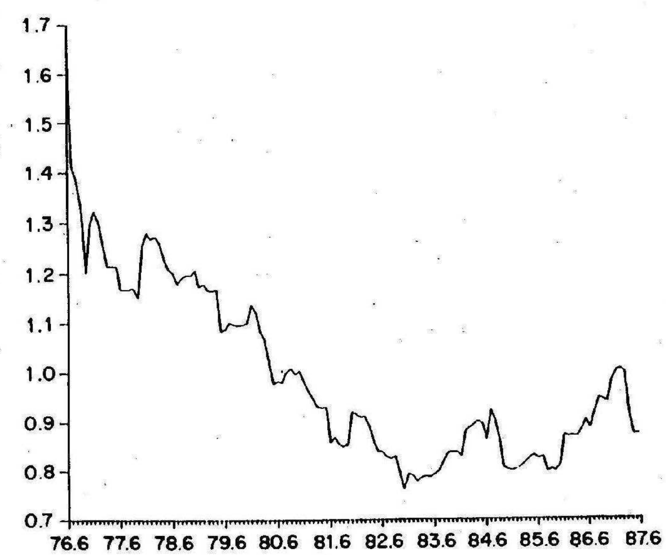

Fig. 6.4 Financial sector beta.

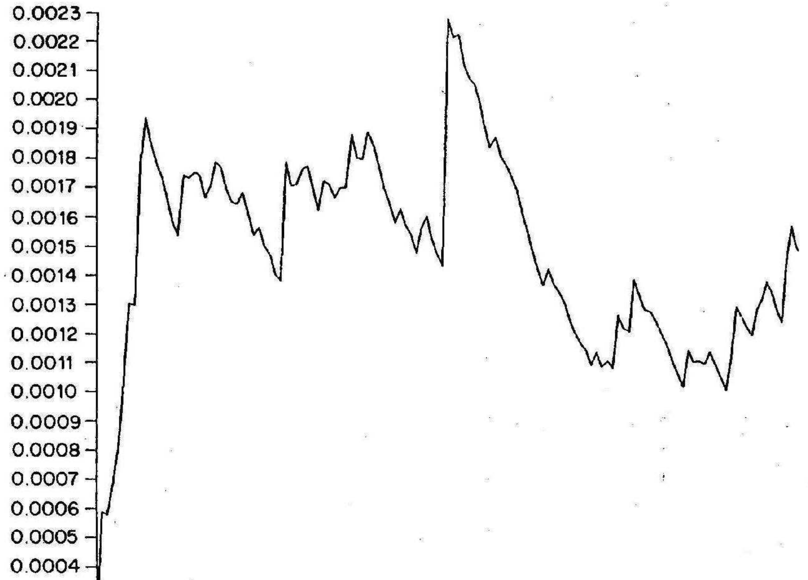

Fig. 6.5 The variance of the market rate.

sector portfolio is more easily diversified than with the other sectors. This may reflect the recent importance of shocks specific to the financial sector—e.g. less developed country (LDC) debt developments—which have little impact on the non-financial sectors of the economy.

6.6 MORE GENERAL MEASURES OF RISK

The CAPM model estimated above takes the covariance of the return on a portfolio with the market return as a sufficient measure of the risk attached to that portfolio. This is only valid if the market portfolio is *the* efficiently diversified portfolio. In practice, it is unlikely that the volatility of the stock market index is a good proxy for the risk of an efficiently diversified portfolio; we noted above that many assets in personal sector portfolios are omitted from the stock market portfolio (e.g. housing and human capital). Breeden (1979) has developed a model in which a proxy for the volatility of

the (unobserved) value of the portfolio of wealth of individuals is used. His insight is that, with efficient capital markets, intertemporal optimizing decisions by individuals would make consumption expenditure closely linked to the present value of the total wealth of individuals. (Here wealth would include human capital and the expected productivity of investments at future dates.) With diminishing marginal utility of consumption, individuals will want to smooth expenditure through time, and changes in real consumption will be linked to changes in the perceived value of wealth. Indeeed, under certain strong conditions, there is an exact relation between changes in comsumption and unexpected fluctuations in the value of portfolios of assets (see Breeden (1979) for further details). A natural measure of the risk of a particular asset then becomes the covariance between the value of that asset and changes in consumption.

Breeden derives an equilibrium condition, analogous to our Equation 6.7, which we can write as

$$E(R_{it}|\Omega_{t-1}) = r_{t-1} + \lambda \operatorname{cov}(R_{it}, \Delta c_t|\Omega_{t-1}) \tag{6.19}$$

where Δc_t is the change in the natural logarithm of aggregate real consumption and λ is now a weighted average of individual coefficients of relative risk aversion whose weights depend on consumption levels. The major difference between 6.19, which we might call a consumption-based CAPM, and the model discussed in the previous section is in the measure of risk.

We estimated a general model of returns on assets using both measures of risk. This model can be written

$$R_{it} = r_{t-1} + \lambda_m E\{\operatorname{cov}(R_{it}, R_{mt}|\Omega_{t-1})\} + \lambda_c E\{\operatorname{cov}(R_{it}, \Delta c_t|\Omega_{t-1})\} + \varepsilon_{it} \tag{6.20}$$

were λ_m is the price of market-related risk (which is the same as λ above) and λ_c is the price of consumption-related risk.

In practice, neither covariance term in 6.20 will be likely to prove a sufficient measure of risk; we can interpret the coefficients λ_m and λ_c as revealing the contribution of each proxy to predicting actual, but unobservable, perceived risk. We model expected covariances using the same GARCH(1, 1) specification as above. We need to append to our earlier model an equation describing the expected change in consumption so as to derive proxies for conditional covariances between consumption and asset prices. The Hall (1978) consumption function is in the spirit of the consumption CAPM and suggests a natural specification of Δc_t as a random walk with drift:

$$\Delta c_t = c_0 + u_t \tag{6.21}$$

where c_0 is a constant and u_t is the unexpected element of the change in consumption.

We now have a six-equation system with c_0 and the risk parameters λ_c and λ_m to estimate. As above, we impose equality of these risk parameters across equations. We also adopt a similar model of the processes generating expected variances and covariances, with key parameters A_1 and B_1 as above. Writing out the model in full we have

$$R_t = r_{t-1}\iota + \lambda_m H_t e_1 + \lambda_c H_t e_2 + w_t$$

$$\Delta c = c_0 + u_t$$

$$\text{vech}(H_t) = A_0' A_0 + A_1 \text{vech}(\tilde{w}_{t-1}\tilde{w}_{t-1}') + B_1 \text{vech}(H_{t-1})$$

The notation is as before except for the following:

$$e_1 = (0\ 0\ 0\ 0\ 1\ 0)'$$

$$e_2 = (0\ 0\ 0\ 0\ 0\ 1)'$$

$$\tilde{w}_t = (\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\ \varepsilon_{4t}\ v_t\ u_t)'$$

and H_t is a 6×6 matrix. $\tilde{w}_t$ is a vector of residuals with the first four elements the residuals from the asset portfolios, the fifth the total stock market residual and the final element the consumption shock. As before,

$$w_t = (\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\ \varepsilon_{4t}\ v_t)'$$

6.7 EMPIRICAL RESULTS FOR THE GENERAL CAPITAL ASSET PRICING MODEL

Consumption data are not available monthly and we were forced to use the percentage change in an index of real monthly retail sales as a proxy for Δc_t. Other data are as above. The estimated parameter values are as follows:

$$\hat{\lambda}_m = 4.23 \quad (3.5)$$

$$\hat{\lambda}_c = 26.54 \quad (3.6)$$

$$\hat{A}_1 = 0.083 \quad (5.1)$$

$$\hat{B}_1 = 0.390 \quad (5.7)$$

$$\hat{c}_0 = 0.0026 \quad (3.0)$$

$\hat{A}_0 =$

Chemical	Electrical	Mechanical	Financial	Market	Consumption
0.34 (5.6)					
0.15 (2.7)	0.31 (6.6)				
0.19 (3.0)	−0.0003 (0.01)	0.21 (4.8)			
0.02 (0.4)	0.26 (7.5)	0.28 (8.6)	0.12 (1.5)		
0.13 (10.1)	0.14 (9.6)	0.14 (9.3)	0.13 (8.1)	0.24 (14.2)	
0.00 (0)	0.001 (0.2)	−0.004 (0.1)	−0.002 (0.2)	0.01 (2.2)	0.11 (11.7)

where the asymptotic t ratios are in parentheses. The diagnostics for the model residuals are given in Table 6.4.

Table 6.4 Diagnostic tests on general model residuals

	Chemical	*Electrical*	*Mechanical*	*Financial*	*Market*
SK	0.42	0.08	−0.33	−0.05	−0.19
KURT	0.40	0.17	0.27	0.62	0.31
BJ	5.19	0.33	3.14	2.41	1.56
LB(1)	0.13	1.52	1.37	0.03	0.33
LB(2)	1.47	1.59	1.55	1.27	0.79
LB(4)	2.13	3.66	5.10	2.16	3.01
LB(8)	8.42	6.87	17.74	4.41	9.53
LB(16)	13.52	19.33	21.58	8.16	14.65

See Table 6.1 for definitions of diagnostic test statistics.

Both λ_m and λ_c are positive and significantly different from zero; our empirical versions of both the original Sharpe–Lintner CAPM and the consumption CAPM can therefore be rejected in favour of a more general model where the two different measures of risk are both relevant. It is notable that in this extended CAPM model the parameters of the GARCH process are significantly different from those in the earlier models. The weight attached to past history is smaller than in earlier models, whilst the martix of constants A_0 reveals a large number of significant off-diagonal elements implying that unconditional covariances are generally well determined and significantly different from zero. Nonetheless A_1 and B_1 are both significantly different from zero, revealing clear signs of time variations in conditional variances and covariances and hence in risk premiums. Model residuals from the portfolios of assets appear consistent with the hypotheses that they follow serially uncorrelated and normally distributed processes.

6.8. CONCLUSION

In this chapter we have estimated versions of the CAPM which allow expectations of the risk characteristics of portfolios to change over time. The results suggest that perceptions of risk are time varying, but that agents update their perceptions relatively slowly; memories are long and a fairly small weight is placed on the most recent random element in returns. Despite this, and because of the high volatility of one-month returns, natural measures of perceived risk—betas—do show significant short-term variation. Extended versions of the CAPM which allow different measures of undiversifiable risk were estimated and suggested that, whilst significant time variation in risk premiums still exists, no single measure of risk appears adequate.

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